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LIGHTLIKE SUBMERSIONS FROM TOTALLY UMBILICAL SEMI-TRANSVERSAL LIGHTLIKE SUBMANIFOLDS

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Abstract. We study lightlike submersions from a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold onto an indefinite almost Hermitian manifold. We show that if an indefinite almost Hermitian manifold B admits a lightlike submersion $\phi : M \to B$ from a totally umbilical semi-transversal lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ then B is necessarily an indefinite Kaehler manifold. We investigate the condition for a totally umbilical semi-transversal lightlike submanifold M to becomes a product manifold and its fibers become geodesic. Finally, we obtain some characterization theorems related to the sectional curvature of an indefinite Kaehler manifold.

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1. INTRODUCTION

The study of Riemannian submersions $\phi : M \to B$, from a Riemannian manifold M onto a Riemannian manifold B was initiated by O'Neill [10]. A Riemannian submersion naturally yields a vertical distribution, which is always integrable and a horizontal distribution. On the other hand, for a CR-submanifold M of a Kaehler manifold $\bar{M}$ there are two orthogonal complementary distributions D and $D^{\perp}$, such that D is $\bar{J}$-invariant and $D^{\perp}$ is totally real and always integrable (cf. Bejancu [2]), where $\bar{J}$ is almost complex structure of $\bar{M}$. Kobayashi [9] observed the similarity between the total space of a Riemannian submersion and a CR-submanifold of a Kaehler manifold in terms of distributions. Then Kobayashi [9] introduced a submersion $\phi : M \to B$, from a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B such that the distributions D and $D^{\perp}$ of the CR-submanifold become the horizontal and the vertical distributions respectively, as required by the submersions and π restricted to D becomes a complex isometry.

Later, semi-Riemannian submersions were introduced by O'Neill in [11]. As it is known that when M and B are Riemannian manifolds then the fibers are always

Riemannian manifolds. However, when the manifolds are semi-Riemannian manifolds then the fibers may not be Riemannian (hence semi-Riemannian) manifolds, (see [15]). Therefore in [13], Sahin introduced a screen lightlike submersion from a lightlike manifold onto a semi-Riemannian manifold and in [15], Sahin and Gunduzalp introduced a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold. It is well-known that semi-Riemannian submersions are of interest in mathematical physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories [3, 4, 8, 16]. Moreover, the geometry of lightlike submanifolds has potential for applications in mathematical physics, particularly in general relativity (for detail, see [5]) therefore in present paper, we study lightlike submersions from a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold onto an almost Hermitian manifold.

2. Lightlike submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$, (see [5]). For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$ and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ respectively. Then $T\bar{M}\mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp})$.

Theorem 1 ([5]). *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $ltr(TM)\mid_{\mathcal{U}}$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.1}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $Rad(TM)$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ then for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.2}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form, A_U is a linear operator on M and known as a shape operator.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (2.2) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.3}$$

where $h^l(X,Y) = L(h(X,Y)), h^s(X,Y) = S(h(X,Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $ltr(TM)$-valued and $S(TM^{\perp})$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.4}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.3) and (2.4), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y). \tag{2.5}$$

Let $\bar{R}$ and R be the curvature tensors of $\bar{\nabla}$ and ∇, respectively then by straightforward calculations (see [5]), we have

$$\begin{aligned}\bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ & + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ & - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).\end{aligned} \tag{2.6}$$

3. Semi-transversal lightlike submanifolds

Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to the indefinite metric $\bar{g}$. Then $\bar{M}$ is called an indefinite Kaehler manifold [1] if the almost complex structure $\bar{J}$ is parallel with respect to $\bar{\nabla}$, that is $(\bar{\nabla}_X \bar{J})Y = 0$, for any $X, Y \in \Gamma(T\bar{M})$.

Definition 1 ([12]). Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then M is called a semi-transversal lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $Rad(TM)$ is transversal with respect to $\bar{J}$.

(ii) There exists a real non-null distribution $D \subset S(TM)$ such that $S(TM) = D \oplus D^{\perp}$, $\bar{J}(D) = D$, $\bar{J}D^{\perp} \subset S(TM^{\perp})$, where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

Then tangent bundle of a semi-transversal lightlike submanifold is decomposed as $TM = D \perp D'$, where $D' = D^{\perp} \perp Rad(TM)$. We say M is a proper semi-transversal lightlike submanifold if $D \neq \{0\}$ and $D^{\perp} \neq \{0\}$. Therefore $dim(Rad(TM)) \geq 2$ and for a proper M, $dim(D) \geq 2s, s > 1$, $dim(D^{\perp}) \geq 1$ and $dim(Rad(TM)) = dim(ltr(TM))$. Thus $dim(M) \geq 5$ and $dim(\bar{M}) \geq 8$. Next, we give example of semi-transversal lightlike submanifolds.

Example 1. Let M be a 5-dimensional submanifold of $(R_2^{10}, \bar{g})$ given by $x_1 = u_1 cosh\theta$, $x_2 = u_2 cosh\theta$, $x_3 = u_1 sinh\theta$, $x_4 = u_2 sinh\theta$, $x_5 = u_3$, $x_6 = \sqrt{1-u_3^2}$, $x_7 = u_4$, $x_8 = u_8$, $x_9 = u_2$, $x_{10} = u_1$, where $\bar{g}$ is of signature $(-,-,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10}\}$. Then TM is spanned by $Z_1 = cosh\theta \partial x_1 + sinh\theta \partial x_3 + \partial x_{10}$, $Z_2 = cosh\theta \partial x_2 + sinh\theta \partial x_4 + \partial x_9$, $Z_3 = x_6 \partial x_5 - x_5 \partial x_6$, $Z_4 = \partial x_7$, $Z_5 = \partial x_8$. Clearly M is a 2-lightlike submanifold with $Rad(TM) = span\{Z_1, Z_2\}$ and the lightlike transversal bundle is spanned by

$$N_1 = \frac{1}{2}(-cosh\theta \partial x_1 - sinh\theta \partial x_3 + \partial x_{10}),\ N_2 = -\frac{1}{2}(cosh\theta \partial x_2 + sinh\theta \partial x_4 - \partial x_9),$$

and $\bar{J}Z_1 = -2N_2$ and $\bar{J}Z_2 = 2N_1$. Hence $\bar{J}(Rad(TM)) = ltr(TM)$. Since $\bar{J}Z_4 = Z_5$ then $D = span\{Z_4, Z_5\}$ which is an invariant distribution on M. By direct calculations, the transversal screen bundle $S(TM^{\perp})$) is spanned by

$$W_1 = sinh\theta \partial x_1 + cosh\theta \partial x_3,\quad W_2 = sinh\theta \partial x_2 + cosh\theta \partial x_4,\quad W_3 = x_6 \partial x_6 + x_5 \partial x_5.$$

