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CONGRUENCES AND DECOMPOSITIONS OF AG**-GROUPOIDS

WAQAR KHAN, FAISAL YOUSAFZAI, AND KOSTAQ HILA

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Abstract. We introduce the concept of completely left inverse AG**-groupoids and study some basic congruences and a congruence pair by means of the kernel and trace approach of completely left inverse AG**-groupoids. Also, we provide separative and anti-separative decomposition of locally associative AG**-groupoids.

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1. INTRODUCTION

An *Abel-Grassmann's* groupoid (abbreviated as AG-groupoid) or Left Almost Semigroup (briefly LA-semigroup) is a groupoid S satisfying the left invertive law, defined as, $(ab)c = (cb)a$ for all $a, b, c \in S$. Inverse AG-groupoids, their different characterisations and congruences on inverse AG-groupoids using the kernel-normal system and kernel-trace approaches have been studied by many authors which can be found in the literature (see [1–4, 6, 7, 11]).

In this paper, we introduce *completely left inverse* AG**-groupoids and investigate a congruence pair consisting a kernel and trace of a congruence of a completely left inverse AG**-groupoid. In the second section, some preliminaries and basic results on completely inverse AG**-groupoids are mentioned. In Section 3, we introduce completely left inverse AG**-groupoids and investigate some basic congruences using the congruence pair. We show that if ρ is a congruence on a completely left inverse AG**-groupoid, then $(\ker\rho, \mathrm{tr}\rho)$ is a congruence. In Section 4, we discuss separative and anti-separative decompositions of a locally associative AG**-groupoid. Before the proofs of the main results, it is important to recall the basic knowledge and necessary terminology.

2. PRELIMINARIES

An AG-groupoid S is regular if $a \in (aS)a$ for all $a \in S$. If for $a \in S$, there exists an element a' such that $a = (aa')a$ and $a' = (a'a)a'$, then we say that a' is inverse of a. In addition, if inverses commute, that is $a^{'}a = aa^{'}$, then S is called completely regular. If $a \in S$, then

$$V(a) = \{a^{'} \in S : a = (aa^{'})a \text{ and } a^{'} = (a^{'}a)a^{'}\}$$

is called the set of all inverses of $a \in S$. Note that if $a^{'} \in V(a)$ and $b^{'} \in V(b)$, then $a \in V(a')$ and $a^{'}b^{'} \in V(ab)$.

An AG-groupoid S in which every element has a unique inverse is called inverse AG-groupoid. If a^{-1} is the unique inverse of $a \in S$, then a groupoid satisfying the following identities is called a completely inverse AG**-groupoid, that is for all $a, b, c \in S$

$$(ab)c = (cb)a,\ a(bc) = b(ac)$$
$$a = (aa^{-1})a,\ \ a^{-1} = (a^{-1}a)a^{-1} \text{ and } aa^{-1} = a^{-1}a.$$

If S is a completely inverse AG**-groupoid, then $a^{-1}a \in E_S$, where E_S is the set of idempotents of S. If S is a completely inverse AG**-groupoid, then E_S is either empty or a semilattice. For any idempotent e in E_S, $e^{-1} = e$. Moreover, the set E_S of an AG-groupoid S is a rectangular AG-band, that is for all $e, f \in E_S$, $e = (ef)e$. For futher concepts and results, the reader is referred to [3]. The set of idempotents E_S of an AG-groupoid S is called left (respectively; right) regular AG-band if it satisfies

$$(ef)e = ef \quad \text{(respectively; } (ef)e = fe) \text{ for all } e, f \in E_S.$$

Note that if S is an AG**-groupoid, then for $e, f \in E_S$

$$ef = (ee)(ff) = (ff)(ee) = fe$$

which shows left and right AG-bands serve the same purpose.

Lemma 1 ([3])**.** *Let S be a completely inverse* AG***-groupoid and let $a, b \in S$ such that $ab \in E_S$. Then $ab = ba$.*

Lemma 2 ([3])**.** *Completely inverse* AG***-groupoids are idempotent-surjective.*

If ρ is a congruence on a completely inverse AG**-groupoid, then S/ρ is also completely inverse AG**-groupoid. The natural morphism maps S onto S/ρ by the rule $x \to (x)_\rho$ and by the uniqueness of inverses $(x^{-1})_\rho = (x)_\rho^{-1}$. If $(a, b) \in \rho$, then $(a^{-1}, b^{-1}) \in \rho$ and $(aa^{-1}, bb^{-1}) \in \rho$.

3. CONGRUENCES IN COMPLETELY LEFT INVERSE AG**-GROUPOID

In this section, we introduce the notion of completely left inverse AG**-groupoids and study certain congruences by means of their kernel and trace for this class of groupoids. The essential part is to describe such congruence in terms of a congruence pair which comprises of a normal subgroupoid and a congruence of a completely left inverse AG**-groupoids.

Definition 1. A completely inverse AG**-groupoid is called completely left inverse AG**-groupoid if the set E_S of idempotents of S is a left regular AG-band.

Proposition 1. *Let ρ be a congruence on a completely inverse AG**-groupoid S. If $(a,e) \in \rho$ for $e \in E_S$, then $(a,a^{-1}) \in \rho$ and $(a,a^{-1}a) \in \rho$.*

Lemma 3. *Let S be a completely left inverse AG**-groupoid. If ρ is a congruence on S, then S/ρ is a completely left inverse AG**-groupoid.*

Proof. It is straightforward, and so it is omitted. □

Definition 2. A nonempty subset N of a completely left inverse AG**-groupoid S is called normal if

(1) $E_S \subseteq N$,

(2) for every $x \in S$, $x \cdot Nx^{-1} \subseteq N$,

(3) for every $a \in N$, $a^{-1} \in N$.

