# On the coefficients of power sums of arithmetic progressions 

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#### Abstract

We investigate the coefficients of the polynomial $$
S_{m, r}^{n}(\ell)=r^{n}+(m+r)^{n}+(2 m+r)^{n}+\cdots+((\ell-1) m+r)^{n} .
$$

We prove that these can be given in terms of Stirling numbers of the first kind and $r$-Whitney numbers of the second kind. Moreover, we prove a necessary and sufficient condition for the integrity of these coefficients.


Key words: arithmetic progressions, power sums, Stirling numbers, $r$-Whitney numbers, Bernoulli polynomials
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## 1 Introduction

Let $n$ be a positive integer, and let

$$
S_{n}(\ell)=1^{n}+2^{n}+\cdots+(\ell-1)^{n}
$$

be the power sum of the first $\ell-1$ positive integers. It is well known that $S_{n}(\ell)$ is strongly related to the Bernoulli polynomials $B_{n}(x)$ in the following way

$$
S_{n}(\ell)=\frac{1}{n+1}\left(B_{n+1}(\ell)-B_{n+1}\right)
$$

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where the polynomials $B_{n}(x)$ are defined by the generating series
$$
\frac{t e^{t x}}{e^{t}-1}=\sum_{k=0}^{\infty} B_{k}(x) \frac{t^{k}}{k!}
$$
and $B_{n}=B_{n}(0)$ is the $n$th Bernoulli number.
It is possible to find the explicit coefficients of $\ell$ in $S_{n}(\ell)$ 9]:
\[

$$
\begin{equation*}
S_{n}(\ell)=\sum_{i=0}^{n+1} \ell^{i}\left(\sum_{k=0}^{n} S_{2}(n, k) S_{1}(k+1, i) \frac{1}{k+1}\right) \tag{1}
\end{equation*}
$$

\]

where $S_{1}(n, k)$ and $S_{2}(n, k)$ are the (signed) Stirling numbers of the first and second kind, respectively.

Recently, Bazsó et al. [1] considered the more general power sum

$$
S_{m, r}^{n}(\ell)=r^{n}+(m+r)^{n}+(2 m+r)^{n}+\cdots+((\ell-1) m+r)^{n},
$$

where $m \neq 0, r$ are coprime integers. Obviously, $S_{1,0}^{n}(\ell)=S_{n}(\ell)$. They, among other things, proved that $S_{m, r}^{n}(\ell)$ is a polynomial of $\ell$ with the explicit expression

$$
\begin{equation*}
S_{m, r}^{n}(\ell)=\frac{m^{n}}{n+1}\left(B_{n+1}\left(\ell+\frac{r}{m}\right)-B_{n+1}\left(\frac{r}{m}\right)\right) . \tag{2}
\end{equation*}
$$

In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [8] deduced a longer expression which contains already just binomial coefficients and Bernoulli numbers.

For some related diophantine results on $S_{m, r}^{n}(\ell)$ see [3,10, 15, 16, 2 ] and the references given there.

Our goal is to give the explicit form of the coefficients of the polynomial $S_{m, r}^{n}(\ell)$, thus generalizing (1). In this expression the Stirling numbers of the first kind also will appear, but, in place of the Stirling numbers of the second kind a more general class of numbers arises, the so-called $r$-Whitney numbers introduced by the second author [13].

The $r$-Whitney numbers $W_{m, r}(n, k)$ of the second kind are generalizations of the usual Stirling numbers of the second kind with the exponential generating function

$$
\sum_{n=k}^{\infty} W_{m, r}(n, k) \frac{z^{n}}{n!}=\frac{e^{r z}}{k!}\left(\frac{e^{m z}-1}{m}\right)^{k}
$$

For algebraic, combinatoric and analytic properties of these numbers see [5, 14 and [6|7], respectively.

First, we prove the following.
Theorem 1 For all parameters $\ell>1, n, m>0, r \geq 0$ we have

$$
S_{m, r}^{n}(\ell)=\sum_{i=0}^{n+1} \ell^{i}\left(\sum_{k=0}^{n} \frac{m^{k} W_{m, r}(n, k)}{k+1} S_{1}(k+1, i)\right) .
$$

Proof. The formula which connects the power sums and the $r$-Whitney numbers is the next one from [13]:

$$
(m x+r)^{n}=\sum_{k=0}^{n} m^{k} W_{m, r}(n, k) x^{\underline{k}} .
$$

Here $x^{\underline{k}}=x(x-1) \cdots(x-k+1)$ is the falling factorial. We can see that it is enough to sum from $x=0,1, \ldots, \ell-1$ to get back $S_{m, r}^{n}(\ell)$. Hence

$$
S_{m, r}^{n}(\ell)=\sum_{k=0}^{n} m^{k} W_{m, r}(n, k) \sum_{x=0}^{\ell-1} x^{\underline{k}} .
$$