Thus $\bar{J}W_3 = -Z_3$. Hence $D^{\perp} = span\{Z_3\}$ is an anti-invariant distribution on M and $span\{W_1, W_2\}$ is invariant and $span\{W_3\}$ is anti-invariant subbundles of $S(TM^{\perp})$ respectively. Thus it enables us to choose $S(TM) = span\{Z_3, Z_4, Z_5\}$. Hence M is a proper semi-transversal lightlike submanifold.

Let M be a semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Let Q, P_1, P_2 and P be the projection morphisms from TM on D, $Rad(TM)$, $D^{\perp}$ and D' respectively. Then for any $X \in \Gamma(TM)$, we put

$$X = QX + P_1X + P_2X. \tag{3.1}$$

Applying $\bar{J}$ to (3.1), we obtain $\bar{J}X = \bar{J}QX + \bar{J}P_1X + \bar{J}P_2X$, can be written as $\bar{J}X = TQX + wP_1X + wP_2X$. Put $wP_1 = w_1$ and $wP_2 = w_2$, then we have

$$\bar{J}X = TX + w_1X + w_2X, \tag{3.2}$$

where $TX \in \Gamma(D), w_1X \in \Gamma(ltr(TM))$ and $w_2X \in \Gamma(\bar{J}D^{\perp}) \subset S(TM^{\perp})$. Similarly, for any $V \in \Gamma(S(TM^{\perp}))$, we can write

$$\bar{J}V = EV + FV, \tag{3.3}$$

where $EV \in \Gamma(D^{\perp})$ and $FV \in \Gamma(\mu)$, where μ is a complementary bundle of $\bar{J}D^{\perp}$ in $S(TM^{\perp})$. Differentiating (3.2) and using (2.3), (2.4) and (3.3), for any $X \in \Gamma(TM)$, we have the following lemma.

Lemma 1. *Let M be a semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then we have*

$$(\nabla_X T)Y = A_{w_1 Y}X + A_{w_2 Y}X + \bar{J}h^l(X,Y) + Eh^s(X,Y), \tag{3.4}$$

$$(\nabla_X w_1)Y = -h^l(X,TY) - D^l(X,w_2 Y), \tag{3.5}$$

$$(\nabla_X w_2)Y = Fh^s(X,Y) - h^s(X,TY) - D^s(X,w_1 Y), \text{where} \tag{3.6}$$

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla_X w_1)Y = \nabla^l_X w_1 Y - w_1 \nabla_X Y, \tag{3.7}$$

$$(\nabla_X w_2)Y = \nabla^s_X w_2 Y - w_2 \nabla_X Y. \tag{3.8}$$

Definition 2 ([6]). A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is said to be a totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that $h(X,Y) = H\bar{g}(X,Y)$, for $X,Y \in \Gamma(TM)$. Using (2.3), clearly M is a totally umbilical, if and only if, for $X,Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$, on each coordinate neighborhood $\mathcal{U}$ there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$ such that

$$h^l(X,Y) = H^l g(X,Y), \quad h^s(X,Y) = H^s g(X,Y), \quad D^l(X,W) = 0. \tag{3.9}$$

Lemma 2. *Let M be a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then the distribution D' defines a totally geodesic foliation in M.*

Proof. Let $X,Y \in \Gamma(D')$ then using (3.4) and (3.7), we obtain $T\nabla_X Y = -A_{w_1 Y}X - A_{w_2 Y}X - \bar{J}h^l(X,Y) - Eh^s(X,Y)$. On taking inner product both sides with $Z \in \Gamma(D)$, we further obtain

$$\begin{aligned} g(T\nabla_X Y, Z) &= \bar{g}(\bar{\nabla}_X w_1 Y, Z) + \bar{g}(\bar{\nabla}_X w_2 Y, Z) = -\bar{g}(\bar{J}Y, \bar{\nabla}_X Z) \\ &= \bar{g}(Y, \bar{\nabla}_X \bar{J}Z) = g(Y, \nabla_X Z'), \end{aligned} \tag{3.10}$$

where $Z' = \bar{J}Z \in \Gamma(D)$. Since M is a totally umbilical lightlike submanifold then for any $X \in \Gamma(D')$ and $Z \in \Gamma(D)$, with (3.5) and (3.7), we have $w_1\nabla_X Z = h^l(X,TZ) = H^l g(X,TZ) = 0$ and using (3.6) and (3.8), we have $w_2\nabla_X Z = -Fh^s(X,Z) + h^s(X,TZ) = -FH^s g(X,Z) + H^s g(X,TZ) = 0$, these facts imply that $\nabla_X Z \in \Gamma(D)$, for any $X \in \Gamma(D')$ and $Z \in \Gamma(D)$. Therefore (3.10) implies that $g(T\nabla_X Y, Z) = 0$, then the non degeneracy of the distribution D implies that $T\nabla_X Y = 0$. Hence the result follows. □

Theorem 2 ([12]). *Let M be a semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D' is integrable, if and only if $A_{wZ}V = A_{wV}Z$, for any $Z,V \in \Gamma(D')$.*

Theorem 3. *Let M be a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then the distribution D' is integrable.*

Proof. Let $X, Y \in \Gamma(D')$ then using (3.4) and (3.7) with the Lemma 2, we get $A_{wY}X = -\bar{J}h^l(X,Y) - Eh^s(X,Y)$ this implies that $A_{wY}X \in \Gamma(D')$ and moreover the symmetric property of the second fundamental form h gives that $A_{wY}X = A_{wX}Y$. Hence by virtue of the Theorem 2, the result follows. □

4. Semi-transversal lightlike submersions

Let $\phi : M \to B$ be a mapping from a Riemannian manifold M onto a Riemannian manifold B then it is said to be a Riemannian submersion if it satisfies the following axioms:

- A1. ϕ has maximal rank. This implies that for each $b \in B$, $\phi^{-1}(b)$ is a submanifold of M, known as *fiber*, of dimension $dimM - dimB$. A vector field tangent to the fibers is called vertical vector field and orthogonal to fibers is called horizontal vector field.
- A2. ϕ_* preserves the lengths of horizontal vectors.

