

Some formulas for the restricted r -Lah numbers

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Abstract

The r -Lah numbers, which we denote here by $\ell^{(r)}(n, k)$, enumerate partitions of an $(n+r)$ -element set into $k+r$ contents-ordered blocks in which the smallest r elements belong to distinct blocks. In this paper, we consider a restricted version $\ell_m^{(r)}(n, k)$ of the r -Lah numbers in which block cardinalities are at most m . We establish several combinatorial identities for $\ell_m^{(r)}(n, k)$ and obtain as limiting cases for large m analogous formulas for $\ell^{(r)}(n, k)$. Some of these formulas correspond to previously established results for $\ell^{(r)}(n, k)$, while others are apparently new also in the r -Lah case. Some generating function formulas are derived as a consequence and we conclude by considering a polynomial generalization of $\ell_m^{(r)}(n, k)$ which arises as a joint distribution for two statistics defined on restricted r -Lah distributions.

Keywords: restricted Lah numbers, polynomial generalization, r -Lah numbers, combinatorial identities

MSC: 11B73, 05A19, 05A18

1. Introduction

Sequences enumerating certain kinds of finite set partitions in which the smallest r elements are required to belong to distinct blocks are often referred to as r -sequences. Examples that have been studied previously include the r -Stirling numbers [4, 14] of the first and second kind, r -Lah numbers [16] and r -derangement numbers [8, 20]. See also [15] for an r -generalization of the partial Bell polynomials.

A *restricted* version of a counting sequence is one in which the block sizes of the underlying structure are at most a fixed number. Restricted Stirling numbers of both kinds (see, e.g., [6, 10, 11]) have been previously considered in conjunction with incomplete versions of the poly-Cauchy [9] and poly-Bernoulli [10] numbers. See [13] for a common polynomial analogue of the restricted Stirling and Lah numbers.

In this paper, we consider a generalization of the r -Lah numbers, denoted by $\ell_m^{(r)}(n, k)$, by requiring that no block size exceeds m . Note that $\ell_m^{(r)}(n, k)$ reduces to the classical Lah numbers (see, e.g., [12]) when $r = 0$ and $m > n - k$. The limiting case as $m \rightarrow \infty$ coincides with the r -Lah numbers studied in [1, 3, 16] and the configurations enumerated in this case are referred to as r -Lah distributions. See [17] for a related polynomial generalization and also [5] where r -Whitney-Lah numbers are introduced. Here, we will study r -Lah distributions with the added restriction that no block size exceeds m . This restriction causes the underlying counting sequence to behave somewhat differently than in the limiting case, which will be manifested by the formulas in the following sections. See [2] where r -Lah distributions are studied in which the block sizes are bounded from below.

This paper is organized as follows. In the next section, we make some preliminary definitions and find a basic recurrence satisfied by $\ell_m^{(r)}(n, k)$ where only the n and k parameters are changing. In the third section, we find some further recurrence formulas for $\ell_m^{(r)}(n, k)$ in which one or both of m and r are changing as well. Taking m large in these formulas recovers prior r -Lah identities in some cases and apparently new identities for these numbers in others. We make use mostly of combinatorial arguments to establish our results, sometimes drawing upon the inclusion-exclusion principle and other times defining a direct bijection between the related structures. An explicit formula is derived in the fourth section in terms of binomial coefficients from which one obtains as a corollary an expression for the generating function. In the final section, a polynomial generalization of $\ell_m^{(r)}(n, k)$ is introduced and some of its properties discussed.

2. Definition and basic recurrence

If m and n are positive integers, then let $[m, n] = \{m, m + 1, \dots, n\}$ for $m \leq n$, with $[m, n] = \emptyset$ if $m > n$, the $m = 1$ case of which will simply be denoted by $[n]$. Given $n, k, r \geq 0$ and $m \geq 1$, let $\mathcal{L}_m^{(r)}(n, k)$ denote the set of partitions of $[n + r]$ into $k + r$ contents-ordered blocks (i.e., lists) in which the elements of $[r]$ belong to distinct blocks and all blocks have size at most m . For example, if $n = m = 2$ and $k = r = 1$, then

$$\mathcal{L}_2^{(1)}(2, 1) = \{1/23, 1/32, 12/3, 21/3, 13/2, 31/2\}.$$

We will refer to the elements of $[r]$ within a member of $\mathcal{L}_m^{(r)}(n, k)$ as *special* and use this also to describe the blocks to which they belong. Elements of $[r + 1, r + n]$ and also the blocks composed solely of these elements will be referred to as *non-special*.

For example, in $15/62/83/47/9 \in \mathcal{L}_2^{(2)}(7, 3)$, the blocks 15 and 62 are special, while the blocks 83, 47 and 9 are non-special.

Let $|\mathcal{L}_m^{(r)}(n, k)| = \ell_m^{(r)}(n, k)$. We will denote the limiting case of $\ell_m^{(r)}(n, k)$ as $m \rightarrow \infty$ (in particular, if $m > n - k$) by $\ell^{(r)}(n, k)$. Then $\ell^{(r)}(n, k)$ counts the set of all r -Lah distributions of size $n + r$ having $k + r$ blocks (which will be denoted by $\mathcal{L}^{(r)}(n, k)$), as there is no restriction on the block cardinalities when $m > n - k$. When $r = 0$ and $m > n - k$, members of $\mathcal{L}_m^{(r)}(n, k)$ coincide with the usual Lah distributions enumerated by [19, Sequence A008297] as there is no restriction also on block membership.

The recurrences below will be based on the following initial conditions. First, let $\ell_m^{(r)}(n, k) = 0$ if any of the parameters are negative or if $0 \leq n < k$. If $m = 0$, then we will assume $\ell_0^{(r)}(n, k) = 0$ for all n and k if $r > 0$, or if $r = 0$ and n and k are not both zero, with $\ell_0^{(0)}(0, 0) = 1$. If $r = 0$, then an inclusion-exclusion argument gives

$$\ell_m^{(0)}(n, k) = \frac{n!}{k!} \sum_{i=0}^{\lfloor \frac{n-1}{m} \rfloor} (-1)^i \binom{k}{i} \binom{n - mi - 1}{k - 1}, \quad n, k, m \geq 1,$$

with $\ell_m^{(0)}(n, 0) = \delta_{n,0}$ for all $m \geq 1$. If $n = 0$, then $\ell_m^{(r)}(0, k) = \delta_{k,0}$ for all $m, r \geq 1$. Furthermore, the factorial of a negative number will always be taken to be 1 for convenience and the binomial coefficient $\binom{n}{k}$ will be assumed to be zero if n or k is negative or if $k > n \geq 0$.