Let ρ be a congruence on a completely left inverse AG**-groupoid S and E_S be the set of idempotents of S. The restriction of ρ on E_S, that is $\rho|_{E_S}$ is the trace of ρ denoted by trρ. The subset

$$\ker\rho = \{a \in S : (\exists e \in E_S)(a,e) \in \rho\}$$

is the kernel of ρ.

Lemma 4. *Let ρ be a congruence on a completely left inverse AG**-groupoid S.*

(1) $\ker\rho$ *is a normal* AG***-subgroupoid of S.*

(2) *For any $a \in S$, $e \in E_S$, if $ea \in \ker\rho$ such that $(e,\ aa^{-1}) \in \mathrm{tr}\rho$, then $a \in \ker\rho$.*

(3) *For any $a \in S$, if $a \in \ker\rho$, then $(a^{-1}a,\ aa^{-1}) \in \mathrm{tr}\rho$.*

Proof. (1) Let ρ be a congruence. If $a,b \in \ker\rho$, then $(a,e) \in \rho$, $(b,f) \in \rho$ so that $(ab,ef) \in \rho$ for some $e,f \in E_S$. Hence $ab \in \ker\rho$ and $\ker\rho$ is a subgroupoid of S. Obviously, all the idempotents of S lie in $\ker\rho$. Let $a \in \ker\rho$, then $(a,e) \in \rho$ for $e \in E_S$. Therefore for all $x \in S$, $(x^{-1} \cdot ax,\ x^{-1} \cdot ex) \in \rho$. Since $x^{-1} \cdot ex = e \cdot x^{-1}x \in E_S$, thus $x^{-1} \cdot ax \in \ker\rho$. Now if $a \in \ker\rho$, then for $g \in E_S$, $(a,g) \in \rho$. Since S/ρ, is left inverse, it is clear that $(a)_\rho^{-1} \in V((a)_\rho)$. Moreover, if $h \in E_S$, then $a^{-1} \in V(h)$ so that $(a^{-1},h) \in \rho$. That is $a^{-1} \in \ker\rho$ for every $a \in \ker\rho$.

(2) If for $a \in S$, $e \in E_S$ and $ea \in \ker\rho$, then there exists $f \in E_S$ such that $(ea,\ f) \in \rho$. Since $(e,\ aa^{-1}) \in \mathrm{tr}\rho$, then $a = aa^{-1} \cdot a \equiv_\rho ea \equiv_\rho f$. Hence $(a,\ f) \in \rho$ and $a \in \ker\rho$.

(3) Let $a \in \ker\rho$. Then $(a,e) \in \rho$ for some $e \in E_S$. By Proposition 1, we have $(a^{-1}a,\ a^{-1}a) \in \rho$. Since $\mathrm{tr}\rho = \rho|_{E_S}$, it follows that $(a^{-1}a, aa^{-1}) \in \mathrm{tr}\rho$. □

Lemma 5. *Let ρ be a congruence on a completely left inverse AG**-groupoid S. If $a^{-1}b \in \ker\rho$, then $ab^{-1} \in \ker\rho$ and for all $a,b \in S$, $((a^{-1}b \cdot ab^{-1}), a^{-1}b) \in \rho$.*

Proof. Let ρ be a congruence on S. If $a^{-1}b \in \ker\rho$, then $(a^{-1}b,e) \in \rho$ for some $e \in E_S$. Then it is clear that $(ab^{-1})_\rho$ is inverse of $(a^{-1}b)_\rho$ in S/ρ and it follows immediately from the preliminaries and Lemma 3 that $(a^{-1}b)_\rho \in E_{S/\rho}$. Hence $(ab^{-1}, f) \in \rho$

for some $f \in E_S$. Thus $ab^{-1} \in \ker\rho$. Moreover, since S/ρ is inverse and E/ρ is a left regular AG-band, we have

$$(a^{-1}b \cdot ab^{-1})_\rho = (e)_\rho (f)_\rho = ((e)_\rho (f)_\rho)(e)_\rho = (e)_\rho = (a^{-1}b)_\rho.$$

□

Lemma 6. *Let ρ be a congruence on a completely left inverse* AG***-groupoid S and let $a, b \in S$ and $e \in E_S$. If $(aa^{-1},\, bb^{-1}) \in \mathrm{tr}\rho$ and $ab^{-1} \in \ker\rho$, then*

$$(a \cdot ea^{-1},\, b \cdot eb^{-1}) \in \mathrm{tr}\rho.$$

Proof. Let ρ be a congruence on S. Let $a, b \in S$ such that $(aa^{-1},\, bb^{-1}) \in \mathrm{tr}\rho$ and $ab^{-1} \in \ker\rho$. Then for all $e \in E_S$, we have