The Riemannian submersions were introduced by O'Neill in [10] and since then plenty of work on this subject matter has been done (for detail, see [7, 14] and many references therein). In the study of submersions, the vertical distribution $\mathcal{V}$ of M is defined by $\mathcal{V}_p = ker\ d\phi_p, p \in M$, which is always integrable and the orthogonal complementary distribution to $\mathcal{V}$ is defined by $\mathcal{H}_p = (ker\ d\phi_p)^{\perp}$, denoted by $\mathcal{H}$ and called a horizontal distribution. Therefore the tangent bundle TM of M has the following decomposition $TM = \mathcal{V} \oplus \mathcal{H}$.

Since the vertical distribution of the Riemannian submersion $\phi : M \to B$ and the totally real distribution $D^{\perp}$ of the CR-submanifold M of a Kaehler manifold are always integrable. Therefore Kobayashi [9] introduced the submersion $\phi : M \to B$ from a CR-submanifold M of a Kaehler manifold onto an almost Hermitian manifold B such that the distributions D and $D^{\perp}$ of the CR-submanifold become the horizontal and the vertical distributions respectively, required by the submersion and ϕ restricted to D becomes a complex isometry.

We have seen that for a Riemannian submersion, the tangent bundle of the source manifold splits into horizontal and vertical part. On the other hand, the tangent bundle of a lightlike submanifold splits into screen and radical part and these natural splitting of the tangent bundle plays an important role in the study of lightlike submanifolds. Therefore Sahin [13] introduced screen lightlike submersion between a lightlike manifold and a semi-Riemannian manifold. Further in [15], Sahin and Gunduzalp introduced the idea of a lightlike submersion from a semi-Riemannian manifold onto a lightlike manifold.

From Theorem 3, we know that for a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold the distribution D' is integrable. Then a totally umbilical semi-transversal lightlike submanifold meets our requirements to define a submersion on it analogous to a submersion of a CR-submanifold. Significant applications of semi-Riemannian submersions in physics and the growing importance of lightlike submanifolds and hypersurfaces in mathematical physics, especially in relativity (see [5]), motivated us to work on this subject matter.

Definition 3. Let (M, g_M, D) be a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ and (B, g_B) be an indefinite almost Hermitian manifold. Then we say that a smooth mapping $\phi : (M, g_M, D) \to (B, g_B)$ is a lightlike submersion if

(a) at every $p \in M, \mathcal{V}_p = ker(d\phi)_p = D'$.
(b) at each point $p \in M$, the differential $d\phi_p$ restricts to an isometry of the horizontal space $\mathcal{H}_p = D_p$ onto $T_{\phi(p)}B$, that is, $g_D(X, Y) = g_B(d\phi(X), d\phi(Y))$, for every vector fields $X, Y \in \Gamma(D)$.

Obviously from the definition, the restriction of the differential $d\phi_p$ to the distribution $\mathcal{H}_p = D_p$ maps that space isomorphically onto $T_{\phi(p)}B$. Then for any tangent vector $\widetilde{X} \in T_{\phi(p)}B$, we say that the tangent vector $X \in D_p$ is a horizontal lift of $\widetilde{X}$ as for submersions. If $\widetilde{X}$ is a vector field on an open subset U of B then the horizontal lift of $\widetilde{X}$ is the vector field $X \in \Gamma(D)$ on $\phi^{-1}(U)$ such that $d\phi(X) = \widetilde{X} o\phi$ and the vector field X is called a *basic vector field*. Now, we give example of lightlike submersions.

Example 2. Let M be a 5-dimensional semi-transversal lightlike submanifold of R_2^{10} as in Example (1) and $B = R_1^2$ be an indefinite almost Hermitian manifold. Let the metrics be defined as $g_M = -(dx_1)^2 - (dx_2)^2 + (dx_3)^2 + (dx_4)^2 + (dx_5)^2$ and $g_B = -(dy_1)^2 + (dy_2)^2$, where $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ and y_1, y_2 be the canonical co-ordinates of R_2^{10} and R^2, respectively. We define a map $\phi : (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}) \in R_2^{10} \mapsto (x_7, x_8) \in R_1^2$. Then the kernel of $d\phi$ is

$$\begin{aligned} ker(d\phi) = D' = span\{&Z_1 = cosh\theta \partial x_1 + sinh\theta \partial x_3 + \partial x_{10}, \\ &Z_2 = cosh\theta \partial x_2 + sinh\theta \partial x_4 + \partial x_9, Z_3 = x_6 \partial x_5 - x_5 \partial x_6\}, \end{aligned}$$

where $d\phi(Z_1) = 0, d\phi(Z_2) = 0$ and $d\phi(Z_3) = 0$. By direct computation, we obtain $D = span\{Z_4 = \partial x_7, Z_5 = \partial x_8\}$, where $d\phi(Z_4) = \partial y_1, d\phi(Z_5) = \partial y_2$. Then it follows that $g_M(Z_4, Z_4) = g_B(d\phi(Z_4), d\phi(Z_4)) = 1$ and $g_M(Z_5, Z_5) = g_B(d\phi(Z_5), d\phi(Z_5)) = -1$. Hence ϕ is a semi-transversal lightlike submersion.

Theorem 4. *Let $\phi : M \to B$ be a lightlike submersion from a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B. If X and Y are basic vectors ϕ-related to $\widetilde{X}, \widetilde{Y}$ respectively, then*

(i) $g_M(X, Y) = g_B(\widetilde{X}, \widetilde{Y}) o\phi$.

(ii) $[X,Y]^{\mathcal{H}}$ *is the basic vector field and* ϕ*-related to* $[\widetilde{X},\widetilde{Y}]$.

(iii) $(\nabla^M_X Y)^{\mathcal{H}}$ *is the basic vector field and* ϕ*-related to* $(\nabla^B_{\widetilde{X}}\widetilde{Y})$.