We now give perhaps the simplest recurrence satisfied by the $\ell_m^{(r)}(n, k)$.

Proposition 2.1. *If $n, m \geq 1$ and $k, r \geq 0$, then*

$$k\ell_m^{(r)}(n, k) = nk\ell_m^{(r)}(n - 1, k) + n\ell_m^{(r)}(n - 1, k - 1) - \frac{n!}{(n - m - 1)!} \ell_m^{(r)}(n - m - 1, k - 1). \tag{2.1}$$

Proof. Note that we may assume $k \geq 1$ in (2.1), for it clearly holds if $k = 0$. The left side of (2.1) then counts all “marked” members of $\mathcal{L}_m^{(r)}(n, k)$ wherein one of the non-special blocks is marked. Alternatively, in the case when the marked non-special block is not a singleton, one can set aside an element of $[r + 1, r + n]$ and then add it at the beginning of the block that is marked within a member of $\mathcal{L}_m^{(r)}(n - 1, k)$, yielding $nk\ell_m^{(r)}(n - 1, k)$ possibilities. However, one would need to subtract $(m + 1)! \binom{n}{m+1} \ell_m^{(r)}(n - m - 1, k - 1)$ which accounts for the case when adding the extra element results in a block of size $m + 1$. On the other hand, if the marked non-special block is a singleton, then there are $n\ell_m^{(r)}(n - 1, k - 1)$ possibilities, and combining this case with the prior gives (2.1). \square

Remark 2.2. The case of (2.1) when $r = 0$ and $m \rightarrow \infty$ is given in [18, Formula 3.5], where a q -generalization in terms of a statistic on Laguerre configurations is provided.

We observe now the following further special values of $\ell_m^{(r)}(n, k)$. If $m = 1$, then $\ell_1^{(r)}(n, k) = \delta_{n,k}$ for all $n, k, r \geq 0$, so it will often be assumed in proofs that $m \geq 2$. If $k = n$, then we have $\ell_m^{(r)}(n, n) = 1$ for all $m \geq 1, r \geq 0$. If $k = n - 1$ where $n \geq 1$, then considering whether a special or a non-special block has cardinality two implies $\ell_m^{(r)}(n, n - 1) = n(2r + n - 1)$ if $m \geq 2$. Finally, if $k = n - 2$ where $n \geq 2$, then considering several cases concerning the blocks that are not singletons gives the formula

$$\ell_m^{(r)}(n, n - 2) = \begin{cases} 2r(2r + 1)\binom{n}{2} + 6(2r + 1)\binom{n}{3} + 12\binom{n}{4}, & \text{if } m \geq 3; \\ 4r(r - 1)\binom{n}{2} + 12r\binom{n}{3} + 12\binom{n}{4}, & \text{if } m = 2. \end{cases}$$

3. Properties of restricted r -Lah numbers

The $\ell_m^{(r)}(n, k)$ are also defined by the following recurrences, where m and/or r is changing as well.

Proposition 3.1. *If $n, m \geq 1$ and $k, r \geq 0$, then*

$$\begin{aligned} \ell_m^{(r)}(n, k) &= \ell_m^{(r)}(n - 1, k - 1) + (n - 1 + k + 2r)\ell_m^{(r)}(n - 1, k) \\ &\quad - (m + 1)! \binom{n - 1}{m} \ell_m^{(r)}(n - m - 1, k - 1) \\ &\quad - r(m + 1)! \binom{n - 1}{m - 1} \ell_m^{(r-1)}(n - m, k) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\ell_m^{(r)}(n, k) \\ &= n! \sum_{i=0}^k \sum_{j=0}^r \frac{m^j}{i!(n - mi - (m - 1)j)!} \binom{r}{j} \ell_{m-1}^{(r-j)}(n - mi - (m - 1)j, k - i). \end{aligned} \tag{3.2}$$

Proof. To show (3.1), first observe that there are $\ell_m^{(r)}(n - 1, k - 1)$ members of $\mathcal{L}_m^{(r)}(n, k)$ such that $n + r$ comprises its own block. Otherwise, the element $n + r$ directly follows some member of $[n + r - 1]$ or occurs at the very beginning of a block containing at least one other element. This yields altogether $(n - 1 + k + 2r)\ell_m^{(r)}(n - 1, k)$ possibilities if the block sizes were unrestricted. However, one needs to subtract $(m + 1)! \binom{n - 1}{m} \ell_m^{(r)}(n - m - 1, k - 1)$ and also $r(m + 1)! \binom{n - 1}{m - 1} \ell_m^{(r-1)}(n - m, k)$ to account for the cases when $n + r$ is added, respectively, to a non-special or special block already containing m elements. Combining the previous cases then gives (3.1).

To show (3.2), let i and j denote, respectively, the number of non-special and special blocks of size m within $\lambda \in \mathcal{L}_m^{(r)}(n, k)$. Then there are

$$\frac{1}{i!} \binom{n}{m, \dots, m, m - 1, \dots, m - 1, n - mi - (m - 1)j} (m!)^{i+j}$$

$$= \frac{n!m^j}{i!(n - mi - (m - 1)j)!}$$

ways in which to choose and order the elements occupying these blocks of λ , where it is assumed that $mi + (m - 1)j \leq n$, and $\binom{r}{j}$ choices for the special blocks that are to contain m elements. The rest of the blocks of λ all have size at most $m - 1$ and hence there are $\ell_{m-1}^{(r-j)}(n - mi - (m - 1)j, k - i)$ ways to arrange the remaining elements of $[n + r]$. Considering all possible i and j gives (3.2). \square

Remark 3.2. Letting $m > n$ in (3.1) gives [16, Theorem 3.1], while letting $r = 0$ and $m > n$ in (3.2) gives a Lah analogue of the Stirling number identity found in [11, Proposition 3].

From (3.1), we get the following identity for $n > k \geq 0$:

$$\begin{aligned} \ell_m^{(r)}(n, k) &= \sum_{i=0}^k (n + k + 2r - 2i - 1) \ell_m^{(r)}(n - i - 1, k - i) \\ &\quad - (m + 1)! \sum_{i=0}^k \left(\binom{n - i - 1}{m} \ell_m^{(r)}(n - m - i - 1, k - i - 1) \right. \\ &\quad \left. + r \binom{n - i - 1}{m - 1} \ell_m^{(r-1)}(n - m - i, k - i) \right). \end{aligned} \tag{3.3}$$

To show (3.3), one can induct on k (starting with $k = 0$) and use (3.1) to show that the $(n - 1, k)$ case of the identity implies the $(n, k + 1)$ case for all n and k .