$$\begin{aligned}
a \cdot ea^{-1} &\underset{\rho}{\equiv} a(e(a^{-1}a \cdot a^{-1}) \\
&\equiv a(e(b^{-1}b \cdot a^{-1})) && \text{(since } (aa^{-1}, bb^{-1}) \in \mathrm{tr}\rho) \\
&\equiv a(e(a^{-1}b \cdot b^{-1})) \\
&\equiv a(e((a^{-1}b \cdot (a^{-1}b)^{-1})b^{-1})) && \text{(since } (a^{-1}b)\rho \in E_{S/\rho}) \\
&\equiv a((a^{-1}a \cdot bb^{-1})(eb^{-1})) \\
&\equiv a(b^{-1}b \cdot eb^{-1}) && \text{(since } (aa^{-1},\, bb^{-1}) \in \mathrm{tr}\rho) \\
&\equiv e \cdot ab^{-1} \\
&\equiv e(ab^{-1} \cdot a^{-1}b) && \text{(by Lemma 5)} \\
&\equiv e(aa^{-1} \cdot bb^{-1}) \\
&\underset{\rho}{\equiv} b \cdot eb^{-1} && \text{(since } (aa^{-1},\, bb^{-1}) \in \mathrm{tr}\rho).
\end{aligned}$$

Thus $(a \cdot ea^{-1},\, b \cdot eb^{-1}) \in \mathrm{tr}\rho$. □

Definition 3. Let N be normal subgroupoid of a completely left inverse AG**-groupoid S and τ be congruence on a left regular AG-band E_S. Then (N, τ) is a congruence pair of S if for all $a,\, b \in S$ and $e \in E_S$ the following conditions hold.

(1) If $ea \in N$ and $(e,\, a^{-1}a) \in \tau$, then $a \in N$,

(2) If $(aa^{-1},\, bb^{-1}) \in \tau$ and $a^{-1}b \in N$, then $(a \cdot ea^{-1},\, b \cdot eb^{-1}) \in \tau$.

Theorem 1. *Let S be a completely left inverse* AG***-groupoid and $(N,\, \tau)$ is congruence pair on S. Then the relation*

$$\rho_{(N,\,\tau)} = \{(a, b) \in S \times S : \ (aa^{-1}, bb^{-1}) \in \tau \text{ and } a^{-1}b \in N\}$$

is a congruence relation.

Proof. Clearly, $(N,\, \tau)$ is reflexive. $\rho_{(N,\,\tau)}$ is symmetric. In fact: τ is symmetric and by Definition 2 (3), $a^{-1} \in N$ for any $a \in N$. Also, if $(a,\, b),\ (b,\, c) \in \rho$, then $(aa^{-1},\, bb^{-1}) \in \tau$, $(bb^{-1},\, cc^{-1}) \in \tau$ and $a^{-1}b,\ b^{-1}c \in N$. Then by Lemma 1,

$cb^{-1} \in N$. Hence $(aa^{-1},\ cc^{-1}) \in \tau$. Since $(c^{-1}\{(aa^{-1} \cdot bb^{-1})c\}) \in E_S$ and N is normal AG**-subgroupoid, then

$$\begin{aligned}(c^{-1}\{(aa^{-1} \cdot bb^{-1})c\})(a^{-1}c) &= \{c^{-1}(a^{-1}a \cdot c)\}(a^{-1}c) \text{ (since } (a^{-1}a,\ bb^{-1}) \in \tau)\\ &= \{a^{-1}a \cdot c^{-1}c\}(a^{-1}c)\\ &= \{a^{-1}a \cdot b^{-1}b\}(a^{-1}c) \text{ (since } (bb^{-1},\ cc^{-1}) \in \tau)\\ &= (bb^{-1})(a^{-1}c) \text{ (since } (a^{-1}a,\ bb^{-1}) \in \tau)\\ &= (cb^{-1})(a^{-1}b) \in N.\end{aligned}$$

Moreover,

$$\begin{aligned}(a^{-1}c)^{-1}(a^{-1}c) &= c^{-1}(aa^{-1} \cdot c)\\ &= c^{-1}((aa^{-1} \cdot cc^{-1})c)\tau c^{-1}((aa^{-1} \cdot bb^{-1})c)\\ &\quad \text{(since } bb^{-1}\tau cc^{-1}).\end{aligned}$$

Hence by Definition 3 (1), $a^{-1}c \in N$ which implies that $(a,c) \in \rho_{(N,\tau)}$. Thus $\rho_{(N,\ \tau)}$ is equivalence relation.

Let $(a,b) \in \rho_{(N,\ \tau)}$, then $(ac,\ bc) \in \rho_{(N,\ \tau)}$. In fact: if $(aa^{-1},\ bb^{-1}) \in \rho$ and $a^{-1}b \in \ker\rho$, then by Definition 2 (2), $(ac)^{-1}(bc) \in N$. Further, using Definition 3 (2), we have

$$(ac)(ac)^{-1} = (aa^{-1})(cc^{-1}) = (c^{-1}c)(aa^{-1}) = a(c^{-1}c \cdot a^{-1})\tau b(c^{-1}c \cdot b^{-1}).$$

Hence by definition of the relation $\rho_{(N,\ \tau)}$, $(ac,\ bc) \in \rho_{(N,\ \tau)}$.

