Miskolc Mathematical Notes
Vol. 19 (2018), No. 2, pp. 731–740
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.1851

TRACES OF PERMUTING GENERALIZED N-DERIVATIONS OF RINGS

MOHAMMAD ASHRAF, ALMAS KHAN, AND MALIK RASHID JAMAL

Received 03 November, 2015

Abstract. Let $n \geq 1$ be a fixed positive integer and R be a ring. A permuting n-additive map $\Omega : R^n \to R$ is known to be permuting generalized n-derivation if there exists a permuting n-derivation $\Delta : R^n \to R$ such that $\Omega(x_1, x_2, \cdots, x_i x_i^{'}, \cdots, x_n) = \Omega(x_1, x_2, \cdots, x_i, \cdots, x_n)x_i^{'} + x_i \Delta(x_1, x_2, \cdots, x_i^{'}, \cdots, x_n)$ holds for all $x_i, x_i^{'} \in R$. A mapping $\delta : R \to R$ defined by $\delta(x) = \Delta(x, x, \cdots, x)$ for all $x \in R$ is said to be the trace of Δ. The trace ω of Ω can be defined in the similar way. The main result of the present paper states that if R is a $(n+1)!$-torsion free semi-prime ring which admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x), x], x] \in Z(R)$ for all $x \in R$, then δ is commuting on R. Besides other related results it is also shown that in a $n!$-torsion free prime ring if the trace ω of a permuting generalized n-derivation Ω is centralizing on R, then ω is commuting on R.

2010 *Mathematics Subject Classification:* 16W25; 16U80

Keywords: derivation, generalized derivation, permuting n-derivation, centralizing map, prime ring

1. INTRODUCTION

Throughout R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, $xy - yx$ denote the commutator $[x, y]$. A ring R is said to be prime (resp. semi-prime) if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$ (resp. $aRa = \{0\}$ implies $a = 0$). Let $m \geq 1$ be a fixed positive integer. A map $f : R \to R$ is said to be centralizing (resp. commuting) on R if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) holds for all $x \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Following [4], an additive mapping $F : R \longrightarrow R$ is said to be a generalized derivation on R if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Suppose n is a fixed positive integer and $R^n = R \times R \times \cdots \times R$. A map $\Delta : R^n \to R$ is said to be permuting if the relation $\Delta(x_1, x_2, \cdots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, \cdots, x_{\pi(n)})$ holds for all $x_i \in R$ and for every permutation $\{\pi(1), \pi(2), \cdots, \pi(n)\}$. The concept of derivation and symmetric bi-derivation was generalized by Park [7] as follows: a permuting map $\Delta : R^n \to R$ is said to be a permuting n-derivation if Δ is n-additive (i.e.; additive in each coordinate) and $\Delta(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = x_i\Delta(x_1,x_2,\cdots,x_i^{'},\cdots,x_n) + \Delta(x_1,x_2,\cdots,x_i,\cdots,x_n)x_i^{'}$ holds for all $x_i,x_i^{'} \in R$. A 1-derivation is a derivation and a 2-derivation is a symmetric bi-derivation while a 3-derivation is known as permuting tri-derivation.

A well known result due to Posner [8] states that a prime ring R which admits a non-zero centralizing derivation is commutative. In fact, this result initiated the study of centralizing and commuting mappings in rings. Since then, several authors have done a great deal of work concerning commutativity of prime and semi-prime rings admitting different kinds of maps which are centralizing or commuting on some appropriate subsets of R (see [5,6] and [7] for further references). Let $n \geq 2$ be a fixed integer and a map $\delta : R \to R$ defined by $\delta(x) = \Delta(x,x,\cdots,x)$ for all $x \in R$, where $\Delta : R^n \to R$ is a permuting map, be the trace of Δ. Moreover, it can be easily seen that $\Delta(x_1,x_2,\cdots,-x_i,\cdots,x_n) = -\Delta(x_1,x_2,\cdots,x_i,\cdots,x_n)$ for all $x_i \in R, i = 1,2,\cdots,n$.

