On the coefficients of power sums of arithmetic progressions

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Abstract

We investigate the coefficients of the polynomial

$$S_{m,r}^{n}(\ell) = r^{n} + (m+r)^{n} + (2m+r)^{n} + \dots + ((\ell-1)m+r)^{n}.$$

We prove that these can be given in terms of Stirling numbers of the first kind and r-Whitney numbers of the second kind. Moreover, we prove a necessary and sufficient condition for the integrity of these coefficients.

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1 Introduction

Let n be a positive integer, and let

$$S_n(\ell) = 1^n + 2^n + \dots + (\ell - 1)^n$$

be the power sum of the first $\ell - 1$ positive integers. It is well known that $S_n(\ell)$ is strongly related to the Bernoulli polynomials $B_n(x)$ in the following way

$$S_n(\ell) = \frac{1}{n+1}(B_{n+1}(\ell) - B_{n+1}).$$

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where the polynomials $B_n(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and $B_n = B_n(0)$ is the *n*th Bernoulli number.

It is possible to find the explicit coefficients of ℓ in $S_n(\ell)$ [9]:

$$S_n(\ell) = \sum_{i=0}^{n+1} \ell^i \left(\sum_{k=0}^n S_2(n,k) S_1(k+1,i) \frac{1}{k+1} \right), \tag{1}$$

where $S_1(n,k)$ and $S_2(n,k)$ are the (signed) Stirling numbers of the first and second kind, respectively.

Recently, Bazsó et al. [1] considered the more general power sum

$$S_{m,r}^{n}(\ell) = r^{n} + (m+r)^{n} + (2m+r)^{n} + \dots + ((\ell-1)m+r)^{n},$$

where $m \neq 0, r$ are coprime integers. Obviously, $S_{1,0}^n(\ell) = S_n(\ell)$. They, among other things, proved that $S_{m,r}^n(\ell)$ is a polynomial of ℓ with the explicit expression

$$S_{m,r}^{n}(\ell) = \frac{m^{n}}{n+1} \left(B_{n+1}\left(\ell + \frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right) \right).$$

$$\tag{2}$$

In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [8] deduced a longer expression which contains already just binomial coefficients and Bernoulli numbers.

For some related diophantine results on $S_{m,r}^n(\ell)$ see [3,10,15,16,2] and the references given there.

Our goal is to give the explicit form of the coefficients of the polynomial $S_{m,r}^n(\ell)$, thus generalizing (1). In this expression the Stirling numbers of the first kind also will appear, but, in place of the Stirling numbers of the second kind a more general class of numbers arises, the so-called *r*-Whitney numbers introduced by the second author [13].

The r-Whitney numbers $W_{m,r}(n,k)$ of the second kind are generalizations of the usual Stirling numbers of the second kind with the exponential generating function

$$\sum_{n=k}^{\infty} W_{m,r}(n,k) \frac{z^n}{n!} = \frac{e^{rz}}{k!} \left(\frac{e^{mz}-1}{m}\right)^k.$$

For algebraic, combinatoric and analytic properties of these numbers see [5,14] and [6,7], respectively.

First, we prove the following.

Theorem 1 For all parameters $\ell > 1, n, m > 0, r \ge 0$ we have

$$S_{m,r}^{n}(\ell) = \sum_{i=0}^{n+1} \ell^{i} \left(\sum_{k=0}^{n} \frac{m^{k} W_{m,r}(n,k)}{k+1} S_{1}(k+1,i) \right).$$

Proof. The formula which connects the power sums and the r-Whitney numbers is the next one from [13]:

$$(mx+r)^n = \sum_{k=0}^n m^k W_{m,r}(n,k) x^{\underline{k}}.$$

Here $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$ is the falling factorial. We can see that it is enough to sum from $x = 0, 1, \ldots, \ell - 1$ to get back $S^n_{m,r}(\ell)$. Hence

$$S_{m,r}^{n}(\ell) = \sum_{k=0}^{n} m^{k} W_{m,r}(n,k) \sum_{x=0}^{\ell-1} x^{\underline{k}}.$$