(iv) *For any vertical vector field* V, $[X,V]$ *is vertical.*

Proof. Let X and Y be basic vector fields of M then (i) follows immediately from part (b) of the Definition 3. Since P and Q be the projections from TM on the distributions D' and D of a semi-transversal lightlike submanifold of indefinite Kaehler manifold respectively, then $[X,Y] = P[X,Y] + Q[X,Y]$. Therefore the horizontal part $Q[X,Y]$ of $[X,Y]$ is a basic vector field and corresponds to $[\widetilde{X},\widetilde{Y}]$, that is, $d\phi(Q[X,Y]) = [d\phi(X), d\phi(Y)]$. Next, from the Koszul's formula, we have

$$2g_M(\nabla_X Y, Z) = X(g_M(Y,Z)) + Y(g_M(Z,X)) - Z(g_M(X,Y)) - g_M(X,[Y,Z]) + g_M(Y,[Z,X]) + g_M(Z,[X,Y]) \tag{4.1}$$

for any $X,Y,Z \in \Gamma(D)$. Consider X,Y and Z are the horizontal lifts of the vector fields $\widetilde{X},\widetilde{Y}$ and $\widetilde{Z}$ respectively, then $X(g_M(Y,Z)) = \widetilde{X}(g_B(\widetilde{Y},\widetilde{Z}))o\phi$ and $g_M(Z,[X,Y]) = g_B(\widetilde{Z},[\widetilde{X},\widetilde{Y}])o\phi$ then from (4.1), we have

$$\begin{aligned}2g_M(\nabla^M_X Y, Z) &= \widetilde{X}(g_B(\widetilde{Y},\widetilde{Z}))o\phi + \widetilde{Y}(g_B(\widetilde{Z},\widetilde{X}))o\phi - \widetilde{Z}(g_B(\widetilde{X},\widetilde{Y}))o\phi\\ &\quad - g_B(\widetilde{X},[\widetilde{Y},\widetilde{Z}])o\phi + g_B(\widetilde{Y},[\widetilde{Z},\widetilde{X}])o\phi + g_B(\widetilde{Z},[\widetilde{X},\widetilde{Y}])o\phi\\ &= 2g_B(\nabla^B_{\widetilde{X}}\widetilde{Y},\widetilde{Z}).\end{aligned} \tag{4.2}$$

Thus from (4.2), (iii) follows, since ϕ is surjective and $\widetilde{Z}$ is arbitrarily chosen. Finally, let $V \in \Gamma(D')$ then $[X,V]$ is ϕ-related to $[\widetilde{X},0]$, hence (iv) follows and this completes the proof of the theorem. □

Let ∇^B be the covariant differentiation on B then we define the corresponding operator $\widetilde{\nabla}^B$ for basic vector fields of B by assuming $\widetilde{\nabla}^B_X Y = (\nabla^M_X Y)^{\mathcal{H}}$, for any basic vector fields X and Y. Thus from (iii) the Theorem 4, $\widetilde{\nabla}^B_X Y$ is a basic vector field and $d\phi(\nabla^M_X Y)^{\mathcal{H}} = d\phi(\widetilde{\nabla}^B_X Y) = \nabla^B_{\widetilde{X}}\widetilde{Y}$. Thus we define the tensor fields C_1 and C_2, using (3.1) as

$$\nabla^M_X Y = \widetilde{\nabla}^B_X Y + C_1(X,Y) + C_2(X,Y), \tag{4.3}$$

for any $X,Y \in \Gamma(D)$, where $C_1(X,Y)$ and $C_2(X,Y)$ denote the vertical parts of $\nabla^M_X Y$. It is easy to check that C_1 and C_2 are bilinear maps from $D \times D \to Rad(TM)$ and $D \times D \to D^{\perp}$ respectively.

Theorem 5. *Let* $\phi : M \to B$ *be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold* $\bar{M}$ *onto an indefinite almost Hermitian manifold* B *then for any basic vector fields* X *and* Y*, we have*

(i) *the tensor fields* C_1 *and* C_2 *are skew-symmetric, that is,* $C_1(X,Y) = -C_1(Y,X)$ *and* $C_2(X,Y) = -C_2(Y,X)$*;*

(ii) $P_1[X,Y] = 2C_1(X,Y)$ *and* $P_2[X,Y] = 2C_2(X,Y)$,

Proof. (i) Let $Z \in \Gamma(D^{\perp})$ be any vertical vector field then for any basic vector field $X \in \Gamma(D)$, we have

$$0 = Z(g(X,X)) = 2\bar{g}(\bar{\nabla}_Z X, X) = 2g(\nabla^M_X Z - [X,Z], X) = -2\bar{g}(Z, \bar{\nabla}_X X)$$
$$= -2g(Z, \widetilde{\nabla}^B_X X + C_1(X,X) + C_2(X,X)) = -2g(Z, C_2(X,X)),$$

then the non degeneracy of the distribution $D^{\perp}$ implies that $C_2(X,X) = 0$, that is C_2 is skew-symmetric. Similarly, let $\bar{J}N \in \Gamma(Rad(TM))$ be a vertical vector field where $N \in \Gamma(ltr(TM))$, we have

$$0 = \bar{J}N(g(X,X)) = -2\bar{g}(\bar{\nabla}_N X, X) = -2g(\nabla^M_X N - [X,N], X)$$
$$= 2g(N, \widetilde{\nabla}^B_X X + C_1(X,X) + C_2(X,X)) = 2g(N, C_1(X,X)),$$

then using (2.1), we obtain $C_1(X,X) = 0$, that is C_1 is skew-symmetric.
(ii) For basic vector fields $X, Y \in \Gamma(D)$, we have $[X,Y] = \nabla^M_X Y - \nabla^M_Y X$, using (3.1), (4.3) and skew-symmetric property of C_1 and C_2, result follows. □

Next for a basic vector field X and a vertical vector field Z, using (3.1), we define the tensor field T as

$$\nabla^M_X Z = (\nabla^M_X Z)^{\mathcal{H}} + (\nabla^M_X Z)^{\mathcal{V}} = T_X Z + (\nabla^M_X Z)^{\mathcal{V}}, \tag{4.4}$$

where T is a bilinear map from $D \times D' \to D$. Since $[X,Z] = \nabla^M_X Z - \nabla^M_Z X$ and $[X,Z]$ is vertical therefore

$$Q(\nabla^M_X Z) = Q(\nabla^M_Z X) = T_X Z, \quad (\nabla^M_X Z)^{\mathcal{V}} = (\nabla^M_Z X)^{\mathcal{V}}. \tag{4.5}$$