Theorem 3.3. *If $n \geq 1$, $m \geq 2$ and $k, r \geq 0$, then*

$$\begin{aligned} \ell_m^{(r)}(n, k) &= \frac{n!}{k!} \sum_{i=0}^r \sum_{j=0}^i \sum_{s=0}^k \frac{(k - s + j)!}{(n - ms - r + i)!} \binom{k}{s} \binom{r}{j, i - j, r - i} \\ &\quad \times \ell_{m-1}^{(r-i)}(n - ms - r + i, k - s + j). \end{aligned} \tag{3.4}$$

Proof. To enumerate the members of $\mathcal{L}_m^{(r)}(n, k)$, first let i denote the number of special elements that start their respective blocks and let s be the number of non-special blocks of size m . Then there are $\binom{n}{m, \dots, m, n - ms} \frac{(m!)^s}{s!}$ ways in which to select and order the elements within these non-special blocks. Next, let j denote the number of special elements that start non-singleton blocks, where $0 \leq j \leq i$. Now choose i members of $[r]$ to start blocks and from these select j that are not to form singleton blocks, which can be done in $\binom{r}{i} \binom{i}{j}$ ways (note that the other $i - j$ members of $[r]$ are to occur as singletons). Concerning the remaining $r - i$ members of $[r]$ which do not start blocks, we pick ‘‘predecessor’’ elements from the remaining $n - ms$ members of $[r + 1, r + n]$, which can be done in $(n - ms)^{\overline{r-i}}$ ways, where it is implicit that $s \leq \min\{k, \lfloor n/m \rfloor\}$.

At this point, we treat each of these $r - i$ special elements, together with their predecessors, as single (special) elements. We form a partition of these elements,

together with the $n - ms - r + i$ remaining non-special elements, in which there are $k - s + j$ non-special blocks and all blocks are of size less than m , which can be effected in $\ell_{m-1}^{(r-i)}(n - ms - r + i, k - s + j)$ ways. Let λ denote one of the resulting partitions. Then we choose j of the non-special blocks of λ and add, one-per-block, the j special elements that were selected earlier to start non-singleton special blocks, which can be done in $\binom{k-s+j}{j} j!$ ways. We leave unchanged the remaining blocks of λ . Note that all of the special blocks in the resulting partition ρ actually can have size up to m due to the addition of these elements and to the occurrence of the “double” special elements described above, and thus $\rho \in \mathcal{L}_m^{(r-i+j)}(n - ms, k - s)$ (after standardization). Observe that the non-special blocks of ρ are each of size at most $m - 1$ since they correspond to the $k - s$ unselected non-special blocks of λ . Adding the $i - j$ singleton special blocks from above, and also the s non-special blocks of size m , to ρ yields an enumerated member of $\mathcal{L}_m^{(r)}(n, k)$ for the given i, j and s . Note that all members of $\mathcal{L}_m^{(r)}(n, k)$ arise uniquely in this way as i, j and s vary.

Summing over these parameters then implies

$$\begin{aligned} \ell_m^{(r)}(n, k) &= \sum_{i=0}^r \sum_{j=0}^i \sum_{s=0}^k \binom{r}{i} \binom{i}{j} \binom{n}{m, \dots, m, n - ms} \frac{(m!)^s}{s!} (n - ms)^{r-i} \binom{k - s + j}{j} j! \\ &\quad \times \ell_{m-1}^{(r-i)}(n - ms - r + i, k - s + j) \\ &= \sum_{i=0}^r \sum_{j=0}^i \sum_{s=0}^k \frac{n! r!}{(r - i)! j! (i - j)! s!} \cdot \frac{(n - ms)^{r-i}}{(n - ms)!} \cdot \frac{(k - s + j)!}{(k - s)!} \\ &\quad \times \ell_{m-1}^{(r-i)}(n - ms - r + i, k - s + j) \\ &= \frac{n!}{k!} \sum_{i=0}^r \sum_{j=0}^i \sum_{s=0}^k \frac{(k - s + j)!}{(n - ms - r + i)!} \binom{k}{s} \binom{r}{j, i - j, r - i} \\ &\quad \times \ell_{m-1}^{(r-i)}(n - ms - r + i, k - s + j), \end{aligned}$$

as desired. □

Remark 3.4. Letting m be large in (3.4) gives the following apparently new identity for the r -Lah number $\ell^{(r)}(n, k)$:

$$\ell^{(r)}(n, k) = \frac{n!}{k!} \sum_{i=0}^r \sum_{j=0}^i \frac{(k + j)!}{(n - r + i)!} \binom{r}{j, i - j, r - i} \ell^{(r-i)}(n - r + i, k + j). \quad (3.5)$$

Considering whether or not a member of $\mathcal{L}_m^{(r)}(n, k)$ contains any special or non-special singleton blocks leads to the following further recurrences.

Theorem 3.5. *If $n, k \geq 0, r \geq 1$ and $m \geq 2$, then*

$$\ell_m^{(r)}(n, k) = \sum_{i=1}^r (-1)^{i-1} \binom{r}{i} \ell_m^{(r-i)}(n, k)$$

$$+ \frac{n!}{k!} \sum_{i=0}^r \sum_{s=0}^k \frac{(k-s+i)!}{(n-ms-r+i)!} \binom{r}{i} \binom{k}{s} \ell_{m-1}^{(r-i)}(n-ms-r+i, k-s+i) \quad (3.6)$$

and

$$\begin{aligned} \ell_m^{(r)}(n, k) &= \sum_{i=1}^k (-1)^{i-1} \binom{n}{i} \ell_m^{(r)}(n-i, k-i) \\ &+ n! \sum_{i=0}^r \frac{m^i}{(n-k-(m-1)i)!} \binom{r}{i} \ell_{m-1}^{(r-i)}(n-k-(m-1)i, k). \end{aligned} \quad (3.7)$$

Proof. We will assume $n, k \geq 1$, as the formulas will be seen to hold also in the case when n or k is zero. To show (3.6), first note that letting $i = j$ in the proof of (3.4) above gives the cardinality of all members of $\mathcal{L}_m^{(r)}(n, k)$ in which no element of $[r]$ forms its own block and that the double sum expression on the right side of (3.6) corresponds to taking only the $j = i$ term in the j -indexed sum in (3.4). Let $\tilde{\mathcal{L}}_m^{(r)}(n, k)$ denote the subset of $\mathcal{L}_m^{(r)}(n, k)$ whose members contain at least one special singleton block. By subtraction, the difference $\ell_m^{(r)}(n, k) - |\tilde{\mathcal{L}}_m^{(r)}(n, k)|$ gives the cardinality of all members of $\mathcal{L}_m^{(r)}(n, k)$ containing no special singleton blocks. On the other hand, by the principle of inclusion-exclusion, this cardinality is also given by

$$\sum_{i=0}^r (-1)^i \binom{r}{i} \ell_m^{(r-i)}(n, k) = \ell_m^{(r)}(n, k) - \sum_{i=1}^r (-1)^{i-1} \binom{r}{i} \ell_m^{(r-i)}(n, k).$$

Comparing expressions gives $|\tilde{\mathcal{L}}_m^{(r)}(n, k)| = \sum_{i=1}^r (-1)^{i-1} \binom{r}{i} \ell_m^{(r-i)}(n, k)$, which implies (3.6).