Motivated by the concept of generalized derivation in ring, we introduce the notion of permuting generalized n-derivation in ring. Let $n \geq 1$ be a fixed positive integer. A permuting n-additive map $\Omega : R^n \to R$ is known to be permuting generalized n-derivation if there exists a permuting n-derivation $\Delta : R^n \to R$ such that $\Omega(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = \Omega(x_1,x_2,\cdots,x_i,\cdots,x_n)x_i^{'} + x_i\Delta(x_1,x_2,\cdots,x_i^{'},\cdots,x_n)$ holds for all $x_i,x_i^{'} \in R$. For an example of permuting generalized n-derivation, let $n \geq 1$ be a fixed positive integer and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c \in \mathbb{C} \right\}$ where $\mathbb{C}$ is a complex field. Consider permuting n-derivation Δ as above and define $\Omega : R^n \to R$ such that

$$\Omega\left(\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix},\cdots,\begin{pmatrix} 0 & a_n & b_n \\ 0 & 0 & c_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & a_1\cdots a_n \\ 0 & 0 & c_1\cdots c_n \\ 0 & 0 & 0 \end{pmatrix}.$$

Then Ω is a permuting generalized n-derivation on R associated with a permuting n-derivation Δ on R.

Let $\omega : R \to R$ such that $\omega(x) = \Omega(x,x,\cdots,x)$. Then ω is known as the trace of Ω. A permuting n-additive map $\Lambda : R^n \to R$ is said to be a permuting left n-multiplier (resp. permuting right n-multiplier) if $\Lambda(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = \Lambda(x_1,x_2,\cdots,x_i,\cdots,x_n)x_i^{'}$ (resp. $\Lambda(x_1,x_2,\cdots,x_ix_i^{'},\cdots,x_n) = x_i\Lambda(x_1,x_2,\cdots,x_i^{'},\cdots,x_n)$) holds for all $x_i,x_i^{'} \in R$. If Λ is both permuting left n-multiplier as well as right n-multiplier, then Λ is called a permuting n-multiplier.

Motivated by the results due to Posner [8], Vukman obtained some results concerning the trace of symmetric bi-derivation in prime ring (see [9, 10]). Ashraf [1] proved similar results for semi-prime ring. In the year 2009, Park [7] introduced the concept of symmetric permuting n-derivation and obtained some results related to the commuting traces of permuting n-derivations in rings. Further, the first author together with Jamal and Parveen [2, 3] obtained commutativity of rings admitting n-derivations whose traces satisfy certain polynomial conditions.

The main objective of this paper is to find the analogous results for permuting generalized n-derivation in the setting of prime and semi-prime rings. In fact, our theorems present a wide generalization of the results obtained in [1], Theorem 2.1, [7], Theorem 2.3, [7], Theorem 2.5, [9], Theorem 1, [9], Theorem 2, [10], Theorem 2 etc.

2. Results

We begin with the following known results which are frequently used in our discussion.

Lemma 1 (Lemma 2.4 in [7]). *Let n be a fixed positive integer and let R be a $n!$-torsion free ring. Suppose that $y_1, y_2, \cdots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ (or $\in Z(R)$) for $\lambda = 1, 2, \cdots, n$. Then $y_i = 0$ (or $y_i \in Z(R)$) for all i.*

Lemma 2 (Theorem 2.3 in [7]). *Let $n \geq 2$ be a fixed positive integer and R be a non-commutative $n!$-torsion free prime ring. Suppose that there exists a permuting n-derivation $\Delta : R^n \to R$ such that the trace δ of Δ is commuting on R. Then we have $\Delta = 0$.*

Lemma 3 (Theorem 2.6 in [7]). *Let $n \geq 2$ be a fixed positive integer and R be a $n!$-torsion free prime ring. Suppose that there exists a non-zero permuting n-derivation $\Delta : R^n \to R$ such that the trace δ of Δ is centralizing on R then R is commutative.*

As stated in the beginning, there has been a great deal of work concerning centralizing and commuting mappings. The following result shows that if the trace δ of a permuting n-derivation Δ is centralizing on R then it is commuting on R. In fact, we prove rather a more general result:

Theorem 1. *Let $n \geq 2$ be a fixed positive integer and R be a $(n+1)!$-torsion free semi-prime ring admitting a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x), x], x] = 0$ for all $x \in R$. Then δ is commuting on R.*

Proof. From our hypothesis we have

$$[[\delta(x), x], x] = 0 \ \text{ for all } \ x \in R. \tag{2.1}$$

An easy computation shows that the traces δ of Δ satisfies the following relations

$$\delta(x+y) = \delta(x) + \delta(y) + \sum_{r=1}^{n-1} \binom{n}{r} h_r(x, y) \text{ for all } x, y \in R$$

where $h_r(x, y) = \Delta(\underbrace{x, x, \cdots, x}_{(n-r)-\text{times}}, \underbrace{y, y, \cdots, y}_{r-\text{times}})$.