Let X and Y be basic vector fields and Z be a vertical vector field such that $Z \in \Gamma(D^{\perp})$ then using (4.3), the tensor fields T and C_2 are related by

$$g(T_X Z, Y) = \bar{g}(\bar{\nabla}_X Z, Y) = -g(Z, \nabla_X Y) = -g(Z, C_2(X,Y)), \tag{4.6}$$

and if $Z \in \Gamma(Rad(TM))$ then

$$g(T_X Z, Y) = -\bar{g}(Z, h^l(X,Y)). \tag{4.7}$$

Theorem 6. *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B then B is also an indefinite Kaehler manifold. Moreover if $\bar{H}$ and H^B denote the holomorphic sectional curvatures of $\bar{M}$ and B, respectively then for any unit basic vector $X \in \Gamma(\mathcal{H})$ of M, we have*

$$\bar{R}^{\bar{M}}(X, \bar{J}X, X, \bar{J}X) = R^B(\widetilde{X}, \bar{J}\widetilde{X}, \widetilde{X}, \bar{J}\widetilde{X}) + 4\|H^s\|^2.$$

Proof. Let $X, Y \in \Gamma(D)$ be basic vector fields then using (2.3) and (4.3), we have

$$\bar{\nabla}_X Y = \widetilde{\nabla}^B_X Y + C_1(X,Y) + C_2(X,Y) + h^l(X,Y) + h^s(X,Y). \tag{4.8}$$

On applying $\bar{J}$ on both sides of (4.8), we obtain

$$\bar{J}\bar{\nabla}_X Y = \bar{J}\widetilde{\nabla}^B_X Y + \bar{J}C_1(X,Y) + \bar{J}C_2(X,Y) + \bar{J}h^l(X,Y) + Eh^s(X,Y) + Fh^s(X,Y), \tag{4.9}$$

on replacing Y by $\bar{J}Y$ in (4.8), we have

$$\bar{\nabla}_X \bar{J}Y = \widetilde{\nabla}^B_X \bar{J}Y + C_1(X,\bar{J}Y) + C_2(X,\bar{J}Y) + h^l(X,\bar{J}Y) + h^s(X,\bar{J}Y). \tag{4.10}$$

Since $\bar{M}$ is a Kaehler manifold therefore $\bar{\nabla}_X \bar{J}Y = \bar{J}\bar{\nabla}_X Y$, then equating (4.9) and (4.10), we obtain

$$\widetilde{\nabla}^B_X \bar{J}Y = \bar{J}\widetilde{\nabla}^B_X Y \in \Gamma(\mathcal{H}), \tag{4.11}$$

$$C_1(X,\bar{J}Y) = \bar{J}h^l(X,Y) \in \Gamma(Rad(TM)), \tag{4.12}$$

$$C_2(X,\bar{J}Y) = Eh^s(X,Y) \in \Gamma(D^\perp), \tag{4.13}$$

$$h^s(X,\bar{J}Y) = \bar{J}C_2(X,Y) + Fh^s(X,Y) \in \Gamma(S(TM^\perp)), \tag{4.14}$$

$$h^l(X,\bar{J}Y) = \bar{J}C_1(X,Y) \in \Gamma(ltr(TM)). \tag{4.15}$$

From (4.11), we see that almost complex structure $\bar{J}$ of B is parallel and hence B is also an indefinite Kaehler manifold.

From (3.3), it is clear that $U \in \Gamma(\bar{J}D^\perp) \subset S(TM^\perp)$, if and only if, $FU = 0$ then $\bar{J}U = EU$ and $U \in \Gamma(\mu = (\bar{J}D^\perp)^\perp) \subset S(TM^\perp)$, if and only if, $EU = 0$ then $\bar{J}U = FU$. Therefore from (4.13), (4.14) and skew-symmetric property of C_2, we obtain $C_2(X,\bar{J}Y) = C_2(Y,\bar{J}X)$, $C_2(\bar{J}X,Y) = C_2(\bar{J}Y,X)$, $C_2(\bar{J}X,\bar{J}Y) = C_2(X,Y)$ and $h^s(X,\bar{J}Y) + h^s(Y,\bar{J}X) = 2Fh^s(X,Y)$. On the other hand, since M is a totally umbilical semi-transversal lightlike submanifold then we have $h^s(X,\bar{J}Y) + h^s(Y,\bar{J}X) = g(X,\bar{J}Y)H^s + g(Y,\bar{J}X)H^s = 0$. Therefore $Fh^s(X,Y) = 0$ and this implies that $h^s(X,Y) \in \Gamma(\bar{J}D^\perp)$, for any $X,Y \in \Gamma(D)$. By virtue of totally umbilical property of M, we also have $h^s(\bar{J}X,\bar{J}Y) = h^s(X,Y)$. Similarly using (4.12) and (4.15), we obtain $C_1(X,\bar{J}Y) = C_1(Y,\bar{J}X)$, $C_1(\bar{J}X,Y) = C_1(\bar{J}Y,X)$, $C_1(\bar{J}X,\bar{J}Y) = C_1(X,Y)$ and $h^l(\bar{J}X,\bar{J}Y) = h^l(X,Y)$, $h^l(\bar{J}X,Y) + h^l(X,\bar{J}Y) = 0$. Now, for any $X,Y,Z \in \Gamma(D)$, using (4.3) and (4.4), we have

$$\nabla_X\nabla_Y Z = \widetilde{\nabla}^B_X\widetilde{\nabla}^B_Y Z + T_X C_1(Y,Z) + T_X C_2(Y,Z) + vertical, \tag{4.16}$$

$$\nabla_Y\nabla_X Z = \widetilde{\nabla}^B_Y\widetilde{\nabla}^B_X Z + T_Y C_1(X,Z) + T_Y C_2(X,Z) + vertical, \tag{4.17}$$

$$\nabla_{[X,Y]}Z = \widetilde{\nabla}^B_{Q[X,Y]}Z + 2T_Z C_1(X,Y) + 2T_Z C_2(X,Y) + vertical. \tag{4.18}$$