To show (3.7), note that by similar reasoning, the first sum on the right-hand side counts all members of $\mathcal{L}_m^{(r)}(n, k)$ containing at least one non-special singleton block. To enumerate those $\lambda \in \mathcal{L}_m^{(r)}(n, k)$ that do not contain any, first suppose that exactly i of the special blocks of λ have size m . Then there are $\binom{r}{i} \binom{n}{m-1, \dots, m-1, n-(m-1)i} (m!)^i = \frac{n!}{(n-(m-1)i)!} \binom{r}{i} m^i$ ways to select and order the elements that comprise these blocks of λ , where $i \leq \lfloor n/(m-1) \rfloor$. Next, we pick k of the remaining elements of $[r+1, r+n]$, which can be done in $\binom{n-(m-1)i}{k}$ ways, and set them aside. We then arrange the rest of the $n-k-(m-1)i$ elements of $[r+1, r+n]$, together with the $r-i$ unchosen elements of $[r]$, according to a member of $\mathcal{L}_{m-1}^{(r-i)}(n-k-(m-1)i, k)$. Then we add the k non-special elements that were set aside to the beginning of the non-special blocks of this partition, one-per-block, according to an arbitrary permutation of $[k]$, to obtain λ . Thus, we get

$$\begin{aligned} &n! \sum_{i=0}^r \frac{m^i}{(n-(m-1)i)!} \binom{r}{i} \binom{n-(m-1)i}{k} k! \ell_{m-1}^{(r-i)}(n-k-(m-1)i, k) \\ &= n! \sum_{i=0}^r \frac{m^i}{(n-k-(m-1)i)!} \binom{r}{i} \ell_{m-1}^{(r-i)}(n-k-(m-1)i, k) \end{aligned}$$

members of $\mathcal{L}_m^{(r)}(n, k)$ that do not contain a non-special singleton block, which gives (3.7). \square

We were unable to find in the literature the r -Lah identities corresponding to the limiting cases of (3.6) and (3.7) as $m \rightarrow \infty$. Before stating the next result, let

$$L_m^{(r)}(n, k) = \sum_{j=k}^n \binom{j}{k} \ell_m^{(r)}(n, j) \quad \text{and} \quad x^{\bar{k}} = x(x+1) \cdots (x+k-1),$$

for a variable x . We have the following relationship between $\ell_m^{(r)}(n, k)$ and its upper binomial transform $L_m^{(r)}(n, k)$.

Theorem 3.6. *If $n, k, m, p \geq 1$ and $r \geq 0$, then*

$$\sum_{j=1}^n (j+p+2r)^{\bar{k}} \ell_m^{(r)}(n, j) = \sum_{i=0}^n \sum_{s=0}^p (i+s)! \binom{p}{s} \ell^{(r)}(k, i+s) L_m^{(r)}(n, i). \quad (3.8)$$

Proof. First note that $(j+p+2r)^{\bar{k}}$ counts partitions of $[k]$ into $j+p+2r$ labeled, contents-ordered blocks in which some of the blocks may be empty. Let $\mathcal{A}_{n,k}^{(p)}$ denote the set of ordered pairs (α, β) in which $\alpha \in \mathcal{L}_m^{(r)}(n, j)$ and β is a partition enumerated by $(j+p+2r)^{\bar{k}}$ for some $1 \leq j \leq n$. Then $|\mathcal{A}_{n,k}^{(p)}|$ is given by the left-hand side of (3.8). Let $\mathcal{B}_{n,k}^{(p)}$ denote the set of triples $(\gamma, \delta, \epsilon)$ such that $\gamma \in \mathcal{L}_m^{(r)}(n, j)$, where i of the non-special blocks of γ are circled for some $0 \leq i \leq j \leq n$, δ is a subset of $[p]$ of size s for some s , and ϵ is a member of $\mathcal{L}^{(r)}(k, i+s)$ in which the non-special blocks can occur in any order. Then $|\mathcal{B}_{n,k}^{(p)}|$ is given by

$$\sum_{j=1}^n \sum_{s=0}^p \sum_{i=0}^j (i+s)! \binom{j}{i} \binom{p}{s} \ell^{(r)}(k, i+s) \ell_m^{(r)}(n, j),$$

which can be rewritten to give the right-hand side of (3.8).

To complete the proof, we define a bijection between the sets $\mathcal{A}_{n,k}^{(p)}$ and $\mathcal{B}_{n,k}^{(p)}$. To do so, let $(\alpha, \beta) \in \mathcal{A}_{n,k}^{(p)}$ and we construct a member of $\mathcal{B}_{n,k}^{(p)}$. Consider the labels of the non-empty blocks among the first j blocks of β (starting from the left) and then those among the non-empty of the next p blocks of β . This determines (possibly empty) subsets S_1 and S_2 of $[j]$ and $[p]$, respectively. Let $\delta = S_2$ and γ be obtained from α by circling the non-special blocks of α corresponding to the subset S_1 , where we assume that the non-special blocks of γ are arranged left-to-right in increasing order of smallest elements. To form ϵ , we first create its non-special blocks using the non-empty blocks among the first $j+p$ blocks of β where each element of β is increased by r (note that all of β 's blocks are labeled in increasing order from left to right, including the empty ones). To create the q -th special block of ϵ where $1 \leq q \leq r$, we form the word $\rho_1 q \rho_2$, where ρ_1 and ρ_2 denote respectively the ordered contents of the $(j+p+2q-1)$ -st and $(j+p+2q)$ -th blocks of β (and ρ_1 and ρ_2

are represented using letters in $[r + 1, r + k]$. Note that there is no restriction on the block cardinalities of ϵ and that the non-special blocks of ϵ are themselves ordered (since the blocks of β were labeled), whence there are $(i + s)! \ell^{(r)}(k, i + s)$ possibilities for ϵ where $i = |S_1|$ and $s = |S_2|$. One may verify that the mapping $(\alpha, \beta) \mapsto (\gamma, \delta, \epsilon)$ defined by the above construction is a bijection between $\mathcal{A}_{n,k}^{(p)}$ and $\mathcal{B}_{n,k}^{(p)}$, which completes the proof. \square

We have the following further r -dependent recurrence.

