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AN IDENTITY WITH DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with center $Z(R)$, and d a derivation of R. Suppose that $(d[x,y]_k)^n - m[x,y]_k \in Z(R)$ for all $x, y \in R$, where $m \neq n, k \geq 1$ are fixed integers. Then $d = 0$ or R satisfies s_4, the standard identity in four variables. In the case $(d[x,y]_k)^n - m[x,y]_k = 0$ for all $x, y \in R$, then $d = 0$ or R is commutative.

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1. INTRODUCTION

In all that follows, unless stated otherwise, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring and Utumi quotient ring U. The center of Q, denoted by C, is called the extended centroid of R (we refer the reader to [3] for related symbols). For any $x, y \in R$, the symbol $[x,y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. Recall that a ring R is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. By a derivation on R we mean an additive mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular d is called an inner derivation induced by an element $a \in R$, if $d(x) = [a,x]$ for all $x \in R$. For any $x, y \in R$, we set $[x,y]_1 = [x,y] = xy - yx$, and $[x,y]_k = [[x,y]_{k-1}, y]$, where $k > 1$ is an integer. Note that $[x,y]_k = \sum_{i=0}^{k}(-x)^i y x^{k-i}$ and $d([x,y]_k) = [d(x), y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, d(y)], y]_{k-i}$.

Many results in literature indicate that global structure of a ring R is often lightly connected to the behavior of additive mappings defined on R. The first classic result on this topic is due to Divinsky [7] who proved that a simple Artinian ring is commutative if it has a commuting non-identity automorphism. Over the last few decades, a number of authors have investigated the relationship between the commutativity of

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the ring R and certain specific types of derivations of R. In [2], Ashraf and Rehman proved that if R is a prime ring, I a nonzero ideal of R and d is a derivation of R such that $d(x \circ y) = x \circ y$ for all $x, y \in I$, then R is commutative. In [1], Argaç and Inceboz generalized the above result as following: Let R be a prime ring, I a nonzero ideal of R and n a fixed positive integer, if R admits a derivation d with the property $(d(x \circ y))^n = x \circ y$ for all $x, y \in I$, then R is commutative. On the other hand, Daif and Bell [6] showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, if $I = R$ then R is commutative. Motivated by the above-cited results, our purpose in this article is to obtain some information of the prime ring R involving a central identity $(d[x, y]_k)^n - m[x, y]_k \in Z(R)$ for all $x, y \in R$, where $m, n, k \geq 1$ are fixed integers.

The standard identity s_4 in four variables is defined as follows:

$$s_4 = \sum (-1)^\tau X_{\tau(1)} X_{\tau(2)} X_{\tau(3)} X_{\tau(4)}$$

where $(-1)^\tau$ is the sign of a permutation τ of the symmetric group of degree 4. As is well known, prime rings satisfying s_4 can be characterized by the following:

Fact 1 ([4])**.** Let R be a prime ring with the extended centroid C. Then the following are equivalent:

(1) $dim_C RC \leq 4$;

(2) R satisfies s_4;

(3) R is commutative or R embeds in $M_2(F)$;

(4) R is algebraic of bounded degree 2 over C;

(5) R satisfies $[[x^2, y], [x, y]]$.

2. Results

We begin with the following lemmas which are crucial for proving our main results.

Lemma 1. *Let $R = M_2(F)$ be the ring of all 2×2 matrices over a field F. If $0 \neq a \in R$ such that*

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n = m[x,y]_k$$

for all $x \in R$ where $m,n,k \geq 1$ are fixed integers, then $a \in Z(R)$.

Proof. Let $a = \sum_{i,j=0}^{2} a_{ij}e_{ij}$ with $a_{ij} \in F$, where e_{ij} is the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. Let $x = e_{12}$, $y = e_{11}$. Then $m[x,y]_k = 0$ and $([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n = (-1)^{kn}(-e_{12}a + e_{11}ae_{12})^n$. By assumption we have $(-1)^{kn}(-e_{12}a + e_{11}ae_{12})^n = 0$. Right multiplying by e_{12}, it yields that $(-1)^{(k+1)n}a_{21}^n e_{12} = 0$, which implies $a_{21} = 0$. Similarly, we have $a_{12} = 0$. Thus a must be a diagonal matrix. Now set $a = \sum_t a_{tt}e_{tt}$, with $a_{tt} \in F$. Let φ be the inner automorphism of R given by $\varphi(x) = (1+e_{ij})a(1-e_{ij})$. Thus $([[a^{\varphi},x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a^{\varphi},y]],y]_{k-i})^n = m[x,y]_k$ for all $x,y \in R$. By above argument,

$$a^{\varphi} = (1+e_{ij})a(1-e_{ij}) = \Sigma_{i=1}^{k} a_{ii}e_{ii} + (a_{jj}-a_{ii})e_{ij}$$

must be diagonal. Therefore $a_{jj} = a_{ii}$ and so $a \in Z(R)$. □

Lemma 2. *Let R be a non-commutative prime ring with center $Z(R)$. If $0 \neq a \in R$ such that*

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n = m[x,y]_k$$

for all $x,y \in R$, where $n,m,k \geq 1$ are fixed integers, then $a \in Z(R)$.