The inner sum can be determined easily (see [9]):

$$
\sum_{x=0}^{\ell-1} x^{\underline{k}}=\frac{\ell \frac{k+1}{k+1}}{k+\delta_{k, 0}}
$$

The Kronecker delta will never appear, because if $k=0$ then the $r$-Whitney number is zero (unless the trivial case $n=0$, which we excluded). Therefore, as an intermediate formula, we now have that

$$
S_{m, r}^{n}(\ell)=\sum_{k=0}^{n} m^{k} W_{m, r}(n, k) \frac{\ell \frac{k+1}{k+1}}{k+. ~}
$$

The falling factorial $\ell \frac{k+1}{}$ is a polynomial of $\ell$ with Stirling number coefficients:

$$
\ell \frac{k+1}{}=\sum_{i=0}^{k+1} S_{1}(k+1, i) \ell^{i} .
$$

Substituting this to the formula above, we obtain:

$$
S_{m, r}^{n}(\ell)=\sum_{k=0}^{n} \frac{m^{k} W_{m, r}(n, k)}{k+1} \sum_{i=0}^{k+1} S_{1}(k+1, i) \ell^{i}
$$

Since $S_{1}(k+1, i)$ is zero if $i>k+1$, we can run the inner summation up to $n+1$ (this is taken when $k=n$ ) to make the inner sum independent of $k$. Altogether, we have that

$$
S_{m, r}^{n}(\ell)=\sum_{i=0}^{n+1} \ell^{i} \sum_{k=0}^{n} \frac{m^{k} W_{m, r}(n, k)}{k+1} S_{1}(k+1, i)
$$

This is exactly the formula that we wanted to prove.
Now we give some elementary consequences of the theorem. The proofs are trivial.

Remark. The next properties of the polynomial $S_{m, r}^{n}(\ell)$ hold true for all parameters $\ell>1, n>0, r, m \geq 0$ :
(i) The constant term of $S_{m, r}^{n}(\ell)$ is 0 ,
(ii) The leading coefficient of $S_{m, r}^{n}(\ell)$ is $m^{n} /(n+1)$,
(iii) $S_{m, r}^{n}(\ell)$ is a polynomial of $\ell$ of degree $n+1$ unless $m=0$; in this latter case the degree is $n$.

The above statements also follow from (21).

## 2 The integer property of the coefficients in $S_{m, r}^{n}(\ell)$

The coefficients of the polynomial $S_{m, r}^{n}(\ell)$ are not integer in the overwhelming majority of the cases:

$$
\begin{aligned}
S_{1,0}^{1}(\ell) & =\frac{\ell(\ell-1)}{2} \\
S_{2,5}^{2}(\ell) & =\frac{1}{3} \ell\left(47+24 \ell+4 \ell^{2}\right) \\
& \text { etc. }
\end{aligned}
$$

However, we revealed that in special cases the polynomial $S_{m, r}^{n}(\ell)$ has integer coefficients. Several parameters are in the next table.

| $m$ | $r$ | $n$ |
| :--- | :--- | :--- |
| 2 | 1 | 3 |
| 2 | 3 | 3 |
| 2 | 5 | 3 |
| 4 | 3 | 3 |
| 4 | 5 | 3 |

For example,

$$
S_{2,1}^{3}(\ell)=\ell^{2}\left(2 \ell^{2}-1\right),
$$

or

$$
S_{2,3}^{3}(\ell)=\ell(2+\ell)\left(2 \ell^{2}+4 \ell+3\right) .
$$

From the formula of Theorem 1 it can be seen that if

$$
(k+1) \mid m^{k} W_{m, r}(n, k) \quad(k=1,2, \ldots, n)
$$

then we get integer coefficients.
To find another condition which is necessary and sufficient for the integrity of the coefficients in $S_{m, r}^{n}(\ell)$, we recall the following well known properties of Bernoulli polynomials and Bernoulli numbers.

$$
\begin{gather*}
B_{n}(x+y)=\sum_{k=0}^{n}\binom{n}{k} B_{k}(x) y^{n-k}=\sum_{k=0}^{n}\binom{n}{k} B_{k}(y) x^{n-k}  \tag{3}\\
B_{n}(x)=\sum_{k=0}^{n}\binom{n}{k} B_{k} x^{n-k}  \tag{4}\\
B_{3}=B_{5}=B_{7}=\ldots=0 \tag{5}
\end{gather*}
$$

By the denominator of a rational number $q$ we mean the smallest positive integer $d$ such that $d q$ is an integer. We recall also the von Staudt theorem

$$
\begin{equation*}
\Lambda_{2 n}=\prod_{\substack{(p-1) \mid 2 n \\ p \text { prime }}} p, \tag{6}
\end{equation*}
$$

where $\Lambda_{n}$ is the denominator of $B_{n}$. In particular, $\Lambda_{n}$ is a square-free integer, divisible by 6. For the proofs of (3)-(5) see e.g. the work of Brillhart [4].

