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# REFINEMENT OF HERMITE–HADAMARD TYPE INEQUALITIES FOR S–CONVEX FUNCTIONS

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*Abstract.* We give refinement of few inequalities from [13], related of the left–hand side of the Hermite–Hadamard type inequalities for the class of mappings whose second derivatives at certain powers are $s$-convex in the second sense.

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## 1. PRELIMINARIES

It is a well known fact that convexity and its generalizations play an important role in different parts of mathematics and science, mainly in optimization theory. Also, modern analysis directly or indirectly involves the applications of convexity. Due to its applications and significant importance, the concept of convexity has been extended and generalized in several directions. For example, this concept introduced the classes of $s$-convex functions, Godunova-Levin functions, $P$-functions, all included in class of $h$-convex functions (for more details see [6, 17] and literature therein).

These are reasons of the topmost motivations for examining the properties of generalized convex functions. In this note we will deal with the concepts of $s$-convexity, which were introduced by [1].

**Definition 1.** Let $s$ be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s$-convex (in the second sense), if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\lambda \in [0,1]$.

If $-f$ is $s$-convex, then $f$ is $s$-concave.

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It can be easily seen that convexity means just $s$-convexity when $s = 1$, so this concept is a generalization of convexity. It is an interesting fact that the set of nonnegative $s$-convex functions strictly flares as $s$ strictly decreases. In addition, $s$-convex functions have a good relationship with algebraic operations just like the convex ones. For example, the sum of two $s$-convex functions is $s$-convex and by multiplying an $s$-convex function with a non-negative scalar we get an $s$-convex function again (see [7] and [3]). Also, there is a nice correspondence between convex and $s$-convex functions. Namely, the class of $s$-convex functions belongs to the class of locally $s$-Hölder functions (see [2] and [12]), the class of 1-Hölder functions coincides with the class of locally Lipschitz functions and it is a known fact that a convex function is a locally Lipschitz one if it is locally bounded from above at a point of an open domain (see [10]).

Many interesting inequalities are proved for convex functions in the literature. For example, extensively studied result is Hermite-Hadamard's inequality. Hermite (1883) and Hadamard (1896) independently have shown that the convex functions are related to an integral inequality, and this inequality is known as Hermite-Hadamard inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality is Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if $f$ is concave. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$. For further generalizations and new inequalities related to Hermite-Hadamard inequality interested readers are referred to [4, 5, 8, 9, 11, 15, 16].

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose second derivatives at certain powers are $s$-convex functions in the second sense. For this class of functions, Sarikaya and Kiris got estimates of the expression $\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right|$ (see [13]). Our results represent improvement of the results given in [13].

## 2. Main results

In this section we will present some estimates of the expression on the left–hand side of Hermite–Hadamard inequality.

**Theorem 1.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^\circ$, $a, b \in I$ with $a < b$. If $|f''|^q$ is $s$-convex in second sense on $[a,b]$, for some fixed $s \in (0,1]$*

*and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big)\right| \le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\cdot\Big(\frac{|f''(a)|^q+|f''(b)|^q}{2}\Big)^{\frac{1}{q}}, \tag{2.1}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* If $\alpha \le \beta$ and $A,B>0$, we have $\big(\frac{A^\alpha+B^\alpha}{2}\big)^{\frac{1}{\alpha}} \le \big(\frac{A^\beta+B^\beta}{2}\big)^{\frac{1}{\beta}}$ (power means inequality). Specially, for $A,B\ge 0$ and $q\ge 1$, we have $\big(\frac{A^{\frac{1}{q}}+B^{\frac{1}{q}}}{2}\big)^q \le \frac{A+B}{2}$ (the case $AB=0$ is trivial). For $A=|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q$ and $B=(2^{s+1}-1)|f''(a)|^q+|f''(b)|^q$, according to Theorem 4 in [13], we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big)\right| \\
&\quad\le \frac{(b-a)^2}{2^{s+4}}\cdot\Big(\frac{2^s}{2p+1}\Big)^{\frac{1}{p}}\cdot\Bigg[\Big(\frac{|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q}{s+1}\Big)^{\frac{1}{q}} \\
&\qquad+\Big(\frac{(2^{s+1}-1)|f''(a)|^q+|f''(b)|^q}{s+1}\Big)^{\frac{1}{q}}\Bigg] \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\Big(\frac{2^{-s}}{s+1}\Big)^{\frac{1}{q}}\cdot\frac{1}{2}\cdot\Big[\big(|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q\big)^{\frac{1}{q}} \\
&\qquad+\big((2^{s+1}-1)|f''(a)|^q+|f''(b)|^q\big)^{\frac{1}{q}}\Big] \\
&\quad\le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\Big(\frac{2^{-s}}{s+1}\Big)^{\frac{1}{q}}\cdot\Bigg\{\frac{1}{2}\cdot\Big[\big(|f''(a)|^q+(2^{s+1}-1)|f''(b)|^q\big) \\
&\qquad+\big((2^{s+1}-1)|f''(a)|^q+|f''(b)|^q\big)\Big]\Bigg\}^{\frac{1}{q}} \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\Big(\frac{2^{-s}}{s+1}\Big)^{\frac{1}{q}}\cdot\Big[2^s\big(|f''(a)|^q+|f''(b)|^q\big)\Big]^{\frac{1}{q}} \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\cdot\Big(\frac{|f''(a)|^q+|f''(b)|^q}{2}\Big)^{\frac{1}{q}}.
\end{aligned}$$