Similarly, since τ is a congruence, then $(aa^{-1},\ bb^{-1}) \in \tau$ implies that

$$(ca)(ca)^{-1} = ca \cdot c^{-1}a^{-1} = a^{-1}a \cdot cc^{-1} = c(aa^{-1} \cdot c^{-1})\tau c(bb^{-1} \cdot c^{-1}).$$

It remains to show that $(ca)^{-1}(cb) \in N$. Therefore

$$\begin{aligned}(b^{-1}((c^{-1}c \cdot aa^{-1})b))((ca)^{-1}(cb)) &= ((c^{-1}c \cdot aa^{-1})(b^{-1}b))(c^{-1}a^{-1} \cdot cb)\\ &= ((c^{-1}c \cdot aa^{-1})(b^{-1}b))(c^{-1}c \cdot a^{-1}b)\\ &= ((c^{-1}c)(aa^{-1} \cdot b^{-1}b)(c^{-1}c))(a^{-1}b)\\ &= ((c^{-1}c)(aa^{-1} \cdot b^{-1}b))(a^{-1}b)\\ &\quad \text{(since } E_S \text{ is left regular)}\\ &= (b^{-1}((aa^{-1} \cdot cc^{-1})b))(a^{-1}b) \in N.\end{aligned}$$

Moreover,

$$\begin{aligned}b^{-1}((c^{-1}c \cdot aa^{-1})b) &= ((c^{-1}c \cdot aa^{-1})(c^{-1}c))(b^{-1}b) \ \text{ (since } E_S \text{ is left regular)}\\ &= ((a^{-1}c \cdot ac^{-1})(c^{-1}c))(b^{-1}b)\\ &= ((ac \cdot c^{-1})(a^{-1}c^{-1} \cdot c))(b^{-1}b)\end{aligned}$$

$$= (b(cc^{-1} \cdot a^{-1}))(b^{-1}(c^{-1}c \cdot a)).$$

Thus $(b^{-1}((c^{-1}c \cdot aa^{-1})b))\tau(b(cc^{-1} \cdot a^{-1}))(b^{-1}(c^{-1}c \cdot a))$. Hence by Definition 3 (1) it follows that $a^{-1}(c^{-1}c \cdot b) \in N$. Thus $(ca,\ cb) \in \rho_{(N,\ \tau)}$.

□

Corollary 1. *Let S be a completely left inverse* AG***-groupoid and $(N,\ \tau)$ is congruence pair on S. Then the relation*

$$\rho_{(N,\ \tau)} = \{(a,b) \in S \times S :\ (aa^{-1}, bb^{-1}) \in \tau \text{ and } ba^{-1} \in N\}$$

is a congruence relation.

Theorem 2. *Let S be a completely left inverse* AG***-groupoid. If ρ is a congruence on S, then* (kerρ, trρ) *is a congruence of S. Conversely, if $(N,\ \tau)$ is congruence pair on S, then the relation*

$$\rho_{(N,\ \tau)} = \{(a,b) \in S \times S :\ (aa^{-1}, bb^{-1}) \in \tau \text{ and } a^{-1}b \in N\}$$

is a congruence relation on S. Furthermore,

$$\ker\rho_{(N,\tau)} = N, \mathrm{tr}\rho_{(N,\tau)} = \tau \text{ and } \rho_{(\ker\rho,\mathrm{tr}\rho)} = \rho.$$

Proof. The proof of the first part can be followed from Lemma 4, 6 and Theorem 1. We show that $\ker\rho_{(N,\ \tau)} = N$ and $\mathrm{tr}\rho_{(N,\tau)} = \tau$. Let $a \in \ker\rho_{(N,\tau)}$, then for some $e \in E_S$, $(a,e) \in \rho_{(N,\ \tau)}$. It follows that $(ee^{-1}, aa^{-1}) \in \tau$ and $ea \in N$. Thus by Definition 3 (1), $a \in N$, that is $\ker\rho_{(N,\ \tau)} \subseteq N$. Conversely, suppose that $a \in N$. Then $a^{-1} \in N$. Let $a^{-1}a = e$, it is clear that $(ee^{-1},\ aa^{-1}) \in \tau$ and $e^{-1}a = ea = a^{-1}a \cdot a = a \in N$. Thus $(e,a) \in \rho_{(N,\tau)}$. Hence $a \in (e)_{\rho_{(N,\ \tau)}} \subseteq \ker\rho_{(N,\tau)}$. Thus $\ker\rho_{(N,\ \tau)} = N$.

Similarly, we show that $\mathrm{tr}\rho_{(N,\ \tau)} \subseteq \tau$ and $\tau \subseteq \mathrm{tr}\rho_{(N,\ \tau)}$. Let $e, f \in E_S$ such that $(e,f) \in \mathrm{tr}\rho_{(N,\tau)}$. Then since E_S is left regular, therefore $e = (ee^{-1})e = ee^{-1} \equiv_\tau ff^{-1} = (ff^{-1})f = f$ and hence $\mathrm{tr}\rho_{(N,\ \tau)} \subseteq \tau$. Conversely, if $e \equiv_\tau f$, then $ee^{-1} = e \equiv_\tau f = ff^{-1}$ and $e^{-1}f \in E_S \subseteq N$. Thus by definition of $\rho_{(N,\tau)}$, it follows $(e,\ f) \in \rho_{(N,\ \tau)} \cap E_S \times E_S = \mathrm{tr}\rho$. Thus $\mathrm{tr}\rho_{(N,\ \tau)} = \tau$.