Consider a positive integer k, $1 \le k \le n+1$. Replacing x by $x+ky$ in equation (2.1), we obtain

$$kQ_1(x,y)+k^2Q_2(x,y)+...+k^{n+1}Q_{n+1}(x,y)=0 \text{ for all } x,y\in R,$$

where $Q_i(x,y)$ denotes the sum of the terms in which y appears i times. By (2.1) and Lemma 1, we have for all $x,y\in R$,

$$[[\delta(x),x],y]+[[\delta(x),y],x]+n[[\Delta(x,x,...,y),x],x]=0. \tag{2.2}$$

Replacing y by xy in (2.2) we get

$$\begin{aligned}0 &= [[\delta(x),x],xy]+[[\delta(x),xy],x]+n[[\Delta(x,x,...,xy),x],x]\\ &= [[\delta(x),x],xy]+[[\delta(x),xy],x]+n[[x\Delta(x,x,...,y),x],x]\\ &\quad +n[[\delta(x)y,x],x]\\ &= [[\delta(x),x],xy]+[x[\delta(x),y],x]+[[\delta(x),x]y,x]\\ &\quad +n[x[\Delta(x,x,...,y),x],x]+n[\delta(x)[y,x],x]+n[[\delta(x),x]y,x]\\ &= [[\delta(x),x],x]y+x[[\delta(x),x],y]+x[[\delta(x),y],x]+[x,x][\delta(x),y]\\ &\quad +[\delta(x),x][y,x]+[[\delta(x),x],x]y+nx[[\Delta(x,x,...,y),x],x]\\ &\quad +n\delta(x)[[y,x],x]+n[\delta(x),x][y,x]+n[\delta(x),x][y,x]+n[[\delta(x),x],x]y.\end{aligned}$$

Using (2.1) and (2.2) we find that

$$(2n+1)[\delta(x),x][y,x]+n\delta(x)[[y,x],x]=0 \text{ for all } x,y\in R. \tag{2.3}$$

Similarly, replacing y by yx in (2.2), one can get

$$(2n+1)[y,x][\delta(x),x]+n[[y,x],x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.4}$$

Replacing y by yz in (2.3), we have

$$\begin{aligned}0 &= (2n+1)[\delta(x),x][yz,x]+n\delta(x)[[yz,x],x]\\ &= (2n+1)\{[\delta(x),x][y,x]z+[\delta(x),x]y[z,x]\}+n\delta(x)y[[z,x],x]\\ &\quad +n\delta(x)[y,x][z,x]+n\delta(x)[y,x][z,x]+n\delta(x)[[y,x],x]z.\end{aligned}$$

Using equation (2.3)

$$(2n+1)[\delta(x),x]y[z,x]+n\delta(x)y[[z,x],x]+2n\delta(x)[y,x][z,x]=0.$$

Replacing y by $\delta(x)$ in the above relation we find that

$$(2n+1)[\delta(x),x]\delta(x)[z,x]+n\delta(x)^2[[z,x],x]+2n\delta(x)[\delta(x),x][z,x]=0. \tag{2.5}$$

From (2.3) we have $n\delta(x)^2[[y,x],x]=-(2n+1)\delta(x)[\delta(x),x][y,x]$. Now using this relation in (2.5) we get

$$\begin{aligned}0 &= (2n+1)[\delta(x),x]\delta(x)[z,x]-(2n+1)\delta(x)[\delta(x),x][z,x]\\ &\quad +2n\delta(x)[\delta(x),x][z,x] \qquad (2.6)\\ &= \{(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\}[z,x].\end{aligned}$$

Similarly using (2.4) one can easily obtain

$$\{(2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)\}[z,x]=0. \tag{2.7}$$