□

*Remark* 1. If we take $s=1$ in Theorem 1, then inequality (2.1) becomes inequality obtained in Theorem 4 in [14].

**Theorem 2.** *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on $I^\circ$, $a,b\in I$ with $a<b$. If $|f''|^q$ is $s$-convex in second sense on $[a,b]$, for some fixed $s\in(0,1]$*

*and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\left(|f''(a)|+|f''(b)|\right), \tag{2.2}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Let $g(x) = (1+x)^q - x^q - 1$ for $x \ge 0$ and $q > 1$. Since $g'(x) = q[(1+x)^{q-1} - x^{q-1}] \ge 0$ and $g(0) = 0$, it follows that $g$ is nondecreasing on $[0,\infty)$, so we have $A^q + B^q \le (A+B)^q$ for $A, B \ge 0$ (the case $AB = 0$ is trivial). By Theorem 1 it follows

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\cdot\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}} \\
&\quad= \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\left(|f''(a)|^q+|f''(b)|^q\right)^{\frac{1}{q}} \\
&\quad\le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\left(|f''(a)|+|f''(b)|\right).
\end{aligned}$$

□

*Remark* 2. Note that the expression occurring on the right-hand side of the inequality proven in Theorem 4 in [13] is equal to the product of $2^{\frac{s}{p}}$ and the expression (2.2), so the inequality from Theorem 2 is refinement of that inequality. Also, if we take $s = 1$ in Theorem 2, then the right–hand side of inequality (2.2) becomes $R = \dfrac{(b-a)^2}{2^{\frac{1}{q}+3}(2p+1)^{\frac{1}{p}}}\cdot\left(|f''(a)|+|f''(b)|\right)$ and the right–hand side of inequality from Corollary 1 in [13] is $\dfrac{(b-a)^2}{2^{\frac{2}{q}+2}(2p+1)^{\frac{1}{p}}}\cdot\left(|f''(a)|+|f''(b)|\right) = 2^{\frac{1}{p}}\cdot R \ge R$, so we get refinement of this result, too.

## 3. Applications to special means

Of course, since the results of Section 2 improve the results from Section 2 in [13], results from Section 3 in [13] which are consequences of those results also can be improved. We will present some of them and we will present few new applications to special means.

Let $g : I \to I_1 \subseteq [0,\infty)$ be a convex function on $I$. Then $g^s$ is $s$-convex on $I$, $0 < s < 1$. We will use this in proofs of our propositions, that will be presented in further text.

For arbitrary positive real numbers $a, b$, we consider the applications of our theorems to the following special means:

(a) The arithmetic mean: $A = A(a,b) := \dfrac{a+b}{2}$;

(b) The geometric mean: $G = G(a,b) := \sqrt{ab}$;

(c) The harmonic mean: $H = H(a,b) := \dfrac{2ab}{a+b}$;

(d) The logarithmic mean: $L = L(a,b) := \begin{cases} a, & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a}, & \text{if } a \neq b \end{cases}$;

(e) The $p$–logarithmic mean: $L_p = L_p(a,b) := \begin{cases} a, & \text{if } a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b \end{cases}$, $p \in \mathbb{R} \setminus \{-1, 0\}$;

(f) The identric mean:

$$I = I(a,b) := \begin{cases} a, & \text{if } a = b \\ \frac{1}{e} \cdot \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a \neq b \end{cases}.$$