Finally, suppose that $(a,b) \in \rho$. Then $(a^{-1}a, a^{-1}b) \in \rho$ so that $a^{-1}b \in \ker\rho$. If a^{-1} is the inverse of a and since S/ρ is completely left inverse, then $(a^{-1})_\rho \in V((a)_\rho) = V((b)_\rho)$. Also, $(b^{-1})_\rho \in V((b)_\rho) = V((a)_\rho)$. It is clear that $(a)_\rho(a^{-1})_\rho = (a)_\rho(b^{-1})_\rho = (b)_\rho(b^{-1})_\rho$ which further implies that
$(aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho$. Thus $(a,\ b) \in \rho_{(\ker\rho,\mathrm{tr}\rho)}$ and $\rho \subseteq \rho_{(\ker\rho,\ \mathrm{tr}\rho)}$. Conversely, let $(a,\ b) \in \rho_{(\ker\rho,\mathrm{tr}\rho)}$. Then $(aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho$ and $a^{-1}b \in \ker\rho$. By Lemma 5, $ab^{-1} \in \ker\rho$ and $(ab^{-1})_\rho \in E_{S/\rho}$. Then there exists $e \in E_S$ such that $(ab^{-1})_\rho = (e)_\rho$, where $(e)_\rho \in E|\rho$. Since $E_{S/\rho}$ is left regular, thus by Lemma 5, we have

$$(ab^{-1})_\rho = (ab^{-1})_\rho((ab^{-1})^{-1})_\rho.$$

Then

$$a \equiv_\rho aa^{-1} \cdot a = bb^{-1} \cdot a \quad (\text{since } (aa^{-1},\ bb^{-1}) \in \mathrm{tr}\rho)$$

$$\begin{aligned}
&\equiv_\rho ab^{-1}\cdot b\\
&\equiv_\rho ((ab^{-1})(ab^{-1})^{-1})b\\
&\equiv_\rho (aa^{-1}\cdot b^{-1}b)b\\
&\equiv_\rho b^{-1}b\cdot b \qquad\qquad (\text{since } (aa^{-1},\ bb^{-1})\in \mathrm{tr}\rho)\\
&\equiv_\rho b.
\end{aligned}$$

Hence $\rho_{(\ker\rho,\mathrm{tr}\rho)}=\rho$. This completes the proof. □

4. Decompositions of locally associative AG**-groupoids

An AG-groupoid has many characteristics similar to that of a commutative semigroup. Let us consider $x^2y^2 = y^2x^2$, which holds for all x, y in a commutative semigroup. On the other hand one can easily see that it holds in an AG**-groupoid. This simply gives that how an AG**-groupoid has closed connections with commutative algebra. In this section, we generalize the results of Hewitt and Zuckerman for commutative semigroups [5].

An AG-groupoid S is called a locally associative AG-groupoid if $a\cdot aa = aa\cdot a$ for all $a\in S$ [8].

Note that a locally associative AG-groupoid does not necessarily have associative powers. For example, in a locally associative AG-groupoid $S=\{a,b,c\}$, defined by the following table [8]:

$\cdot$	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

$$(a\cdot aa)a = b \neq c = a(a\cdot aa).$$

Definition 4. A locally associative AG**-groupoid is an AG**-groupoid S satisfying an identity $a\cdot aa = aa\cdot a$ for all $a\in S$.

Example 1. Let us consider an AG**-groupoid $S=\{a,b,c,d,e\}$ in the following multiplication table.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

It is easy to verify that S is a locally associative AG***-groupoid.*

Proposition 2. *The following statements hold:*

(1) *Every locally associative* AG***-groupoid has associative powers, that is* $aa^n = a^n a$ *for all* $a \in S$ *and positive integer* n [8].
(2) *In an* AG***-groupoid* S, $a^m a^n = a^{m+n}$ *for all* $a \in S$ *and positive integers* m, n [8].
(3) *In a locally associative* AG***-groupoid* S, $(a^m)^n = a^{mn}$ *for all* $a \in S$ *and positive integers* m, n [10].
(4) *If* S *is a locally associative* AG***-groupoid and* $a, b \in S$, *then* $(ab)^n = a^n b^n$ *for any* $n \geq 1$ *and* $(ab)^n = b^n a^n$ *for any* $n \geq 2$ [9].
(5) *Let* S *be a locally associative* AG***-groupoid. Then* $a^n = a^{n-1}a = aa^{n-1}$ *for all* $a \in S$ *and* $n > 1$ [10].
(6) *If* S *is a locally associative* AG***-groupoid and* $a, b \in S$, *then* $a^n b^m = b^m a^n$ *for* $m, n > 1$ [8].

Note that $a^{n-1}a = ((((aa)a)a)...a)a$ and $aa^{n-1} = a((((aa)a)a)...a)$.

4.1. *Separative decomposition*

If S is a locally associative AG**-groupoid, then $ab^n \cdot c = a \cdot b^n c$ is not generally true for all $a, b, c \in S$, that is $(Sx^n)S \neq S(x^n S)$ for some $x \in S$.

Let us define the relations λ and μ in a locally associative AG**-groupoid S as follows:

for all $a, b \in S$, $a\lambda b \Longleftrightarrow$ there exists $n \in \mathbb{N}$, such that $a^n \in S(b^n S)$ and $b^n \in S(a^n S)$.

for all $a, b \in S$, $a\mu b \Longleftrightarrow$ there exists $n \in \mathbb{N}$, such that $a^n \in (Sb^n)S$ and $b^n \in (Sa^n)S$.

Theorem 3. *λ is equivalent to μ on a locally associative* AG***-groupoid* S.