The inner sum can be determined easily (see [9]):

$$\sum_{x=0}^{\ell-1} x^{\underline{k}} = \frac{\ell^{\underline{k+1}}}{k+1} + \delta_{k,0}.$$

The Kronecker delta will never appear, because if k = 0 then the *r*-Whitney number is zero (unless the trivial case n = 0, which we excluded). Therefore, as an intermediate formula, we now have that

$$S_{m,r}^{n}(\ell) = \sum_{k=0}^{n} m^{k} W_{m,r}(n,k) \frac{\ell^{k+1}}{k+1}$$

The falling factorial $\ell^{\underline{k+1}}$ is a polynomial of ℓ with Stirling number coefficients:

$$\ell^{\underline{k+1}} = \sum_{i=0}^{k+1} S_1(k+1,i)\ell^i.$$

Substituting this to the formula above, we obtain:

$$S_{m,r}^{n}(\ell) = \sum_{k=0}^{n} \frac{m^{k} W_{m,r}(n,k)}{k+1} \sum_{i=0}^{k+1} S_{1}(k+1,i)\ell^{i}.$$

Since $S_1(k+1,i)$ is zero if i > k+1, we can run the inner summation up to n+1 (this is taken when k = n) to make the inner sum independent of k. Altogether, we have that

$$S_{m,r}^{n}(\ell) = \sum_{i=0}^{n+1} \ell^{i} \sum_{k=0}^{n} \frac{m^{k} W_{m,r}(n,k)}{k+1} S_{1}(k+1,i).$$

This is exactly the formula that we wanted to prove.

Now we give some elementary consequences of the theorem. The proofs are trivial.

Remark. The next properties of the polynomial $S_{m,r}^n(\ell)$ hold true for all parameters $\ell > 1, n > 0, r, m \ge 0$:

- (i) The constant term of $S_{m,r}^n(\ell)$ is 0,
- (ii) The leading coefficient of $S_{m,r}^n(\ell)$ is $m^n/(n+1)$,
- (iii) $S_{m,r}^n(\ell)$ is a polynomial of ℓ of degree n+1 unless m=0; in this latter case the degree is n.

The above statements also follow from (2).

2 The integer property of the coefficients in $S^n_{m,r}(\ell)$

The coefficients of the polynomial $S_{m,r}^n(\ell)$ are not integer in the overwhelming majority of the cases:

$$S_{1,0}^{1}(\ell) = \frac{\ell(\ell-1)}{2},$$

$$S_{2,5}^{2}(\ell) = \frac{1}{3}\ell(47 + 24\ell + 4\ell^{2}),$$

etc.

However, we revealed that in special cases the polynomial $S_{m,r}^n(\ell)$ has integer coefficients. Several parameters are in the next table.

For example,

$$S_{2,1}^3(\ell) = \ell^2 (2\ell^2 - 1),$$

or

$$S_{2,3}^3(\ell) = \ell(2+\ell)(2\ell^2 + 4\ell + 3).$$

From the formula of Theorem 1 it can be seen that if

$$(k+1) \mid m^k W_{m,r}(n,k) \quad (k=1,2,\ldots,n),$$

then we get integer coefficients.

To find another condition which is necessary and sufficient for the integrity of the coefficients in $S_{m,r}^n(\ell)$, we recall the following well known properties of Bernoulli polynomials and Bernoulli numbers.

$$B_n(x+y) = \sum_{k=0}^n \binom{n}{k} B_k(x) y^{n-k} = \sum_{k=0}^n \binom{n}{k} B_k(y) x^{n-k};$$
 (3)

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}; \tag{4}$$

$$B_3 = B_5 = B_7 = \dots = 0. \tag{5}$$

By the *denominator* of a rational number q we mean the smallest positive integer d such that dq is an integer. We recall also the von Staudt theorem

$$\Lambda_{2n} = \prod_{\substack{(p-1)|2n\\p \text{ prime}}} p,\tag{6}$$

where Λ_n is the denominator of B_n . In particular, Λ_n is a square-free integer, divisible by 6. For the proofs of (3)-(5) see e.g. the work of Brillhart [4].