It is well known that $L_p$ is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A.$$

The following propositions hold:

**Proposition 1.** *Let* $0 < a < b$, $s \in (0,1]$ *and* $p > 1$. *Then we have*

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(b-a)^2}{4} \cdot \left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} \cdot A(a^{s-1}, b^{s-1}).$$

*Proof.* The proof is immediate from Theorem 2 applied for function $f : [a,b] \to \mathbb{R}$, $f(x) = x^{s+1}$, $s \in (0,1]$. □

*Remark* 3. Note that the left–hand side of the inequality in previous proposition does not depend on $p$, so we can conclude that

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \inf_{p>1} \left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} \cdot \frac{s(b-a)^2}{4} \cdot A(a^{s-1}, b^{s-1})$$

for any $s \in (0,1]$, $p > 1$, $a < b$. If $g(p) = \ln\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} = \frac{1}{p} \ln \frac{s+1}{2p+1}$ for $p \geq 1$ and $s \in (0,1]$, then we have $g'(p) = -\frac{1}{p^2}\left(\ln \frac{s+1}{2p+1} + \frac{2p}{2p+1}\right)$. If $h(p) = \ln \frac{s+1}{2p+1} + \frac{2p}{2p+1}$ for $p \geq 1$ and $s \in (0,1]$, then we have $h'(p) = -\frac{4p}{(2p+1)^2}$, so $h(p)$ decreases. Since $\lim\limits_{p\to\infty} h(p) = -\infty$ and $\lim\limits_{p\to 1+} h(p) = \ln \frac{s+1}{3} + \frac{2}{3}$ we have:

(a) if $0 < s \leq 3e^{-\frac{2}{3}} - 1 \approx 0.54$, then $h(p) \leq 0$ for $p \in [1, \infty)$, so $g'(p) = -\frac{1}{p^2} h(p) \geq 0$ for $p \in [1, \infty)$. It follows that $\inf\limits_{p>1}\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}} = \frac{s+1}{3}$, so, if

$p > 1$, $s \in (0, 3e^{-\frac{2}{3}} - 1]$ we have

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(s+1)}{12} \cdot (b-a)^2 \cdot A(a^{s-1}, b^{s-1});$$

(b) if $3e^{-\frac{2}{3}} - 1 < s \leq 1$, then $h(p) > 0$ for $p \in [1, p_0(s))$ and $h(p) < 0$ for $p \in (p_0(s), \infty)$, where $p_0(s)$ is an unique solution of equation $\ln \frac{s+1}{2p+1} + \frac{2p}{2p+1} = 0$. It follows that $g(p) = -\frac{1}{p^2} h(p)$ decreases on $(1, p_0(s))$ and increases on $(p_0(s), \infty)$, so for $p > 1$ and $s \in (3e^{-\frac{2}{3}} - 1, 1]$ we have

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(b-a)^2}{4} \cdot \left(\frac{s+1}{2p_0(s)+1}\right)^{\frac{1}{p_0(s)}} \cdot A(a^{s-1}, b^{s-1}).$$

**Proposition 2.** *Let* $0 < a < b$, $s \in (0,1]$ *and* $p > 1$. *Then we have*

$$\left|L_{s+1}^{s+1}(a,b) - A^{s+1}(a,b)\right| \leq \frac{s(b-a)^2}{4} \cdot \left(\frac{s+1}{4p+2}\right)^{\frac{1}{p}} \cdot \left[A\left(a^{q(s-1)}, b^{q(s-1)}\right)\right]^{\frac{1}{q}},$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* The proof is immediate from Theorem 1 applied for function $f : [a,b] \to \mathbb{R}$, $f(x) = x^{s+1}$, $s \in (0,1]$. $\square$