Theorem 3.7. *If $n, k \geq 0, m \geq 1$ and $1 \leq s \leq r$, then*

$$\ell_m^{(r)}(n, k) = n! \sum_{i=0}^{(m-1)s} \frac{\ell_m^{(r-s)}(n-i, k)}{(n-i)!} \left[\sum_{j=0}^{\lfloor \frac{i}{m} \rfloor} \sum_{p=0}^j (-1)^j \binom{s}{p, j-p, s-j} \times \binom{i+2s-(m+1)j-1}{2s-p-1} (m+1)^p \right]. \quad (3.9)$$

Proof. We start by considering the number t of elements in the first s special blocks within a member of $\mathcal{L}_m^{(r)}(n, k)$, where $s \leq t \leq ms$. Then there are $\binom{n}{t-s}$ choices for the non-special elements within these blocks and $\ell_m^{(r-s)}(n-t+s, k)$ ways in which to arrange elements within the non-special and the final $r-s$ special blocks. Finally, there are

$$(t-s)! \sum_{\substack{\lambda_1 + \dots + \lambda_s = t \\ 1 \leq \lambda_i \leq m}} \lambda_1 \cdots \lambda_s$$

ways in which to arrange the elements in the first s special blocks. To realize this, note that the multi-indexed sum counts all compositions of t with s parts, where the i -th part for each i has size $\lambda_i, 1 \leq \lambda_i \leq m$, and is colored in one of λ_i ways. The i -th special block for each $i \in [s]$ is then to have cardinality λ_i , within which i is to occupy the b_i -th position from the left, where b_i denotes the color assigned to the part λ_i . The $t-s$ non-special elements within the first s special blocks can occur in any order in a left-to-right scan of their contents, which accounts for the $(t-s)!$ factor. Combining the above observations gives

$$\ell_m^{(r)}(n, k) = \sum_{t=s}^{ms} \left(\sum_{\substack{\lambda_1 + \dots + \lambda_s = t \\ 1 \leq \lambda_i \leq m}} \lambda_1 \cdots \lambda_s \right) \frac{n!}{(n-t+s)!} \ell_m^{(r-s)}(n-t+s, k). \quad (3.10)$$

We now simplify the multi-sum in (3.10). To do so, we make use of the inclusion-exclusion principle and sieve out from the set of all (colored) compositions of t having s parts those whose parts are at most m . Consider the number j of parts of size exceeding m ; note that $(m+1)j + (s-j) \leq t$ gives $j \leq \lfloor \frac{t-s}{m} \rfloor$, which we denote by u . Then $t \leq ms$ implies $u \leq s$. Thus, we have

$$\sum_{\substack{\lambda_1 + \dots + \lambda_s = t \\ 1 \leq \lambda_i \leq m}}$$

$$\begin{aligned}
 &= \sum_{j=0}^u (-1)^j \binom{s}{j} \sum_{\substack{\lambda_1 + \dots + \lambda_s = t - (m+1)j \\ \lambda_i \geq 0}} (\lambda_1 + m + 1) \cdots (\lambda_j + m + 1) \lambda_{j+1} \cdots \lambda_s \\
 &= \sum_{j=0}^u (-1)^j \binom{s}{j} \sum_{p=0}^j \binom{j}{p} (m + 1)^p \sum_{\substack{\lambda_1 + \dots + \lambda_s = t - (m+1)j \\ \lambda_i \geq 0}} \lambda_{p+1} \cdots \lambda_s \\
 &= \sum_{j=0}^u (-1)^j \binom{s}{j} \sum_{p=0}^j \binom{j}{p} (m + 1)^p \binom{s + t - (m + 1)j - 1}{2s - p - 1},
 \end{aligned}$$

where we have used [2, Formula 26] in the last equality. Substituting this into (3.10), and replacing t with $i + s$, gives (3.9). \square

When m is large in (3.9), note that $j = p = 0$ is required in the two inner sums, which yields the following recurrence for $\ell^{(r)}(n, k)$.

Corollary 3.8 (Nyul and Rácz [16]). *If $n, k \geq 0$ and $1 \leq s \leq r$, then*

$$\ell^{(r)}(n, k) = \sum_{i=0}^{n-k} (2s)^{\bar{i}} \binom{n}{i} \ell^{(r-s)}(n - i, k). \tag{3.11}$$

We conclude this section with the following recurrences which are obtained by considering the nature of the singleton blocks within a member of $\mathcal{L}_m^{(r)}(n, k)$.

Theorem 3.9. *If $n, k \geq 0$, $m \geq 1$ and $r \geq 0$, then*

$$\ell_m^{(r)}(n, k) = \ell_m^{(0)}(n, k) + \sum_{j=1}^r \sum_{i=1}^{m-1} (i + 1)! \binom{n}{i} \ell_m^{(r-j)}(n - i, k) \tag{3.12}$$

and

$$\begin{aligned}
 \ell_m^{(r)}(n, k) &= \sum_{j=0}^r \sum_{i=0}^k \sum_{t=0}^j \frac{(n - k + i)!(i + 1)^{\bar{i}}}{(n - k - j + t)!} \binom{n}{k - i} \binom{r}{t, j - t, r - j} \\
 &\quad \times \ell_{m-1}^{(j-t)}(n - k - j + t, i + t). \tag{3.13}
 \end{aligned}$$

Proof. To show (3.12), consider within a member of $\mathcal{L}_m^{(r)}(n, k)$ the smallest $j \in [r]$ if it exists such that the singleton block $\{j\}$ does not occur (note that there are $\ell_m^{(0)}(n, k)$ possibilities if no such j exists). If $i + 1$ denotes the cardinality of the block containing j , where $1 \leq i \leq \min\{m - 1, n - k\}$, then there are $\binom{n}{i}(i + 1)!$ ways in which to select and order the elements belonging to this block. There are thus $\ell_m^{(r-j)}(n - i, k)$ ways in which to arrange the remaining $r - j$ special and $n - i$ non-special elements. Considering all i and j gives (3.12).