Let $2 \leq j \leq n$ be an even number and put

$$
\begin{align*}
f(n, j):=\operatorname{lcm}\left(\frac{\Lambda_{j}}{\operatorname{gcd}\left(\Lambda_{j},\binom{n+1}{j}\binom{j}{j}\right)},\right. & \frac{\Lambda_{j}}{\operatorname{gcd}\left(\Lambda_{j},\binom{n+1}{j+1}\binom{j+1}{j}\right)}, \ldots, \\
& \left.\frac{\Lambda_{j}}{\operatorname{gcd}\left(\Lambda_{j},\binom{n+1}{n}\binom{n}{j}\right)}\right) . \tag{7}
\end{align*}
$$

Further, we define

$$
F(n):= \begin{cases}\operatorname{lcm}(\operatorname{rad}(n+1), f(n, 2), f(n, 4), \ldots, f(n, n)) & \text { if } n \text { is even }  \tag{8}\\ \operatorname{lcm}(\operatorname{rad}(n+1), f(n, 2), f(n, 4), \ldots, f(n, n-1)) & \text { if } n \text { is odd }\end{cases}
$$

where

$$
\operatorname{rad}(n)=\prod_{\substack{p \mid n \\ p \text { prime }}} p
$$

Theorem 2 The polynomial $S_{m, r}^{n}(\ell)$ has integer coefficients if and only if $F(n) \mid m$.

Proof. By relations (2), (3) and (4) we can rewrite $S_{m, r}^{n}(\ell)$ as follows:

$$
\begin{align*}
S_{m, r}^{n}(\ell) & =\frac{m^{n}}{n+1}\left(B_{n+1}\left(\ell+\frac{r}{m}\right)-B_{n+1}\left(\frac{r}{m}\right)\right)=  \tag{9}\\
& =\frac{m^{n}}{n+1}\left(\left(\sum_{k=0}^{n+1}\binom{n+1}{k} B_{k}\left(\frac{r}{m}\right) \ell^{n+1-k}\right)-B_{n+1}\left(\frac{r}{m}\right)\right)=  \tag{10}\\
& =\frac{m^{n}}{n+1} \sum_{k=0}^{n}\binom{n+1}{k} B_{k}\left(\frac{r}{m}\right) \ell^{n+1-k}=  \tag{11}\\
& =\frac{m^{n}}{n+1} \sum_{k=0}^{n}\binom{n+1}{k}\left(\sum_{j=0}^{k}\binom{k}{j} B_{j} \cdot\left(\frac{r}{m}\right)^{k-j}\right) \ell^{n+1-k} \tag{12}
\end{align*}
$$

We denote the common denominator of the coefficients of $S_{m, r}^{n}(\ell)$ by $Q$. One can see from (9) that the polynomial has integral coefficients if and only if $m$ is divisible by $Q$. Thus we have to determine $Q$.

By (12) we observe that neither $m$ nor $r$ occurs in $Q$. Moreover, the only algebraic expressions which may affect $Q$ in (12) are on one hand $n+1$ and on the other hand, the denominators of the Bernoulli numbers involved, which are $2, \Lambda_{j}(2 \leq j \leq n$ even $)$ by (5) and the von Staudt theorem.

It can easily be seen that $n+1 \mid m^{n}$ if $\operatorname{rad}(n+1) \mid m$. Indeed, supposing the contrary, i.e., that $\operatorname{rad}(n+1) \mid m$ and $n+1 \nmid m^{n}$, it implies that there is a prime factor $p$ of $n+1$ such that $p^{n+1}$ divides $n+1$. Hence $2^{n+1} \leq p^{n+1} \leq n+1$, which is a contradiction.

Let $2 \leq j \leq n$ be an even index. It follows from (12) that the contribution of $\Lambda_{j}$ to the common denominator $Q$ is precisely $f(n, j)$ defined in (77). In other words, if $f(n, j) \mid m$, then every term of (12) containing the factor $B_{j}$ has integer coefficients.

In conclusion, we obtained that $Q$ is the least common multiple of $\operatorname{rad}(n+1)$ and $f(n, j)$ for all even $j \in[2, n]$, which number we denoted in (8) by $F(n)$. The theorem is proved.

Remark. An easy consequence of our Theorem 2 is that $S_{n}(\ell)=S_{1,0}^{n}(\ell) \notin \mathbb{Z}[x]$ for any positive integer $n$.

Some small values of $F(n)$ are listed in the following table. These are results of an easy computation in MAPLE.

| $n$ | $F(n)$ | $n$ | $F(n)$ | $n$ | $F(n)$ | $n$ | $F(n)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 6 | 42 | 11 | 6 | 16 | 510 |
| 2 | 6 | 7 | 6 | 12 | 2730 | 17 | 30 |
| 3 | 2 | 8 | 30 | 13 | 210 | 18 | 3990 |
| 4 | 30 | 9 | 10 | 14 | 30 | 19 | 210 |
| 5 | 6 | 10 | 66 | 15 | 6 | 20 | 2310 |

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