*Remark* 4. Under conditions of Proposition 2, for $s = 1$ we derive

$$\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{4} \cdot \left(\frac{1}{2p+1}\right)^{\frac{1}{p}},$$

which is refinement of the inequality derived in part 1(b) of Section 3 in [13] (note that there is a misprint on this place in [13], the right–hand side of inequality obtained there should be $\frac{(b-a)^2}{4} \cdot \left(\frac{4}{2p+1}\right)^{\frac{1}{p}}$). Also, the left–hand side of the inequality in derived inequality does not depend on $p$, so we have $\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{4} \cdot \inf\limits_{p>1} \left(\frac{1}{2p+1}\right)^{\frac{1}{p}}$. If $g(p) = \ln\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} = \frac{1}{p} \ln \frac{1}{2p+1}$ for $p \geq 1$, then we have $g'(p) = -\frac{1}{p^2}\left(\ln \frac{1}{2p+1} + \frac{2p}{2p+1}\right)$. Since it holds $\ln x < x - 1$ for $x > 1$, it follows that $\ln \frac{1}{2p+1} + \frac{2p}{2p+1} < \frac{1}{2p+1} - 1 + \frac{2p}{2p+1} = 0$, so we have $g'(p) > 0$ for $p > 1$. It follows that $g(p)$ increases on $[1, \infty)$. Therefore $\inf\limits_{p>1} e^{g(p)} = e^{g(1)} = \frac{1}{3}$, so we have $\left|L_2^2(a,b) - A^2(a,b)\right| \leq \frac{(b-a)^2}{12}$. This is the result derived in part 1(a) of Section 3 in [13], and since $L_2^2(a,b) - A^2(a,b) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(a-b)^2}{12}$ for $a \neq b$, we can conclude that inequality derived in Proposition 2 is sharp.

**Proposition 3.** *Let* $0 < a < b$ *and* $p > 1$. *Then we have*

$$\left|\frac{b^2 \ln b - a^2 \ln a}{b-a} - A(a,b)\left[1 + \ln A^2(a,b)\right]\right| \leq \frac{(b-a)^2}{2(4p+2)^{\frac{1}{p}}} \cdot \left[A\left(a^{-q}, b^{-q}\right)\right]^{\frac{1}{q}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* The proof is immediate from Proposition 2, after multiplying both side of the inequality with $\frac{2}{s}$ and passing to the limit $s \to 0+$. □

**Proposition 4.** *Let* $0<a<b$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|L_{-s}^{-s}(a,b)-A^{-s}(a,b)\right| \le \frac{s(b-a)^2}{4}\cdot\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}}\cdot A(a^{-s-2},b^{-s-2}).$$

*Proof.* The proof is immediate from Theorem 2 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=\dfrac{1}{x^s}$, $s\in(0,1]$. □

**Proposition 5.** *Let* $0<a<b$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|\ln\frac{I(a,b)}{A(a,b)}\right| \le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\cdot\frac{\left[A(a^{2q},b^{2q})\right]^{\frac{1}{q}}}{a^2b^2},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* The proof is immediate from Theorem 1 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=\ln x$. □

For instance, if $s=1$ then we have

$$\left|\ln\frac{I(a,b)}{A(a,b)}\right| \le \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\cdot\frac{\left[A(a^{2q},b^{2q})\right]^{\frac{1}{q}}}{a^2b^2}.$$

**Proposition 6.** *Let* $0<a<b$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|\ln\frac{I(a,b)}{A(a,b)}\right| \le \frac{(b-a)^2}{4(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\frac{A(a^2,b^2)}{a^2b^2},$$

$$\text{i.e.}\quad \left|\ln\frac{I(a,b)}{A(a,b)}\right| \le \frac{(b-a)^2}{4(2p+1)^{\frac{1}{p}}}\cdot\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\cdot\frac{1}{H(a^2,b^2)},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* The proof is immediate from Theorem 2 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=\ln x$. □

**Proposition 7.** *Let* $0<\alpha<\beta$, $s\in(0,1]$ *and* $p>1$. *Then we have*

$$\left|L_s^s(\alpha,\beta)-G^{s+1}(\alpha,\beta)\right| \le \frac{(s+1)\cdot\ln^2\frac{\beta}{\alpha}}{4}\cdot\left(\frac{s+1}{2p+1}\right)^{\frac{1}{p}}\cdot A(\alpha^{s+1},\beta^{s+1}).$$

*Proof.* The proof is immediate from Theorem 2 applied for function $f:[a,b]\to\mathbb{R}$, $f(x)=e^{(s+1)x}$, $s\in(0,1]$ and $\alpha=e^a$, $\beta=e^b$. □

For instance, if $s=1$ then we have

$$\left|L_1(\alpha,\beta)-G^2(\alpha,\beta)\right| \le \frac{\ln^2\frac{\beta}{\alpha}}{2^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\cdot A(\alpha^2,\beta^2),$$

where $\frac{1}{p}+\frac{1}{q}=1$.

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