Proof. Let $a^n \in S(b^n S)$. Then by using Proposition 2(3), we get

$$\begin{aligned} a^{2n} &= (a^n)^2 \in (S \cdot b^n S)^2 = (S \cdot b^n S)(S \cdot b^n S) = (SS)(b^n S \cdot b^n S) \\ &= (SS)(b^n b^n \cdot SS) \\ &= (b^n b^n)(SS \cdot SS) = (SS \cdot SS)(b^n b^n) = (b^n b^n \cdot SS)(SS) \\ &= (SS \cdot b^n b^n)(SS) \\ &\subseteq (Sb^{2n})S. \end{aligned}$$

Similarly, we can show that $b^n \in S(a^n S)$ implies $b^{2n} \in (Sa^{2n})S$.

Conversely, assume that $a^n \in (Sb^n)S$. Then by using Proposition 2(3), we get

$$\begin{aligned} a^{2n} &= (a^n)^2 \in (Sb^n \cdot S)^2 = (Sb^n \cdot S)(Sb^n \cdot S) = (Sb^n \cdot Sb^n)(SS) \\ &= (SS \cdot b^n b^n)(SS) \\ &= (SS)(b^n b^n \cdot SS) \subseteq S(b^{2n} S). \end{aligned}$$

Similarly, we can show that $b^n \in (Sa^n)S$ implies $b^{2n} \in S(a^{2n} S)$. Thus λ is equivalent to μ. $\square$

Theorem 4. *The relation λ on a locally associative* AG**-*groupoid S is a congruence relation.*

Proof. Clearly λ is reflexive and symmetric. For transitivity, let us suppose that $a\lambda b$ and $b\lambda c$, such that $a^n \in S(b^n S)$ and $b^n \in S \cdot c^n S$ for all $a,b,c \in S$ with assumption that $n > 1$. By using Proposition 2(3), we get

$$a^n \in S(b^n S) = b^n(SS) \subseteq (S \cdot c^n S)S = (c^n \cdot SS)S \subseteq (c^n S)S = (SS)c^n$$
$$= SS \cdot c^{n-1}c = cc^{n-1} \cdot SS = c^n(SS) = S(c^n S).$$

Similarly, we can show that $c^n \in S(a^n S)$. Hence λ is an equivalence relation. To show that λ is compatible, assume that $a\lambda b$ such that for $n > 1$, $a^n \in S(b^n S)$ and $b^n \in S(a^n S)$ for all $a,b \in S$. Let $c \in S$, then

$$(ac)^n = a^n c^n \in (S \cdot b^n S)c^n = (b^n \cdot SS)c^n = (b^{n-1}b \cdot SS)c^n = (SS \cdot bb^{n-1})c^n$$
$$= (SS \cdot b^n)c^n = c^n b^n \cdot SS = b^n c^n \cdot SS = S(b^n c^n \cdot S) = S \cdot (bc)^n S.$$

Similarly, we can show that $(ca)^n \in S((cb)^n S)$. Hence λ is a congruence relation on S. □

Definition 5. A congruence σ is said to be separative congruence in S, if $ab \equiv_\sigma a^2$ and $ab \equiv_\sigma b^2$ implies that $a \equiv_\sigma b$.

Theorem 5. *The relation λ on a locally associative* AG**-*groupoid S is separative.*

Proof. Let $a,b \in S$ such that $ab \equiv_\lambda a^2$ and $ab \equiv_\lambda b^2$. Then for a positive integer n,

$$(ab)^n \in S \cdot (a^2)^n S, \;\; (a^2)^n \in S \cdot (ab)^n S$$

and

$$(ab)^n \in S \cdot (b^2)^n S, \;\; (b^2)^n \in S \cdot (ab)^n S.$$

Now

$$a^{2n} = (a^2)^n \in S \cdot (ab)^n S \in S \cdot (S \cdot (b^2)^n S)S = (S \cdot (b^2)^n S)(SS)$$
$$= ((b^2)^n \cdot SS)(SS)$$
$$= (SS \cdot SS)(b^n b^n) = (b^n b^n)(SS \cdot SS) = (SS)(b^n b^n \cdot SS) \subseteq S(b^{2n} S).$$

Similarly we can show that $b^{2n} \in S(a^{2n} S)$. Hence λ is separative. □

Proposition 3. *If S is a locally associative* AG**-*groupoid, then $ab \equiv_\lambda ba$ for all $a,b \in S$, that is λ is commutative.*

Proof. Let $a,b \in S$ such that $a \equiv_\lambda b$ and n be a positive integer. Then by using Proposition 2(4), we get

$$(ab)^n = a^n b^n \in (S \cdot b^n S)(S \cdot a^n S) = (SS)(b^n S \cdot a^n S)$$
$$= (SS)(b^n a^n \cdot SS) \subseteq S(ba)^n \cdot S.$$

Similarly, we can show that $(ba)^n \in S(ab)^n \cdot S$. Hence $ab\lambda ba$. □

Corollary 2. *Let S be a locally associative* AG***-groupoid. Then S/λ is a separative commutative image of S.*

Let us define a relation γ on a locally associative AG**-groupoid S as follows: for all $x, y \in S$, $x\gamma y \iff$ there exists $n \in \mathbb{N}$, such that $(xa)^n \in (ya)^n S$ and $(ya)^n \in (xa)^n S$, for some $a \in S$.