To show (3.13), first note that we may assume $m \geq 2$ since the formula is seen to hold when $m = 1$. Let $r - j$ and $k - i$ denote the number of special and non-special

singleton blocks, respectively, within a member of $\mathcal{L}_m^{(r)}(n, k)$, where $0 \leq j \leq r$ and $0 \leq i \leq k$. We select the elements comprising these blocks in $\binom{r}{j} \binom{n}{k-i}$ ways and set them aside. At this point, let us refer to non-singleton special blocks that are to start with a special (non-special, resp.) element as being of type 1 (of type 2, resp.), with type 3 referring to non-singleton non-special blocks. Let t denote the number of type 1 blocks, where $0 \leq t \leq j$. The special elements in these blocks may be chosen in $\binom{j}{t}$ ways, the set of which we denote by S . We now construct $\lambda \in \mathcal{L}_m^{(r)}(n, k)$ meeting the above specifications with respect to i, j and t . To do so, we first choose $i + j - t$ of the remaining elements of $[r + 1, r + n]$ that are to start either one of the $j - t$ blocks of λ of type 2 or one of its i blocks of type 3, which can be done in $\binom{n-k+i}{i+j-t}$ ways, the set of which we denote by T . We place the elements of T aside and then arrange all elements of $[n + r]$ not chosen thus far according to some partition ρ , where ρ (when standardized) belongs to $\mathcal{L}_{m-1}^{(j-t)}(n - k - j + t, i + t)$.

We now choose t of the non-special blocks of ρ in one of $\binom{i+t}{t}$ ways and then add a member of S to the beginning of each of these blocks according to any permutation of S . This produces the t type 1 blocks of λ . Next, we add the elements of T , one-per-block, to the beginning of the remaining $i + j - t$ blocks of ρ (i.e., those that did not receive an element of S), which can be done in $(i + j - t)!$ ways. This gives all of the blocks of λ of type 2 or 3. Appending as singleton blocks the $r - j$ special and the $k - i$ non-special elements set aside above completes the construction of the enumerated partition λ . One may verify that all λ satisfying the given requirements arise uniquely in this manner. By the preceding, the cardinality of such λ is given by

$$\begin{aligned} & \binom{r}{j} \binom{n}{k-i} \binom{j}{t} \binom{n-k+i}{i+j-t} \binom{i+t}{t} t!(i+j-t)! \ell_{m-1}^{(j-t)}(n-k-j+t, i+t) \\ &= \frac{(n-k+i)!(i+1)^{\bar{t}}}{(n-k-j+t)!} \binom{n}{k-i} \binom{r}{t, j-t, r-j} \ell_{m-1}^{(j-t)}(n-k-j+t, i+t). \end{aligned}$$

Summing over i, j and t yields all members of $\mathcal{L}_m^{(r)}(n, k)$. □

Remark 3.10. Letting $m \rightarrow \infty$ in (3.12) and (3.13) gives further identities for the r -Lah numbers. Letting $r = 0$ in the second of these identities implies

$$\ell^{(0)}(n, k) = \sum_{i=0}^k \frac{(n-k+i)!}{(n-k)!} \binom{n}{k-i} \ell^{(0)}(n-k, i), \quad n, k \geq 0,$$

which can also be shown directly using the $r = 0$ case of (4.4) below. Note that by using the formula from (4.4) in the limiting case of (3.13), one obtains an interesting family of binomial coefficient identities indexed by r .

4. Explicit formula for $\ell_m^{(r)}(n, k)$

We provide a combinatorial proof of the following expression for $\ell_m^{(r)}(n, k)$ in terms of binomial coefficients.

Theorem 4.1. *If $n, k, m \geq 1$ and $r \geq 0$, then*

$$\ell_m^{(r)}(n, k) = \frac{n!}{k!} \sum_{i=0}^{k+r} \sum_{t=0}^r (-1)^i \binom{r}{t} \binom{k+r-t}{i-t} \binom{n+2r-mi-t-1}{k+2r-t-1} m^t. \quad (4.1)$$

Proof. Let $C_{n,k,r}^{(m)}$ denote the set of compositions of $n+r$ having $k+r$ parts each of size at most m in which the first r parts are colored such that a part of size x is colored in one of x ways (the remaining k parts are not uncolored). Then we have $\ell_m^{(r)}(n, k) = \frac{n!}{k!} |C_{n,k,r}^{(m)}|$. To see this, given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{r+k}) \in C_{n,k,r}^{(m)}$, we distribute the elements of $[n+r]$ in blocks such that $\lambda_1, \dots, \lambda_r$ correspond to the cardinalities of the special and $\lambda_{r+1}, \dots, \lambda_{r+k}$ to the cardinalities of the non-special blocks (written in any order), where the element $i \in [r]$ is to occupy the position $a \in [\lambda_i]$ (from the left) within its block if a is the color assigned to the part λ_i of λ . Then there are $n!$ ways to arrange the elements of $[n+r]$ as described once λ is specified, and we divide by $k!$ since the non-special blocks are themselves not to be ordered.

Next, we determine the cardinality of $C_{n,k,r}^{(m)}$ and first show

$$\begin{aligned} |C_{n,k,r}^{(m)}| &= \sum_{i=0}^{k+r} (-1)^i \sum_{j=0}^i \binom{r}{j} \binom{k}{i-j} \\ &\quad \times \sum_{\lambda_1 + \dots + \lambda_{k+r} = n+r-mi} (\lambda_1 + m) \cdots (\lambda_j + m) \lambda_{j+1} \cdots \lambda_r, \end{aligned} \quad (4.2)$$

where the λ_i are positive in the innermost sum. To do so, first let $C_{n,k,r}^*$ denote the set of compositions of $n+k$ having $k+r$ parts in which the first r parts are colored just as members of $C_{n,k,r}^{(m)}$ were above, where now part sizes are unrestricted and where a (possibly empty) subset of the parts of size $m+1$ or more is circled. Let $C_{n,k,r}^*(i)$ denote the subset of $C_{n,k,r}^*$ containing exactly i circled parts. Then we have $|C_{n,k,r}^{(m)}| = \sum_{i=0}^{k+r} (-1)^i |C_{n,k,r}^*(i)|$. To see this, let members of $C_{n,k,r}^*(i)$ have sign $(-1)^i$. Define a sign-changing involution on $\cup_{i=0}^{k+r} C_{n,k,r}^*(i)$ by identifying the leftmost part of size greater than m and either circling or uncircling it (where the color is preserved, if the part is among the first r). The survivors of this involution comprise the set $C_{n,k,r}^{(m)}$, so to complete the proof of (4.2), it suffices to show

$$\begin{aligned} &|C_{n,k,r}^*(i)| \\ &= \sum_{j=0}^i \binom{r}{j} \binom{k}{i-j} \sum_{\lambda_1 + \dots + \lambda_{k+r} = n+r-mi} (\lambda_1 + m) \cdots (\lambda_j + m) \lambda_{j+1} \cdots \lambda_r. \end{aligned} \quad (4.3)$$

To establish (4.3), consider the number j of parts among the first r that are circled within a member of $C_{n,k,r}^*(i)$. Then there are $\binom{r}{j}\binom{k}{i-j}$ possible ways to select the parts that are to be circled. Note that the number of possible ways in which to color the first r parts depends on how many and not which of these parts are circled. Thus, once j is given, we may assume that it is the first j parts that are circled. Once the circled parts of $\lambda \in C_{n,k,r}^*(i)$ are specified, it follows that there are

$$\sum_{\lambda_1+\dots+\lambda_{k+r}=n+r-mi} (\lambda_1+m)\cdots(\lambda_j+m)\lambda_{j+1}\cdots\lambda_r$$

ways to determine the sizes of all the parts of λ together with the colors of the first r parts. Considering all possible j then implies (4.3) and thus (4.2), as desired.