Adding (2.6) and (2.7) we arrive at

$$2n\{[\delta(x),x]\delta(x)+2n\delta(x)[\delta(x),x]\}[z,x]=0.$$

Since $2n$ divides $(n+1)!$, we find that R is $2n$-torsion free and hence for all $x,z\in R$,

$$\{[\delta(x),x]\delta(x)+\delta(x)[\delta(x),x]\}[z,x]=0. \tag{2.8}$$

Using (2.8) in (2.6) we obtain $(2n+2)[\delta(x),x]\delta(x)[z,x]=0$ for all $x,z\in R$. Since $2(n+1)$ divides $(n+1)!$, we find that R is $2(n+1)$-torsion free and hence for all $x,z\in R$,

$$[\delta(x),x]\delta(x)[z,x]=0 \ \text{ for } \ \text{ all } \ x,z\in R.$$

Substituting yz for z we get $[\delta(x),x]\delta(x)y[z,x]=0$ for all $x,y,z\in R$. Replacing z by $\delta(x)$ we obtain $0=[\delta(x),x]\delta(x)y[\delta(x),x]\delta(x)$. Semiprimeness of R yields

$$[\delta(x),x]\delta(x)=0 \ \text{ for } \ \text{ all } \ x\in R. \tag{2.9}$$

Similarly application of (2.7) and (2.8) yields that

$$\delta(x)[\delta(x),x]=0 \ \text{ for } \ \text{ all } \ x\in R. \tag{2.10}$$

Replacing x by $x+ky$ in equation (2.10) where $1\le k\le 2n$ and implementing Lemma 1

$$\delta(x)[\delta(x),y]+n\delta(x)[\Delta(x,x,\ldots,y),x]+n\Delta(x,x,\ldots,y)[\delta(x),x]=0. \tag{2.11}$$

Replacing y by yx

$$\begin{aligned}0&=\delta(x)[\delta(x),yx]+n\delta(x)[y\delta(x)+\Delta(x,x,\ldots,y)x,x]\\&\quad+n\{y\delta(x)+\Delta(x,x,\ldots,y)x\}[\delta(x),x]\\&=\delta(x)y[\delta(x),x]+\delta(x)[\delta(x),y]x+n\delta(x)y[\delta(x),x]\\&\quad+n\delta(x)[y,x]\delta(x)+n\delta(x)[\Delta(x,x,\ldots,y),x]x\\&\quad+ny\delta(x)[\delta(x),x]+n\Delta(x,x,\ldots,y)x[\delta(x),x].\end{aligned}$$

From (2.11) we have

$$-n\Delta(x,x,\ldots,y)[\delta(x),x]x=\delta(x)[\delta(x),y]x+n\delta(x)[\Delta(x,x,\ldots,y),x]x. \tag{2.12}$$

Using (2.10) and (2.12) in the above relation, we get

$$\begin{aligned}0&=(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)+n\Delta(x,x,\ldots,y)x[\delta(x),x]\\&\quad-n\Delta(x,x,\ldots,y)[\delta(x),x]x\\&=(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)-n\Delta(x,x,\ldots,y)[[\delta(x),x]x].\end{aligned}$$

This gives that

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)=0 \ \text{ for } \ \text{ all } \ x,y\in R. \tag{2.13}$$

Substituting xy for y in (2.13)

$$(n+1)\delta(x)xy[\delta(x),x]+n\delta(x)x[y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.14}$$

Left multiply (2.13) by x, we obtain

$$(n+1)x\delta(x)y[\delta(x),x]+nx\delta(x)[y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.15}$$

Combining (2.14) and (2.15), we get

$$(n+1)[\delta(x),x]y[\delta(x),x]+n[\delta(x),x][y,x]\delta(x)=0 \text{ for all } x,y\in R. \tag{2.16}$$

Replacing y by yz in (2.4), we obtain

$$\begin{aligned}0&=(2n+1)[yz,x][\delta(x),x]+n[[yz,x],x]\delta(x)\\&=(2n+1)[yz,x][\delta(x),x]+n[y[z,x],x]\delta(x)+n[[y,x]z,x]\delta(x)\\&=(2n+1)y[z,x][\delta(x),x]+(2n+1)[y,x]z[\delta(x),x]+ny[[z,x],x]\delta(x)\\&\quad+n[y,x][z,x]\delta(x)+n[y,x][z,x]\delta(x)+n[[y,x],x]z\delta(x).\end{aligned}$$