Theorem 6. *The relation γ is a congruence relation on a locally associative* AG***-groupoid S.*

Proof. Clearly γ is reflexive and symmetric. For transitivity let us suppose that $x\gamma y$ and $y\gamma z$, then there exist positive integers m,n such that $(xa)^n \in (ya)^n S$, $(ya)^n \in (xa)^n S$ and $(ya)^m \in (za)^m S$ and $(za)^m \in (ya)^m S$, for some $a \in S$. More specifically, there exists $t_1 \in S$ such that $(xa)^n = (ya)^n t_1$. Assume that $m,n > 1$. Now by using Proposition 2(3) and Proposition 2(4), we get

$$\begin{aligned}(xa)^{mn} &= \left((xa)^n\right)^m = \left((ya)^n t_1\right)^m = \left((ya)^m\right)^n t_1^m \subseteq \left((za)^m S\right)^n S \\ &= (za)^{mn} S^n \cdot S \\ &= (SS^n)(za)^{mn} = (SS^n)\cdot(za)^{mn-1}(za) = (za)(za)^{mn-1}\cdot(S^n S) \\ &\subseteq (za)^{mn} S.\end{aligned}$$

Similarly we can show that $(za)^{mn} \in (xa)^{mn} S$. Hence γ is an equivalence relation on S.

To show compatibility, let $x\gamma y$, then there exists a positive integer n such that $(xa)^n \in (ya)^n S$ and $(ya)^n \in (xa)^n S$. Hence there exists $t_3 \in S$ such that $(xa)^n = (ya)^n t_3$. Now using Proposition 2(3), Proposition 2(4) and Proposition 2(6) with assumption that $n > 1$, we get

$$\begin{aligned}(xz\cdot a)^{2n} &= \left((xz\cdot a)^n\right)^2 = \left((xz)^n a^n\right)^2 = (x^n z^n \cdot a^n)^2 = (z^n x^n \cdot a^n)^2 \\ &= (a^n x^n \cdot z^n)^2 = (x^n a^n \cdot z^n)^2 = \left((xa)^n z^n\right)^2 = \left((ya)^n t_3 \cdot z^n\right)^2 \\ &= \left((z^n t_3)(ya)^n\right)^2 = (z^{2n} t_3^2)(ya)^{2n} = (z^n z^n \cdot t_3 t_3)(ya)^{2n} \\ &= (t_3 t_3 \cdot z^n z^n)(ya)^{2n} \\ &= (t_3^2 z^{2n})(ya)^{2n} = \left((t_3 z^n)(ya)^n\right)^2 = \left((ya)^n z^n \cdot t_3\right)^2 \\ &= \left((y^n a^n) z^n \cdot t_3\right)^2 \\ &= \left((a^n y^n) z^n \cdot t_3\right)^2 = \left((z^n y^n) a^n \cdot t_3\right)^2 = \left((y^n z^n) a^n \cdot t_3\right)^2 \\ &= \left((yz\cdot a)^n t_3\right)^2 \\ &= (yz\cdot a)^{2n} t_3^2 \in (yz\cdot a)^{2n} S.\end{aligned}$$

Similarly, we can show that $(yz\cdot a)^{2n} \in (xz\cdot a)^{2n} S$. Therefore $xz\gamma yz$. Similarly we can show that γ is left compatible. Hence γ is a congruence relation on S. □

4.2. *Anti-separative decomposition*

In this section, we show that S/τ is a maximal anti-separative commutative image of a locally associative AG**-groupoid S, where τ is defined as follows:

$a\tau b$ if and only if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for all $a,b \in S$ and a positive integer n.

Lemma 7. *Let S be a locally associative* AG***-groupoid. If $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$ for $a,b \in S$ and positive integers m,n, then $a\tau b$.*

Proof. Without loss of generality let us suppose that $n > m$. Thus by using Proposition 2(2), we get

$$b^{n-m}b^{m+1} = b^{n-m} \cdot ab^m = a \cdot b^{n-m}b^m = ab^n.$$

Hence $a\tau b$. □

Theorem 7. *The relation τ on a locally associative* AG***-groupoid S is a congruence relation.*

Proof. Clearly τ is reflexive and symmetric. For transitivity, let $a\tau b$ and $b\tau c$, so there exist positive integers m,n such that $ab^n = b^{n+1}$, $ba^n = a^{n+1}$ and $bc^m = c^{m+1}$, $cb^m = b^{m+1}$. Let $k = (n+1)(m+1)-1$, that is $k = n(m+1)+m$. Now by using Proposition 2(3) and Proposition 2(6), we get

$$\begin{aligned}
ac^k &= ac^{n(m+1)+m} = a \cdot c^{n(m+1)}c^m = a \cdot (c^{m+1})^n c^m = a \cdot (bc^m)^n c^m \\
&= a \cdot (b^n c^{mn})c^m \\
&= b^n c^{mn} \cdot ac^m = b^n a \cdot c^{mn}c^m = c^m c^{mn} \cdot ab^n = c^m c^{mn} \cdot b^{n+1} \\
&= c^{m(n+1)}b^{n+1} \\
&= c^{m(n+1)-1}c \cdot b^n b = bb^n \cdot cc^{m(n+1)-1} = b^{n+1}c^{m(n+1)} = (bc^m)^{n+1} \\
&= (c^{m+1})^{n+1} \\
&= c^{(m+1)(n+1)} = c^{k+1}.
\end{aligned}$$

Similarly, we can show that $ca^k = a^{k+1}$. Thus τ is an equivalence relation. To show that τ is compatible, assume that $a\tau b$ such that for some integer n,

$$ab^n = b^{n+1} \text{ and } ba^n = a^{n+1}.$$

Let $c \in S$. By using Proposition 2(4), we get

$$(ac)(bc)^n = ac \cdot b^n c^n = ab^n \cdot cc^n = b^{n+1}c^{n+1} = (bc)^{n+1}.$$

Similarly, we can show that $(bc)(ac)^n = (ac)^{n+1}$. Hence τ is a congruence relation on S. □

Definition 6. A congruence σ is said to be anti-separative congruence in S, if $ab \equiv_\sigma a^2$ and $ba \equiv_\sigma b^2$ implies that $a \equiv_\sigma b$.