Proof. By assumption, R satisfies the generalized polynomial identity

$$p(x,y) = ([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k.$$

By Chuang [5], this generalized polynomial identity (GPI) is also satisfied by U. If $a \notin C$ then $p(x,y) = 0$ is a nontrivial (GPI) for U. In case C is infinite, we have $p(x,y) = 0$ for all $x,y \in U \bigotimes_C \overline{C}$ where $\overline{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \overline{C}$ are prime and centrally closed [8], we may replace R by U or $U \bigotimes_C \overline{C}$ according to C finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and $p(x,y) = 0$ for all $x,y \in R$. By Martindale's theorem [13], R is then a primitive ring having nonzero $\mathrm{soc}(R)$ with C as the associated division ring. Hence by Jacobson's theorem [9], R is isomorphic to a dense ring of linear transformations of a vector space V over C. If $\dim_C V = k$, then the density of R on V implies that $R \cong M_k(C)$, where $k = \dim_C V$. Since R is noncommutative, $k \geq 2$. If $\dim_C V = 2$, then by Lemma 1 we have $a \in Z(R)$. Now suppose that $\dim_C V \geq 3$. We want to show that for any $v \in V$, v and av are linearly C-dependent. Suppose on contrary that v and av are linearly C-independent for some $v \in V$. Since $\dim_C V \geq 3$, there exists $w \in V$ such that $\{v, av, w\}$ are linearly C-independent set of vectors. By the density of R on V, there exist $x, y \in R$ such that $xv = v, xw = v + w, xav = w; yv = v, yw = 0, yav = v$. Then $0 = p(x,y)v = (-1)^{kn}v$, a contradiction. Therefore v, av are linearly C-dependent for any $v \in V$. Hence we can write $av = v\alpha_v$ for all $v \in V$ and $\alpha_v \in C$. Then by a standard argument, it is very easy to prove that α_v is independent of the choice of $v \in V$. In fact, Since $dim_C V \geq 3$, then there exist u, v, w which are linearly independent, and so $\alpha_u, \alpha_v, \alpha_w \in C$ such that $au = u\alpha_u$, $av = v\alpha_v$, $aw = w\alpha_w$, that is $a(u+v+w) = u\alpha_u + v\alpha_v + w\alpha_w$. Moreover $a(u+v+w) = (u+v+w)\alpha_{u+v+w}$ for some $\alpha_{u+v+w} \in C$. Then $0 = (\alpha_{u+v+w} - \alpha_u)u + (\alpha_{u+v+w} - \alpha_v)v + (\alpha_{u+v+w} - \alpha_w)w$ and hence $\alpha_u = \alpha_v = \alpha_w = \alpha_{u+v+w}$, that is α does not depend on the choice of v. Thus we can write $av = v\alpha$ for all $v \in V$ and $\alpha \in C$ fixed. Now, let $r \in R$, $v \in V$. Since $av = v\alpha$, we have $[a,r]v = (ar)v - (ra)v = a(rv) - r(av) = (rv)\alpha - r(v\alpha) = 0$, that is $[a,r]V = 0$. Hence $[a,r] = 0$ for all $r \in R$, implying $a \in Z(R)$. □

Theorem 1. *Let R be a prime ring and d a derivation of R. Suppose that $(d[x,y]_k)^n = m[x,y]_k$ for all $x, y \in R$, where $m \neq n, k \geq 1$ are fixed integers. Then $d = 0$ or R is commutative.*

Proof. Using the identity $d([x,y]_k) = [d(x),y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, d(y)], y]_{k-i}$ and the hypothesis, we have

$$([d(x),y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, d(y)], y]_{k-i})^n = m[x,y]_k$$

for all $x, y \in R$. Assume first that d is Q-inner, that is, $d(x) = [a,x]$ for all $x \in Q$, where a is a non-central element in Q. Then

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, [a,y]], y]_{k-i})^n = m[x,y]_k$$

for all $x, y \in U$. Thus by Lemma 2, $a \in Z(R)$ that gives $d = 0$.