Let $2 \leq j \leq n$ be an even number and put

$$f(n,j) := \operatorname{lcm}\left(\frac{\Lambda_j}{\operatorname{gcd}\left(\Lambda_j, \binom{n+1}{j}\binom{j}{j}\right)}, \frac{\Lambda_j}{\operatorname{gcd}\left(\Lambda_j, \binom{n+1}{j+1}\binom{j+1}{j}\right)}, \dots, \frac{\Lambda_j}{\operatorname{gcd}\left(\Lambda_j, \binom{n+1}{n}\binom{n}{j}\right)}\right). \quad (7)$$

Further, we define

$$F(n) := \begin{cases} \operatorname{lcm} (\operatorname{rad}(n+1), f(n,2), f(n,4), \dots, f(n,n)) & \text{if } n \text{ is even,} \\ \operatorname{lcm} (\operatorname{rad}(n+1), f(n,2), f(n,4), \dots, f(n,n-1)) & \text{if } n \text{ is odd,} \end{cases}$$
(8)

where

$$\operatorname{rad}(n) = \prod_{\substack{p \mid n \\ p \text{ prime}}} p.$$

Theorem 2 The polynomial $S_{m,r}^n(\ell)$ has integer coefficients if and only if $F(n) \mid m$.

Proof. By relations (2), (3) and (4) we can rewrite $S_{m,r}^n(\ell)$ as follows:

$$S_{m,r}^{n}(\ell) = \frac{m^{n}}{n+1} \left(B_{n+1}\left(\ell + \frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right) \right) = \tag{9}$$

$$=\frac{m^{n}}{n+1}\left(\left(\sum_{k=0}^{n+1}\binom{n+1}{k}B_{k}\left(\frac{r}{m}\right)\ell^{n+1-k}\right)-B_{n+1}\left(\frac{r}{m}\right)\right)=\qquad(10)$$

$$=\frac{m^n}{n+1}\sum_{k=0}^n \binom{n+1}{k} B_k\left(\frac{r}{m}\right)\ell^{n+1-k} =$$
(11)

$$=\frac{m^n}{n+1}\sum_{k=0}^n \binom{n+1}{k} \left(\sum_{j=0}^k \binom{k}{j}B_j \cdot \left(\frac{r}{m}\right)^{k-j}\right)\ell^{n+1-k}$$
(12)

We denote the common denominator of the coefficients of $S_{m,r}^n(\ell)$ by Q. One can see from (9) that the polynomial has integral coefficients if and only if m is divisible by Q. Thus we have to determine Q.

By (12) we observe that neither m nor r occurs in Q. Moreover, the only algebraic expressions which may affect Q in (12) are on one hand n+1 and on the other hand, the denominators of the Bernoulli numbers involved, which are $2, \Lambda_j (2 \leq j \leq n \text{ even})$ by (5) and the von Staudt theorem.

It can easily be seen that $n + 1 \mid m^n$ if $\operatorname{rad}(n + 1) \mid m$. Indeed, supposing the contrary, i.e., that $\operatorname{rad}(n + 1) \mid m$ and $n + 1 \nmid m^n$, it implies that there is a prime factor p of n+1 such that p^{n+1} divides n+1. Hence $2^{n+1} \leq p^{n+1} \leq n+1$, which is a contradiction.

Let $2 \leq j \leq n$ be an even index. It follows from (12) that the contribution of Λ_j to the common denominator Q is precisely f(n, j) defined in (7). In other words, if $f(n, j) \mid m$, then every term of (12) containing the factor B_j has integer coefficients.

In conclusion, we obtained that Q is the least common multiple of rad(n+1)and f(n, j) for all even $j \in [2, n]$, which number we denoted in (8) by F(n). The theorem is proved.

Remark. An easy consequence of our Theorem 2 is that $S_n(\ell) = S_{1,0}^n(\ell) \notin \mathbb{Z}[x]$ for any positive integer n.

Some small values of F(n) are listed in the following table. These are results of an easy computation in MAPLE.

n	F(n)	n	F(n)	n	F(n)	n	F(n)
1	2	6	42	11	6	16	510
2	6	7	6	12	2730	17	30
3	2	8	30	13	210	18	3990
4	30	9	10	14	30	19	210
5	6	10	66	15	6	20	2310

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