Now observe that

$$\begin{aligned} & \sum_{\substack{\lambda_1+\dots+\lambda_{k+r}=n+r-mi \\ \lambda_\ell \geq 1}} (\lambda_1+m)\cdots(\lambda_j+m)\lambda_{j+1}\cdots\lambda_r \\ &= \sum_{t=0}^j \binom{j}{t} m^t \sum_{\substack{\lambda_1+\dots+\lambda_{k+r}=n+r-mi \\ \lambda_\ell \geq 1}} \lambda_{t+1}\lambda_{t+2}\cdots\lambda_r \\ &= \sum_{t=0}^j \binom{j}{t} m^t \sum_{\substack{\lambda_1+\dots+\lambda_{k+r}=n-k+r-mi-t \\ \lambda_{t+1},\dots,\lambda_r \geq 1 \\ \lambda_\ell \geq 0 \text{ otherwise}}} \lambda_{t+1}\lambda_{t+2}\cdots\lambda_r \\ &= \sum_{t=0}^j \binom{j}{t} \binom{n+2r-mi-t-1}{k+2r-t-1} m^t, \end{aligned}$$

where we have used [2, Formula 26] in the last equality. Thus, by (4.2) and [7, Identity 5.23], we get

$$\begin{aligned} |C_{n,k,r}^{(m)}| &= \sum_{i=0}^{k+r} (-1)^i \sum_{j=0}^i \binom{r}{j} \binom{k}{i-j} \sum_{t=0}^j \binom{j}{t} \binom{n+2r-mi-t-1}{k+2r-t-1} m^t \\ &= \sum_{i=0}^{k+r} (-1)^i \sum_{t=0}^r \binom{r}{t} \binom{n+2r-mi-t-1}{k+2r-t-1} m^t \sum_{j=t}^i \binom{k}{i-j} \binom{r-t}{j-t} \\ &= \sum_{i=0}^{k+r} (-1)^i \sum_{t=0}^r \binom{r}{t} \binom{k+r-t}{i-t} \binom{n+2r-mi-t-1}{k+2r-t-1} m^t, \end{aligned}$$

which implies (4.1). □

Allowing m to be large in (4.1) recovers the following explicit formula for $\ell^{(r)}(n,k)$.

Corollary 4.2 (Nyul and Rácz [16]). *If $n, k \geq 1$ and $r \geq 0$, then*

$$\ell^{(r)}(n, k) = \frac{n!}{k!} \binom{n + 2r - 1}{k + 2r - 1}. \tag{4.4}$$

Let $f_{k,m}^{(r)}(x) = \sum_{n \geq k} \ell_m^{(r)}(n, k) \frac{x^n}{n!}$. Multiplying both sides of (4.1) by $\frac{x^n}{n!}$, summing over $n \geq k$ and interchanging summation, we have

$$\begin{aligned} f_{k,m}^{(r)}(x) &= \frac{x^k}{k!(1-x)^{k+2r}} \sum_{i=0}^{k+r} \sum_{t=0}^i (-1)^i \binom{r}{t} \binom{k+r-t}{i-t} m^t x^{mi} (1-x)^t \\ &= \frac{x^k}{k!(1-x)^{k+2r}} \sum_{t=0}^r \binom{r}{t} m^t (x^{m+1} - x^m)^t \sum_{i=0}^{k+r-t} (-1)^i \binom{k+r-t}{i} x^{mi} \\ &= \frac{x^k (1-x^m)^{k+r}}{k!(1-x)^{k+2r}} \sum_{t=0}^r \binom{r}{t} \left[\frac{mx^m(x-1)}{1-x^m} \right]^t \\ &= \frac{(x-x^{m+1})^k}{k!(1-x)^{k+2r}} (1 - (m+1)x^m + mx^{m+1})^r. \end{aligned}$$

Let

$$f_m^{(r)}(x, y) = \sum_{n \geq 0} \left(\sum_{k=0}^n \ell_m^{(r)}(n, k) y^k \right) \frac{x^n}{n!}.$$

Multiplying the last equality by y^k , and summing over $k \geq 0$, yields the following result.

Corollary 4.3. *If $m \geq 1$ and $r \geq 0$, then*

$$f_m^{(r)}(x, y) = \left(\frac{1 - (m+1)x^m + mx^{m+1}}{(1-x)^2} \right)^r \exp \left(\frac{x(1-x^m)}{1-x} y \right). \tag{4.5}$$

Consider $L_{n,m}^{(r)}$ defined by

$$L_{n,m}^{(r)} = \sum_{k=0}^n \ell_m^{(r)}(n, k) \frac{(-1)^{n-k}}{k+1}, \quad m \geq 1, r \geq 0. \tag{4.6}$$

Note that when $r = 0$, the $L_{n,m}^{(r)}$ provide a Lah analogue to the restricted Cauchy numbers studied in [11], which reduce to the classical Cauchy numbers as $m \rightarrow \infty$. Let $L_m^{(r)}(x) = \sum_{n \geq 0} L_{n,m}^{(r)} \frac{x^n}{n!}$. Our next result in the $r = 0$ case is analogous to the one from [11] for restricted Cauchy numbers.

Proposition 4.4. *If $m \geq 1$ and $r \geq 0$, then*

$$L_m^{(r)}(-x) = \frac{(1 - (m+1)x^m + mx^{m+1})^r \left(1 - \exp \left(\frac{x(1-x^m)}{x-1} \right) \right)}{x(1-x^m)(1-x)^{2r-1}}. \tag{4.7}$$