Using (2.4) we get,

$$(2n+1)[y,x]z[\delta(x),x]+2n[y,x][z,x]\delta(x)+n[[y,x],x]z\delta(x)=0.$$

Replacing y by $\delta(x)$ in the above relation we get

$$(2n+1)[\delta(x),x]z[\delta(x),x]+2n[\delta(x),x][z,x]\delta(x)=0 \text{ for all } x,z\in R. \tag{2.17}$$

Combining equations (2.16) and (2.17) we find that

$$\begin{aligned}0&=(2n+1)[\delta(x),x]z[\delta(x),x]-2(n+1)[\delta(x),x]z[\delta(x),x]\\&=[\delta(x),x]z[\delta(x),x] \text{ for all } x,z\in R.\end{aligned}$$

Since R is semi-prime, we get $[\delta(x),x]=0$, for all $x\in R$. □

Theorem 2. *Let $n\geq 2$ be a fixed positive integer and R be a $(n+1)!$-torsion free semi-prime ring admitting a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x),x],x]\in Z(R)$ for all $x\in R$. Then δ is commuting on R.*

Proof. Replace x by $x+ky$ for $1\leq k\leq n+1$ in the given condition to find that

$$kQ_1(x,y)+k^2Q_2(x,y)+...+k^{n+1}Q_{n+1}(x,y)\in Z(R) \text{ for all } x,y\in R,$$

where $Q_i(x,y)$ denotes the sum of the terms in which y appears i times. By Lemma 1, we have for all $x,y\in R$,

$$[[\delta(x),x],y]+[[\delta(x),y],x]+n[[\Delta(x,x,...,y),x],x]\in Z(R). \tag{2.18}$$

Again replacing y by xy in the above expression, we get

$$\begin{aligned}&x([[\delta(x),x],y]+[[\delta(x),y],x]+n[[\Delta(x,x,...,y),x],x])+(n+2)[[\delta(x),x],x]y\\&\quad+(2n+1)[\delta(x),x][y,x]+n\delta(x)[[y,x],x]\in Z(R).\end{aligned}$$

Combining (2.18) with the latter relation, we find that

$$(3n+3)[[\delta(x),x],x][y,x]+(3n+1)[\delta(x),x][[y,x],x]+n\delta(x)[[[y,x],x],x]=0. \tag{2.19}$$

Further replace y by $\delta(x)$ in (2.19) to get

$$(6n+4)[[\delta(x),x],x][\delta(x),x]=0.$$

On commuting with x, we find that

$$(6n+4)[[\delta(x),x],x]^2=0. \tag{2.20}$$

Next, on replacing y by $[\delta(x),x]$ in (2.19) and using the given condition, we have

$$(3n+3)[[\delta(x),x],x]^2=0. \tag{2.21}$$

Now combine (2.20) and (2.21) to get $2[[\delta(x),x],x]^2=0$.
Since, R is (n+1)!-torsion free and also the center of semi-prime ring is free from nilpotent element, we have $[[\delta(x),x],x]=0$. From Theorem 1, δ is commuting on R. □

Combining Theorem 2 with Lemma 3, we can prove the following:

Corollary 1. *Let $n\geq 2$ be a fixed positive integer and R be a $(n+1)!$-torsion free semi-prime ring admitting a non-zero permuting n-derivation Δ such that the trace δ satisfies $[[\delta(x),x],x]\in Z(R)$ for all $x\in R$. Then R is commutative.*

Theorem 3. *Let $n\geq 1$ be a fixed positive integer and R be a non-commutative $n!$-torsion free prime ring admitting a permuting generalized n-derivation Ω with associated n-derivation Δ such that the trace ω of Ω is commuting on R. Then Ω is a left n-multiplier on R.*

Proof. Our hypothesis yields that

$$[\omega(x),x]=0 \text{ for all } x\in R. \tag{2.22}$$

It can be easily seen that

$$\omega(x+y)=\omega(x)+\omega(y)+\sum_{r=1}^{n-1}\binom{n}{r}p_r(x,y) \text{ for all } x,y\in R$$

where $p_r(x,y)=\Omega(\underbrace{x,x,\cdots,x}_{(n-r)-\text{times}},\underbrace{y,y,\cdots,y}_{r-\text{times}})$.