Further using (4.16)-(4.18), we obtain

$$R^M(X,Y)Z = (R^B(\widetilde{X},\widetilde{Y})\widetilde{Z})^* + T_X C_1(Y,Z) + T_X C_2(Y,Z) - T_Y C_1(X,Z) - T_Y C_2(X,Z) - 2T_Z C_1(X,Y) - 2T_Z C_2(X,Y) + vertical, \tag{4.19}$$

where $(R^B(\widetilde{X},\widetilde{Y})\widetilde{Z})^*$ denotes the basic vector field of M corresponding to $R^B(\widetilde{X},\widetilde{Y})\widetilde{Z}$. Using (4.19) in (2.6), we obtain

$$\begin{aligned}\bar{R}^{\bar{M}}(X,Y)Z = &(R^B(\widetilde{X},\widetilde{Y})\widetilde{Z})^* + T_X C_1(Y,Z) + T_X C_2(Y,Z) - T_Y C_1(X,Z)\\ &- T_Y C_2(X,Z) - 2T_Z C_1(X,Y) - 2T_Z C_2(X,Y) + A_{h^l(X,Z)}Y\\ &- A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z)\\ &- (\nabla_Y h^l)(X,Z) + D^l(X,h^s(Y,Z)) - D^l(Y,h^s(X,Z))\\ &+ (\nabla_X h^s)(Y,Z) - (\nabla_Y h^s)(X,Z) + D^s(X,h^l(Y,Z))\\ &- D^s(Y,h^l(X,Z)) + vertical.\end{aligned}$$

Now, for basic vector field $W \in \Gamma(D)$ with (2.4), (2.5), (4.4)-(4.7), we obtain

$$\begin{aligned}\bar{R}^{\bar{M}}(X,Y,Z,W) = &R^B(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) - \bar{g}(C_1(Y,Z),h^l(X,W))\\ &- g(C_2(Y,Z),C_2(X,W)) + \bar{g}(C_1(X,Z),h^l(Y,W))\\ &+ g(C_2(X,Z),C_2(Y,W)) + 2\bar{g}(C_1(X,Y),h^l(Z,W))\\ &+ 2g(C_2(X,Y),C_2(Z,W)) + g(A_{h^l(X,Z)}Y,W)\\ &- g(A_{h^l(Y,Z)}X,W) + \bar{g}(h^s(X,Z),h^s(Y,W))\\ &- \bar{g}(h^s(Y,Z),h^s(X,W)).\end{aligned} \tag{4.20}$$

Now, using (2.4) and (4.3), we have $g(A_{h^l(X,Z)}Y,W) = \bar{g}(h^l(X,Z),\bar{\nabla}_Y W) = \bar{g}(h^l(X,Z),C_1(Y,W))$ and similarly $g(A_{h^l(Y,Z)}X,W) = \bar{g}(h^l(Y,Z),C_1(X,W))$. Using these expressions with (4.15) in (4.20), we obtain

$$\begin{aligned}\bar{R}^{\bar{M}}(X,Y,Z,W) = &R^B(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{W}) + \bar{g}(\bar{J}h^l(Y,\bar{J}Z),h^l(X,W))\\ &- g(C_2(Y,Z),C_2(X,W)) - \bar{g}(\bar{J}h^l(X,\bar{J}Z),h^l(Y,W))\\ &+ g(C_2(X,Z),C_2(Y,W)) - 2\bar{g}(\bar{J}h^l(X,\bar{J}Y),h^l(Z,W))\\ &+ 2g(C_2(X,Y),C_2(Z,W)) - \bar{g}(\bar{J}h^l(Y,\bar{J}W),h^l(X,Z))\\ &+ \bar{g}(\bar{J}h^l(X,\bar{J}W),h^l(Y,Z)) + \bar{g}(h^s(X,Z),h^s(Y,W))\\ &- \bar{g}(h^s(Y,Z),h^s(X,W)).\end{aligned} \tag{4.21}$$

To compare holomorphic sectional curvature of $\bar{M}$ with that of B, set $Y = \bar{J}X$, $Z = X$ and $W = \bar{J}X$ in (4.21) and then using the hypothesis that M is a totally umbilical semi-transversal lightlike submanifold, we obtain $\bar{R}^{\bar{M}}(X,\bar{J}X,X,\bar{J}X) = R^B(\widetilde{X},\widetilde{\bar{J}X},\widetilde{X},\widetilde{\bar{J}X}) + \|C_2(X,X)\|^2 + 3\|C_2(X,\bar{J}X)\|^2 + \|h^s(X,X)\|^2$. Since $Fh^s(X,Y) = 0$ therefore (4.14) implies $\|h^s(X,X)\|^2 = \|C_2(X,\bar{J}X)\|^2$ and by virtue of the totally umbilical property of M, (4.14) implies that $C_2(X,X) = -\bar{J}h^s(X,\bar{J}X) = -\bar{J}(H^S g(X,\bar{J}X)) = 0$. Thus the holomorphic sectional curvature of $\bar{M}$ is given as

$$\begin{aligned}\bar{R}^{\bar{M}}(X,\bar{J}X,X,\bar{J}X) &= R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X})+4\|C_2(X,\bar{J}X)\|^2\\ &= R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X})+4\|h^s(X,X)\|^2\\ &= R^B(\widetilde{X},\bar{J}\widetilde{X},\widetilde{X},\bar{J}\widetilde{X})+4\|H^s\|^2.\end{aligned}$$

This completes the proof. □

Theorem 7. *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B. If the distribution D is integrable, then M is a lightlike product manifold.*

Proof. Let the distribution D be an integrable therefore $P_1[X,Y]=0$ and $P_2[X,Y] = 0$, for any $X,Y \in \Gamma(D)$, where P_1 and P_2 are the projection morphisms from TM to $Rad(TM)$ and $D^\perp$, respectively. Therefore using the Theorem 5, we have $C_1(X,Y)=0$ and $C_2(X,Y)=0$. Hence using (4.3), we obtain that $\nabla^M_X Y \in \Gamma(D)$, for any $X,Y \in \Gamma(D)$, consequently the distribution D defines a totally geodesic foliation in M. Moreover, from the Lemma 2, the distribution D' also defines a totally geodesic foliation in M. Thus using the De Rham's theorem, M is a product manifold $M_1 \times M_2$, where M_1 and M_2 are the leaves of the distributions of D and D'. □