Proof. Multiplying both sides of (4.6) by $\frac{(-x)^n}{n!}$, and summing over $n \geq 0$, implies

$$\begin{aligned} L_m^{(r)}(-x) &= \sum_{k \geq 0} \frac{(-1)^k f_{k,m}^{(r)}(x)}{k+1} = \int_0^1 \sum_{k \geq 0} f_{k,m}^{(r)}(x) (-y)^k dy = \int_0^1 f_m^{(r)}(x, -y) dy \\ &= \frac{1-x}{x(1-x^m)} \left(1 - \exp\left(\frac{x(1-x^m)}{x-1}\right) \right) \left(\frac{1-(m+1)x^m + mx^{m+1}}{(1-x)^2} \right)^r, \end{aligned}$$

by (4.5), which gives (4.7). □

5. Polynomial generalization

In this section, we briefly discuss a polynomial generalization of the sequence $\ell_m^{(r)}(n, k)$ based on a pair of statistics on $\mathcal{L}_m^{(r)}(n, k)$. Given a block \mathcal{B} of $\lambda \in \mathcal{L}_m^{(r)}(n, k)$, an element $i \in \mathcal{B}$ is said to be a *left-to-right minimum* if there exists no j to the left of i in \mathcal{B} with $j < i$. If \mathcal{B} is a special block of λ containing say $b \in [r]$, then we will say that i is a *special block record low* if (i) i occurs to the left of b in \mathcal{B} , with no element $j < i$ to the left of i , or (ii) i occurs to the right of b in \mathcal{B} , with no $j < i$ occurring between b and i . Let $rec'(\lambda)$ denote the total number of special block record lows in all of its special blocks. Let $nmin(\lambda)$ denote the number of elements of $[r+1, r+n]$ either (a) belonging to a non-special block and not a left-to-right minimum, or (b) belonging to a special block and not a special block record low. For example, if $n = 12, k = 2, r = 3, m = 5$ and

$$\lambda = \{10, 7, 1, 12\}, \{2\}, \{5, 15, 3, 6, 8\}, \{4, 14, 11\}, \{13, 9\} \in \mathcal{L}_5^{(3)}(12, 2),$$

then $rec'(\lambda) = 5$ (the enumerated elements being 10, 7, 12, 5 and 6) and $nmin(\lambda) = 4$ (the elements being 15, 8, 14 and 11). Note that minimal elements in all blocks and left-to-right minima in non-special blocks are among those excluded from the counts of both statistics, whence $0 \leq nmin(\lambda) + rec'(\lambda) \leq n - k$ with all values in this range being realized. Define the joint distribution polynomial for the $nmin$ and rec' statistics on $\mathcal{L}_m^{(r)}(n, k)$ by

$$\ell_m^{(r)}(n, k; a, b) = \sum_{\lambda \in \mathcal{L}_m^{(r)}(n, k)} a^{nmin(\lambda)} b^{rec'(\lambda)}.$$

See [13] for a related generalization of the Lah numbers.

Let $[a, b]_j = \prod_{i=1}^m (aj + b)$ if $j \geq 1$, with $[a, b]_0 = 1$. Considering whether or not the element $n + r$ forms its own block within a member of $\mathcal{L}_m^{(r)}(n, k)$, and if not, considering further cases based on whether $n + r$ follows directly a member of $[r + 1, r + n - 1]$ or starts a non-special block or is a special block record low yields the recurrence

$$\ell_m^{(r)}(n, k; a, b) = \ell_m^{(r)}(n - 1, k - 1; a, b) + (a(n - 1) + k + 2br)\ell_m^{(r)}(n - 1, k; a, b)$$

$$\begin{aligned}
 & - [a, 1]_m \binom{n-1}{m} \ell_m^{(r)}(n-m-1, k-1; a, b) \\
 & - 2br[a, 2b]_{m-1} \binom{n-1}{m-1} \ell_m^{(r-1)}(n-m, k; a, b),
 \end{aligned} \tag{5.1}$$

which reduces to (3.1) when $a = b = 1$. Note that it is possible to consider a further polynomial generalization wherein the $k\ell_m^{(r)}(n-1, k; a, b)$ term in (5.1) is multiplied by an indeterminate c . However, the statistic marked by c in this case can be obtained as $n - k - n\min(\lambda) - rec'(\lambda)$ for all λ . Thus, we may assume without loss of generality that one of a, b or c equals 1, and it is most convenient here to take $c = 1$.

Several of the properties shown in prior sections can be extended to the polynomial case. For example, generalizing the arguments used to show (3.2) and (3.8) respectively yields

$$\begin{aligned}
 \ell_m^{(r)}(n, k; a, b) &= n! \sum_{i=0}^k \sum_{j=0}^r \frac{(2b)^j [a, 1]_{m-1}^i [a, 2b]_{m-2}^j}{i!(m!)^i [(m-1)!]^j (n-mi-(m-1)j)!} \binom{r}{j} \\
 &\quad \times \ell_{m-1}^{(r-j)}(n-mi-(m-1)j, k-i; a, b)
 \end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
 & \sum_{j=1}^n (j+p+2br)[a, j+p+2br]_{k-1} \ell_m^{(r)}(n, j; a, b) \\
 &= \sum_{i=0}^n \sum_{s=0}^p (i+s)! \binom{p}{s} \ell^{(r)}(k, i+s; a, b) L_m^{(r)}(n, i; a, b),
 \end{aligned} \tag{5.3}$$

where $L_m^{(r)}(n, k; a, b) = \sum_{j=k}^n \binom{j}{k} \ell_m^{(r)}(n, j; a, b)$. One can also generalize (3.6) if the second sum in (3.6) is expressed instead using multiple indices which yields

$$\begin{aligned}
 \ell_m^{(r)}(n, k; a, b) &= \sum_{j=1}^r (-1)^{j-1} \binom{r}{j} \ell_m^{(r-j)}(n, k; a, b) \\
 &+ \sum_{\substack{i_1+\dots+i_r \leq n-k \\ 1 \leq i_j \leq m-1}} \binom{n - \sum_{j=1}^r i_j}{i_1, \dots, i_r, n - \sum_{j=1}^r i_j} \left((2b)^r \prod_{j=1}^r [a, 2b]_{i_j-1} \right) \\
 &\quad \times \ell_m^{(0)}\left(n - \sum_{j=1}^r i_j, k; a, b\right).
 \end{aligned} \tag{5.4}$$

Note that it is possible to write a recurrence for the multi-sum occurring in (5.4) that is analogous to (5.1) above (here, one would need an extra term $2br\ell_m^{(r-1)}(n-1, k; a, b)$ and assume $m \geq 2$).

We conclude by suggesting some further problems to consider. First, it would be interesting to find polynomial generalizations of formulas (3.4) and (4.1). The

unimodality of $\ell_m^{(r)}(n, k)$ could be considered as well as for what k the maximum value is achieved when n is fixed. While the r -Lah numbers are log-concave (see, e.g., [16]), it seems to be more difficult to establish this fact for $\ell_m^{(r)}(n, k)$ or, more generally, for $\ell_m^{(r)}(n, k; a, b)$ when a and b are positive real numbers, due to the more complicated formulas that are involved. Finally, it would be interesting to find an orthogonality relation for $\ell_m^{(r)}(n, k)$ that has as its limiting case when $m \rightarrow \infty$ the known orthogonality formula for the r -Lah numbers (see [16, Corollary 3.1]).

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