Substituting $x+\lambda y$, where λ $(1\leq\lambda\leq n)$ is a positive integer, in place of x in the above equation we obtain

$$\begin{aligned}0&=[\omega(x+\lambda y),x+\lambda y]\\&=[\omega(x)+\omega(\lambda y)+\sum_{r=1}^{n-1}\binom{n}{r}p_r(x,\lambda y),x+\lambda y].\end{aligned}$$

Using (2.22), we have

$$0 = \lambda\{[\omega(x), y] + \binom{n}{1}[p_1(x,y),x]\} + \lambda^2\{\binom{n}{1}[p_1(x,y),y] + \binom{n}{2}[p_2(x,y),x]\} + \cdots + \lambda^n\{[\omega(y),x] + \binom{n}{n-1}[p_{n-1}(x,y),x]\} \quad \text{for all } x,y \in R.$$

Implementing Lemma 1 we get

$$\begin{aligned} 0 &= [\omega(x), y] + \binom{n}{1}[p_1(x,y),x] \\ &= [\omega(x),y] + n[\Omega(x,x,\cdots,y),x]. \end{aligned}$$

Replacing y by yx we obtain

$$\begin{aligned} 0 &= y[\omega(x),x] + [\omega(x),y]x + n[\Omega(x,x,\cdots,y)x + y\Delta(x,x,\cdots,x),x] \\ &= [\omega(x),y]x + ny[\delta(x),x] + n[y,x]\delta(x) + n[\Omega(x,x,\cdots,y),x]x \\ &= n[y,x]\delta(x) + ny[\delta(x),x]. \end{aligned}$$

Again replacing y by zy for any $z \in R$ we have $[z,x]y\delta(x) = 0$ for all $x,y,z \in R$. Since R is prime we find that for any $x \notin Z(R), \delta(x) = 0$. Now for any $y \in Z(R)$ and $x \notin Z(R)$, $x + \lambda y \notin Z(R)$. Hence,

$$\begin{aligned} 0 &= \delta(x+\lambda y) \\ &= \lambda p_1(x,y) + \cdots + \lambda^{n-1}p_{n-1}(x,y) + \lambda^n\delta(y). \end{aligned}$$

Using Lemma 2 we obtain $\delta(y) = 0$ for any $y \in Z(R)$. Thus, $\delta(x) = 0$ for all $x \in R$. Lemma 1 yields that $\Delta = 0$. This implies that Ω acts as a left n-multiplier. □

Theorem 4. *Let $n \geq 2$ be a fixed positive integer and R be an $n!$-torsion free semi-prime ring admitting a permuting generalized n-derivation Ω with associated n-derivation Δ such that the trace w of Ω is centralizing on R. Then w is commuting on R.*

Proof. It is given that $[w(x),x] \in Z(R)$ for all $x \in R$. Using the similar arguments as used in Theorem 2, we obtain

$$[w(x),y] + n[\Omega(x,x,...,y),x] \in Z(R) \text{ for all } x,y \in R. \tag{2.23}$$

Replacing y by yx, we obtain

$$y[w(x),x] + [w(x),y]x + n[\Omega(x,x,...,y),x]x + ny[\delta(x),x] + n[y,x]\delta(x) \in Z(R).$$

Now in view of (2.23), we find that

$$0=[y,x][w(x),x]+n[y,x][\delta(x),x]+ny[[\delta(x),x],x]$$
$$+n[y,x][\delta(x),x]+n[[y,x],x]\delta(x) \text{ for all } x,y\in R. \tag{2.24}$$

Again replace y by $w(x)y$ to get

$$\begin{aligned}0=&\,w(x)[y,x][w(x),x]+[w(x),x]y[w(x),x]+nw(x)[y,x][\delta(x),x]\\&+n[w(x),x]y[\delta(x),x]+nw(x)y[[\delta(x),x],x]+nw(x)[y,x][\delta(x),x]\\&+n[w(x),x]y[\delta(x),x]+n[w(x),x][y,x]\delta(x)+nw(x)[[y,x],x]\delta(x)\\&+n[w(x),x][y,x]\delta(x)+n[[w(x),x],x]y\delta(x).\end{aligned}$$

Using (2.24) and the given condition, we find that

$$[w(x),x]y[w(x),x]+2n[w(x),x]y[\delta(x),x]+2n[w(x),x][y,x]\delta(x)=0. \tag{2.25}$$