Theorem 8. *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B such that $\bar{J}(D^\perp) = S(TM^\perp)$. Then the fibers are totally geodesic submanifolds of M.*

Proof. Let $U,V \in \Gamma(D')$ and then define

$$\nabla^M_U V = \hat{\nabla}_U V + L(U,V), \tag{4.22}$$

where $\hat{\nabla}_U V = (\nabla^M_U V)^{\mathcal{V}}$ and $L(U,V) = (\nabla^M_U V)^{\mathcal{H}}$. Since the distribution D' is integrable always, then $L(U,V) = L(V,U)$. Now, using the Kaehlerian property of $\bar{M}$, we have $\bar{\nabla}_U \bar{J}V = \bar{J}\bar{\nabla}_U V$, since $\bar{J}(D^\perp) = S(TM^\perp)$, then

$$-A_{\bar{J}V}U + \nabla^t_U \bar{J}V = \bar{J}\hat{\nabla}_U V + \bar{J}L(U,V) + \bar{J}h(U,V).$$

On comparing the horizontal and vertical components both sides, we get

$$\mathcal{H}(A_{\bar{J}V}U) = -\bar{J}L(U,V), \quad \mathcal{V}(A_{\bar{J}V}U) = -\bar{J}h(U,V). \tag{4.23}$$

From (4.22), it is clear that the fibers are totally geodesic submanifolds of M, if and only if, $L(U,V)=0$ or using $(4.23)_1$, if and only if, $A_{\bar{J}V}U \in \Gamma(D')$, for any $U,V \in \Gamma(D')$. Now, particularly choose $V \in D^\perp$ then using the hypothesis of this theorem $\bar{J}V \in \Gamma(S(TM^\perp))$. Let $Y \in \Gamma(D)$ then using (2.5) with the fact that M is a totally umbilical lightlike submanifold, we obtain $g(A_{\bar{J}V}U,Y) = \bar{g}(h^s(U,Y),\bar{J}V) = g(U,Y)\bar{g}(H^s,\bar{J}V) = 0$. Similarly, let $V \in \Gamma(Rad(TM))$ then $g(A_{\bar{J}V}U,Y) = \bar{g}(\bar{J}V,\bar{\nabla}_U Y) = -\bar{g}(V,h^l(U,\bar{J}Y)) = -g(U,\bar{J}Y)\bar{g}(V,H^l) = 0$. Thus $A_{\bar{J}V}U \in \Gamma(D')$ and the assertion follows. □

Theorem 9. *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B. Then the sectional curvature of $\bar{M}$ and of the fiber are related by*

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) + g(A_{h^l(U,U)}V, V) - g(A_{h^l(V,U)}U, V) + g([A_{\bar{J}V}, A_{\bar{J}U}]U, V),$$

for any orthonormal vector fields $U, V \in \Gamma(D^\perp)$.

Proof. Let ∇ and $\hat{\nabla}$ be the connections of semi-transversal lightlike submanifold M and its fiber, respectively. Let R and $\hat{R}$ be the curvature tensors of ∇ and $\hat{\nabla}$, respectively then for any $U, V \in \Gamma(D^\perp)$, using (4.22) we have

$$R(U,V)U = \nabla_U(\hat{\nabla}_V U + L(V,U)) - \nabla_V(\hat{\nabla}_U U + L(U,U)) - (\hat{\nabla}_{[U,V]}U + L([U,V],U)),$$

this further implies that

$$R(U,V,U,V) = g(\nabla_U \hat{\nabla}_V U, V) + g(\nabla_U L(V,U), V) - g(\nabla_V \hat{\nabla}_U U, V) - g(\nabla_V L(U,U), V) - g(\hat{\nabla}_{[U,V]}U, V).$$

Again using (4.22), it leads to

$$R(U,V,U,V) = \hat{R}(U,V,U,V) + g(\nabla_U L(V,U), V) - g(\nabla_V L(U,U), V). \tag{4.24}$$

Now, using the fact that M is totally umbilical lightlike submanifold, we get

$$g(\nabla_U L(V,W), F) = g(\bar{\nabla}_U L(V,W) - g(h^l(U, L(V,W)), F) = -g(L(V,W), \nabla_U F) = -g(L(V,W), L(U,F)),$$

for any $U, V, W, F \in \Gamma(D^\perp)$ therefore (4.24) becomes

$$R(U,V,U,V) = \hat{R}(U,V,U,V) - g(L(U,V), L(U,V)) + g(L(U,U), L(V,V)). \tag{4.25}$$

Using (2.5), (2.6) and M is totally umbilical lightlike submanifold, we have

$$\bar{R}(U,V,U,V) = R(U,V,U,V) + g(A_{h^l(U,U)}V, V) - g(A_{h^l(V,U)}U, V) + \bar{g}(h^s(V,V), h^s(U,U)) - \bar{g}(h^s(U,V), h^s(V,U)).$$

Further using (4.23), (4.25) and the fact $L(U,V) = L(V,U)$, we obtain

$$\begin{aligned}\bar{R}(U,V,U,V) = {} & \hat{R}(U,V,U,V) - g(\mathcal{H}(A_{\bar{J}U}V), \mathcal{H}(A_{\bar{J}U}V)) \\ & + g(\mathcal{H}(A_{\bar{J}U}U), \mathcal{H}(A_{\bar{J}V}V)) + g(A_{h^l(U,U)}V, V) \\ & - g(A_{h^l(V,U)}U, V) + g(\mathcal{V}(A_{\bar{J}V}V), \mathcal{V}(A_{\bar{J}U}U)) \\ & - g(\mathcal{V}(A_{\bar{J}V}U), \mathcal{V}(A_{\bar{J}V}U)).\end{aligned}$$

Since $U, V \in \Gamma(D^\perp)$ and let $X \in \Gamma(D)$ then using (2.3), we get $g(A_{\bar{J}U}V, X) = 0$, which further implies that $A_{\bar{J}U}V \in \Gamma(D^\perp)$ and $A_{\bar{J}U}V = A_{\bar{J}V}U$, then

$$\bar{R}(U,V,U,V) = \hat{R}(U,V,U,V) - g(A_{\bar{J}U}V, A_{\bar{J}U}V) + g(A_{\bar{J}U}U, A_{\bar{J}V}V) + g(A_{h^l(U,U)}V,V) - g(A_{h^l(V,U)}U,V). \tag{4.26}$$