Theorem 8. *The relation τ is anti-separative.*

Proof. Let $a, b \in S$ such that $ab \equiv_\tau a^2$ and $ba \equiv_\tau b^2$. Then by definition of τ there exist positive integers m and n such that

$$(ab)(a^2)^m = (a^2)^{m+1},\ a^2(ab)^m = (ab)^{m+1},$$

and

$$(ba)(b^2)^n = (b^2)^{n+1},\ b^2(ba)^n = (ba)^{n+1}.$$

Now by using Proposition 2(2) and Proposition 2(3), we get

$$\begin{aligned} ba^{2m+1} &= b \cdot a^{2m} a = a^{2m} \cdot ba = a^m a^m \cdot ba = ab \cdot a^m a^m = ab \cdot a^{2m} \\ &= (ab)(a^2)^m = (a^2)^{m+1} = a^{2m+2}, \end{aligned}$$

and

$$\begin{aligned} ab^{2n+1} &= a \cdot b^{2n} b = b^{2n} \cdot ab = b^n b^n \cdot a = ba \cdot b^n b^n = ba \cdot b^{2n} \\ &= (ba)(b^2)^n = (b^2)^{n+1} = b^{2n+2}. \end{aligned}$$

Thus by using Lemma 7, $a \equiv_\tau b$. Hence τ is anti-separative. □

Proposition 4. *If S is a locally* AG***-groupoid, then $ab \equiv_\tau ba$ for all $a, b \in S$, that is τ is commutative.*

Proof. Let $a, b \in S$ and n be a positive integer. Then by using Proposition 2(6), Proposition 2(2) and Proposition 2(4) with assumption that $n > 1$, we get

$$(ab)(ba)^n = ab \cdot b^n a^n = ab \cdot a^n b^n = aa^n \cdot bb^n = b^n b \cdot a^n a = b^{n+1} a^{n+1} = (ba)^{n+1}.$$

Similarly we can show that $(ba)(ab)^n = (ab)^{n+1}$. Hence $ab \equiv_\tau ba$. □

Theorem 9. *Let S be a locally associative* AG***-groupoid. Then S/τ is a maximal anti-separative commutative image of S.*

Proof. By Theorem 8, τ is anti-separative, and hence S/τ is anti-separative. We now show that τ is contained in every anti-separative congruence relation ξ on S. Let $a \equiv_\tau b$ so that there exists a positive integer n such that

$$ab^n = b^{n+1} \text{ and } ba^n = a^{n+1}.$$

We need to show that $a \equiv_\xi b$, where ξ is an anti-separative congruence on S. Let k be a positive integer such that

$$ab^k \equiv_\xi b^{k+1} \text{ and } ba^k \equiv_\xi a^{k+1}.$$

Suppose that $k > 2$. Now by using Proposition 2(2), we get

$$\begin{aligned} (ab^{k-1})^2 &= ab^{k-1} \cdot ab^{k-1} = aa \cdot b^{k-1} b^{k-1} = a^2 b^{2k-2}, \\ a^2 b^{2k-2} &= aa \cdot b^{k-2} b^k = ab^{k-2} \cdot ab^k \xi ab^{k-2} \cdot b^{k+1} \\ &= ab^{k-2} \cdot b^k b = ab^k \cdot b^{k-2} b = ab^k \cdot b^{k-1}, \end{aligned}$$

and from above, we have

$$a^2b^{2k-2} \equiv_\xi ab^k \cdot b^{k-1} = b^{k-1}b^k \cdot a = b^k b^{k-1} \cdot a = ab^{k-1} \cdot b^k.$$

Thus $(ab^{k-1})^2 \equiv_\xi ab^k \cdot b^{k-1}$. Since $ab^k \equiv_\xi b^{k+1}$ implies that $ab^k \cdot b^{k-1} \equiv_\xi b^{k+1} \cdot b^{k-1}$. Hence $(ab^{k-1})^2 \equiv_\xi (b^k)^2$. It further implies that

$$(ab^{k-1})^2 \equiv_\xi a^2b^{2k-2} = b^{2k-2}a^2 \equiv_\xi (b^k)^2.$$

Thus $ab^{k-1} \equiv_\xi b^k$. Similarly we can show that $ba^{k-1} \equiv_\xi a^k$.

By induction down from k, it follows that for $k = 1$, $ab \equiv_\xi b^2$ and $ba \equiv_\xi a^2$. Hence by using anti-separativity and Proposition 4, it follows that S/τ is a maximal anti-separative commutative image of S. □

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Authors' addresses

Waqar Khan

School of Mathematics & Statistics, Southwest University, Beibei, Chongqing 400715, P. R. China, Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad 22060, Pakistan

E-mail address: waqarmaths@gmail.com

Faisal Yousafzai

Military College of Engineering, National University of Sciences & Technology (NUST), Islamabad, Pakistan.

E-mail address: yousafzaimath@gmail.com

Kostaq Hila
Department of Mathematics & Computer Science, University of Gjirokastra, Gjirokastra 6001, Albania, or: Department of Mathematical Engineering, Polytechnic University of Tirana, Tirana, Albania
E-mail address: kostaq_hila@yahoo.com