Further, replacing y by $[w(x),x]^2$ in (2.25) and using the given condition, we have

$$[w(x),x]^4+2n[w(x),x]^3[\delta(x),x]=0 \text{ for all } x\in R. \tag{2.26}$$

Again, replace y by yz in (2.25) and use (2.25), to get

$$2n[w(x),x][y,x]z\delta(x)=0 \text{ for all } x,y,z\in R. \tag{2.27}$$

Next, we replace y by $w(x)$ and z by $[w(x),x]$ to find that $2n[w(x),x]^3\delta(x)=0$ for all $x\in R$. On commuting the latter relation with x and using the given condition, we have

$$2n[w(x),x]^3[\delta(x),x]=0 \text{ for all } x\in R. \tag{2.28}$$

From (2.26) and (2.28), we find that $[w(x),x]^4=0$. Since the center of a semi-prime ring does not contain any nilpotent element, we get $[w(x),x]=0$. □

Corollary 2. *Let $n\geq 2$ be a fixed positive integer and R be a non-commutative $n!$-torsion free semi-prime ring admitting a permuting generalized n-derivation Ω with associated n-derivation Δ such that the trace w of Ω is centralizing on R. Then Ω is a left n-multiplier on R.*

Proof. By Theorem 3 and Theorem 2, we get the required result. □

In conclusion, if we look at Theorem 2 closely, it is tempting to conjecture as follows:

Conjecture 1. *Let R be a semi-prime ring with suitable torsion restrictions and Δ be a non-zero permuting n-derivation. Suppose that for some integer $m\geq 1$, $\delta_m(x)\in Z(R)$ for all $x\in R$ where $\delta_{k+1}(x)=[\delta_k(x),x]$ for $k>1$ and $\delta_1(x)=\delta(x)$ stands for the trace of Δ. Then $[\delta(x),x]=0$ for all $x\in R$.*

ACKNOWLEDGEMENT

The authors are indebted to the referee for his/her useful suggestions and valuable comments.

REFERENCES

[1] M. Ashraf, "On symmetric bi-derivations in rings," *Rend. Istit. Mat. Univ. Trieste*, vol. 31, no. 1-2, pp. 25–36, 1999.

[2] M. Ashraf and M. R. Jamal, "Traces of Permuting n-additive maps and permuting n- derivations of rings," *Mediterr. J. Math*, vol. 11, no. 2, pp. 287–297, 2014, doi: 10.1017/S00009-015-0606-3.

[3] M. Ashraf, N. Parveen, and M. R. Jamal, "Traces of permuting n-derivations and commutativity of rings," *Southeast Asian Bulletin of Mathematics*, vol. 38, pp. 321–332, 2014.

[4] M. Brešar, "On the distance of the composition of two derivations to the generalized derivations," *Glasgow Math. J.*, vol. 33, pp. 89–93, 1991, doi: 10.1017/S0017089500008077.

[5] M. Brešar, "Commuting maps: a survey," *Taiwanese J. Math.*, vol. 8, no. 3, pp. 361–397, 2004.

[6] Q. Deng and H. E. Bell, "On derivations and commutativity in semiprime rings." *Comm. Algebra*, vol. 23, no. 10, pp. 3705–3713, 1995, doi: 10.1080/00927879508825427.

[7] K. Park, "On prime and semiprime rings with symmetric n-derivations," *J. Chungcheong Math. Soc.*, vol. 22, no. 3, pp. 451–458, 2009.

[8] E. Posner, "Derivations in prime rings," *Proc. Amer. Math. Soc.*, vol. 8, pp. 1093–1100, 1957, doi: 10.1090/S0002-9939-1957-0095863-0.

[9] J. Vukman, "Symmetric bi-derivations on prime and semiprime rings," *Aequationes Math.*, vol. 38, pp. 245–254, 1989, doi: 10.1007/BF01840009.

[10] J. Vukman, "Two results concerning symmetric bi-derivations on prime rings," *Aequationes Math.*, vol. 40, pp. 181–189, 1990, doi: 10.1007/BF0211294.

Authors' addresses

Mohammad Ashraf
Aligarh Muslim University, Department of Mathematics Aligarh, India
E-mail address: mashraf80@hotmail.com

Almas Khan
Aligarh Muslim University, Department of Mathematics Aligarh, India
E-mail address: almasleo@gmail.com

Malik Rashid Jamal
Integral University, Department of Mathematics Lucknow, India
E-mail address: rashidmaths@gmail.com