Now, let $W \in \Gamma(S(TM^\perp))$ then for $U, V \in \Gamma(D^\perp)$, using (2.5), we have $g(A_W U, V) = g(U, A_W V)$. Using this fact with $A_{\bar{J}U}V \in \Gamma(D^\perp)$, we get

$$\begin{aligned} g(A_{\bar{J}U}V, A_{\bar{J}U}V) - g(A_{\bar{J}U}U, A_{\bar{J}V}V) &= g(A_{\bar{J}V}U, A_{\bar{J}U}V) - g(A_{\bar{J}U}U, A_{\bar{J}V}V) \\ &= g(A_{\bar{J}U}A_{\bar{J}V}U, V) - g(A_{\bar{J}V}A_{\bar{J}U}U, V) \\ &= -g([A_{\bar{J}V}, A_{\bar{J}U}]U, V). \end{aligned} \tag{4.27}$$

On using (4.27) in (4.26), the assertion follows. □

Now we define O'Neill's tensors [10] for a lightlike submersion. Let ∇ be a connection of M then tensors $\mathcal{T}$ and $\mathcal{A}$ of type $(1,2)$ are given by

$$\mathcal{T}_X Y = \mathcal{H}\nabla_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{V}X}\mathcal{H}Y, \quad \mathcal{A}_X Y = \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y. \tag{4.28}$$

Using (4.28), we have the following lemma.

Lemma 3. *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B. Then we have the following:*

(i) $\nabla_U V = \mathcal{T}_U V + \mathcal{V}\nabla_U V$.
(ii) $\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X$.
(iii) $\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V$.
(iv) $\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y$,

for any $X, Y \in \mathcal{H}$ and $U, V \in \mathcal{V}$.

Theorem 10. *Let $\phi : M \to B$ be a lightlike submersion of a totally umbilical semi-transversal lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ onto an indefinite almost Hermitian manifold B such that $\bar{J}(D^\perp) = S(TM^\perp)$. Then $\bar{K}(X \wedge V) = \|H^s\|^2 - \|T_X V\|^2$, for any unit vector fields $X \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$.*

Proof. Let $X \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$ then using the Theorem 5 and Lemma 3 with (4.3), we obtain

$$g(R(V,X)X,V) = g(\nabla_V \mathcal{H}(\nabla_X X), V) - g(\nabla_X \mathcal{H}(\nabla_V X), V) - g(\nabla_X \mathcal{T}_V X, V) + g(\mathcal{T}_{[X,V]}X, V).$$

It should be noted that $g(\nabla_V \mathcal{H}(\nabla_X X), V) = -g(\mathcal{H}(\nabla_X X), \nabla_V V)$, and similarly $g(\nabla_X \mathcal{H}(\nabla_V X), V) = -g(\mathcal{H}(\nabla_V X), \nabla_X V)$. Therefore we have

$$g(R(V,X)X,V) = -g(\mathcal{H}(\nabla_X X), \nabla_V V) + g(\mathcal{H}(\nabla_V X), \nabla_X V) - g(\nabla_X \mathcal{T}_V X, V) + g(\mathcal{T}_{[X,V]}X, V). \tag{4.29}$$

Since $\bar{J}(D^{\perp}) = S(TM^{\perp})$ then using the Theorem 8, we have $L(U,V) = 0$, for $U,V \in \Gamma(D^{\perp})$. Hence using the definition of $\mathcal{T}$ with (2.3) and (4.22), we get

$$g(\mathcal{T}_V X, U) = -g(\mathcal{T}_V U, X) = -g(L(V,U), X) = 0. \tag{4.30}$$

Now, using (4.22), we have

$$g(\mathcal{H}(\nabla_X X), \nabla_V V) = g(\mathcal{H}(\nabla_X X), L(V,V)) = 0. \tag{4.31}$$

Since M is a totally umbilical then using (4.30), we obtain

$$\begin{aligned} g(\nabla_X \mathcal{T}_V X, V) &= -g(\mathcal{T}_V X, \bar{\nabla}_X V) = -g(\mathcal{T}_V X, \mathcal{V}(\nabla_X V)) \\ &= g(L(V, \mathcal{V}(\nabla_X V)), X) = 0. \end{aligned} \tag{4.32}$$

Since for a vertical vector field V, $[X,V]$ is always vertical therefore again using (4.30), we have

$$g(\mathcal{T}_{[X,V]} X, V) = -g(L([X,V],V), X) = 0. \tag{4.33}$$

Using (4.6) and (4.31)-(4.33) in (4.29), we obtain

$$g(R(V,X)X,V) = g(T_X V, T_X V). \tag{4.34}$$

Since M is a totally umbilical then using (2.6) and (4.34), we get

$$\begin{aligned} \bar{R}(X,V,X,V) = &-g(T_X V, T_X V) + g(h^l(X,X), \nabla_V V) \\ &+ g(h^s(X,X), h^s(V,V)). \end{aligned} \tag{4.35}$$

Now, using Kaehlerian property of $\bar{M}$, we have $\bar{\nabla}_V \bar{J}\xi = \bar{J}\bar{\nabla}_V \xi$, for $V \in \Gamma(D^{\perp})$ and $\xi \in \Gamma(Rad(TM))$. Using the Lemma 3 with (2.4) and then comparing the horizontal components of resulting equation, we obtain

$$A_{\bar{J}\xi} V = -\bar{J}\mathcal{T}_V \xi. \tag{4.36}$$

Since M is semi-transversal lightlike submanifold then for $\xi \in \Gamma(Rad(TM))$, $\bar{J}\xi \in \Gamma(ltr(TM))$ and using (4.28) for any $U,V \in \mathcal{V}$, $\mathcal{T}_U V = \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V \in \mathcal{H}$. Therefore (4.36) implies that $A_{\bar{J}\xi} V \in \mathcal{H}$ or $A_N V \in \mathcal{H}$. Then for $V \in \Gamma(D^{\perp})$ and $N \in \Gamma(ltr(TM))$, we have $g(\nabla_V V, N) = -g(V, \bar{\nabla}_V N) = g(V, A_N V) = 0$. This implies that $\nabla_V V$ has no component in $Rad(TM)$. Using this fact in (4.35) with (3.9), the assertion follows. □

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