Assume next that d is Q-outer. Applying Kharchenko's theorem [10], we get $([v,y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1}, u], y]_{k-i})^n = m[x,y]_k$ for all $x, y, u, v \in U$. In particular, for $u = 0$ and $v = x$ we have $([x,y]_k)^n = m[x,y]_k$, for all $x, y \in U$. Note that, this is a polynomial identity and hence there exists a field F such that $R \subseteq M_t(F)$, the ring of $t \times t$ matrices over a field F, where $t \geq 1$. By Chuang [5], this generalized polynomial identity (GPI) is also satisfied by R as well. Moreover, R and $M_t(F)$ satisfy the same polynomial identity [11, Lemma 1], that is, $([x,y]_k)^n = m[x,y]_k$ for all $x, y \in M_t(F)$. But by choosing $x = e_{12}$, $y = e_{11}$, we get $0 = ([x,y]_k)^n - m[x,y]_k = (-1)^k(e_{12}^n - me_{12})$. This is a contradiction, ending the proof.

□

Lemma 3. *Let $R = M_t(F)$ be the ring of all $t \times t$ matrices over a field F with $t \geq 3$. If $0 \neq a \in R$ and $m,n,k \geq 1$ are fixed integers, such that $([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in F \cdot I_t$, for all $x,y \in R$, then $a \in F \cdot I_t$.*

Proof. We are given that

$$[([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k, z] = 0$$

for all $x,y,z \in R$. Let $a = (a_{ij})_{t\times t}$. By choosing $x = e_{ij}$, $y = e_{ii}$ and $z = e_{ik}$ for any $i \neq j \neq k$, we have

$$[(-1)^{kn}[a,e_{ij}]^n - (-1)^k m e_{ij}, e_{ik}] = (-1)^{kn}((e_{ij}a)^n e_{ik} - e_{ik}(ae_{ij})^n) = 0.$$

Thus $a_{ij} = 0$, and so a is a diagonal matrix. Using the same technique in Lemma 1, we get $a \in F \cdot I_t$, proving the lemma. □

Theorem 2. *Let R be a prime ring with center $Z(R)$, and d a derivation of R. Suppose that $(d[x,y]_k)^n - m[x,y]_k \in Z(R)$ for all $x,y \in R$, where $m \neq n, k \geq 1$ are fixed integers. Then $d = 0$ or R satisfies s_4, the standard identity in four variables.*

Proof. By assumption R satisfies the generalized differential identity

$$0 = [(d[x,y]_k)^n - m[x,y]_k, w] = [[d(x),y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},d(y)],y]_{k-i} - m[x,y]_k, w]$$

for $x,y,w \in R$. By Lee [12], R and U satisfy the same differential identities we may assume that above identity is also satisfied by U. Now we consider the following two cases:

Case 1. Suppose that d is a Q-outer derivation. Then Kharchenko's theorem [10], we have

$$[([v,y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},u],y]_{k-i})^n - m[x,y]_k, w] = 0,$$

for all $x,y,u,v,w \in U$. This is a polynomial identity and hence there exists a field F such that $U \subseteq M_t(F)$ with $t > 1$ and U, $M_t(F)$ satisfy the same polynomial identity [11]. If $t \geq 3$ then by choosing $w = e_{13}$, $v = x = e_{12}$, $y = e_{11}$, $u = 0$, we get $0 = [([v,y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},u],y]_{k-i})^n - m[x,y]_k, w] = (-1)^k e_{13}$. This is a contradiction. Thus $t = 2$ and so R satisfies s_4 by Fact 1.

Case 2. Suppose that d is a Q-inner derivation. In this case there exists $a \in Q$ such that $d(x) = [a,x]$ for all $x \in R$. Then we have

$$[([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k, w] = 0,$$

for all $x, y, w \in R$. By localizing R at $Z(R)$ it follows that

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in Z(R_Z),$$

for all $x, y \in R_Z$. Since R and R_Z satisfy the same polynomial identities, in order to prove that R satisfies s_4, we may assume that R is simple with 1. Hence, $([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in Z(R)$ for all $x, y \in R$. Therefore R satisfies a generalized polynomial identity and it is simple with 1, which implies that $Q = RC = R$ and R has a minimal right ideal, whose commuting ring D is a division ring which is finite dimensional over $Z(R)$. However, since R is a simple ring with 1, R must be Artinian, that is, $R = D_s$, the $s \times s$ matrices over D, for some $s \geq 1$. By [11] there exists a field F such that $R \subseteq M_t(F)$, the ring of $t \times t$ matrices over field F, with $t > 1$, and

$$([[a,x],y]_k + \sum_{i=1}^{k}[[[x,y]_{i-1},[a,y]],y]_{k-i})^n - m[x,y]_k \in Z(M_t(F)) = F \cdot I_t,$$

for all $x, y \in M_t(F)$. If $t \geq 3$, then by Lemma 3, $a \in F \cdot I_t$ and so $d = 0$. If $t = 2$, then R satisfies s_4. □

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