JORDAN TRIPLE ENDOMORPHISMS AND ISOMETRIES OF UNITARY GROUPS

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Abstract. In this paper we present the general form of all continuous endomorphisms of the group $\mathbb{U}_n$ of $n \times n$ complex unitary matrices with respect to the Jordan triple product. These are the continuous maps $\phi : \mathbb{U}_n \to \mathbb{U}_n$ which satisfy

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad V, W \in \mathbb{U}_n.$$

The result is applied to determine the structure of certain isometries of $\mathbb{U}_n$. These include the isometries relative to any metric given by a unitarily invariant norm on the space $\mathbb{M}_n$ of all $n \times n$ complex matrices and also the isometries relative to any member of a new class of metrics on $\mathbb{U}_n$ recently introduced by Chau, Li, Poon and Sze [6].

1. Introduction and statement of the main results

The famous Mazur-Ulam theorem states that every surjective isometry (i.e., surjective distance preserving mapping) between real normed spaces is automatically affine, it preserves the operation of convex combination. Motivated by this important result, in the paper [9] Hatori, Hirasawa, Miura and Molnár made attempts to generalize it for the noncommutative setting, especially for metric groups and for certain substructures of them. The authors managed to obtain results saying that under certain conditions, the surjective isometries of groups equipped with translation and inverse invariant metrics locally preserve an operation called inverted Jordan triple product. In some cases the local preservation of that operation can be shown to extend globally. These results demonstrate that in the considered cases the surjective isometries have a certain remarkable algebraic property, they are some sort of isomorphisms between the underlying groups. In [10] the results given in [9] were utilized to describe the structure of surjective isometries of the unitary group of an arbitrary complex Hilbert space relative to the metric induced by the usual operator norm. In [15] Molnár and Šemrl determined the structure of surjective isometries of the unitary group of a complex infinite dimensional separable Hilbert space with respect to

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any unitarily invariant uniform norm on the full operator algebra over the underlying Hilbert space. By employing different analytical tools but using the same algebraic properties of surjective isometries between groups that were obtained in [9], Hatori and Molnár presented results in [11] on the structure of surjective isometries (relative to the usual norm) of unitary groups in C^*-algebras and in von Neumann algebras. An interesting recent result on the form of certain isometries of the special orthogonal group is to appear in [1].

In [15] the problem of describing the surjective isometries of the unitary group under unitarily invariant norms in the finite dimensional case was left as an open problem, see [15, 4. Remarks, examples, open problems]. One of the aims of this paper is to give a solution of that problem. On the other hand, below we determine the structure of all isometries of the unitary group $\mathbb{U}_n$ relative to a new class of metrics on $\mathbb{U}_n$ that has been introduced by Chau, Li, Poon and Sze very recently [6].

As can be suspected from the discussion above the isometries we are going to consider turn to be isomorphisms under a certain algebraic operation on $\mathbb{U}_n$. Indeed, this operation is the inverted Jordan triple product that we define by the formula $VW^{-1}V$. Morphisms with respect to this product are very closely related to morphisms with respect to a much more common and important operation which is called the Jordan triple product. This is defined by the formula VWV. Transformations preserving this operation or alike operations are extensively investigated in ring theory and its applications. The second main aim of this paper is to obtain the full description of all continuous Jordan triple endomorphisms of the group $\mathbb{U}_n$.

We begin with presenting the notation and definitions that we shall use throughout the paper. We denote by $\mathbb{M}_n$ the space of all $n \times n$ complex matrices, by $\mathbb{H}_n$ the space of all self-adjoint elements of $\mathbb{M}_n$ and by $\mathbb{U}_n$ the group of all unitary elements of $\mathbb{M}_n$. A unitary matrix is called a symmetry if it is self-adjoint (it has eigenvalues ± 1). It is well-known that every unitary matrix is the exponent of a skew-symmetric matrix, i.e., every $U \in \mathbb{U}_n$ can be written of the form $U = e^{iH}$ with some $H \in \mathbb{H}_n$. In what follows $\|.\|$ denotes the usual operator norm (or, in another words, spectral norm) on $\mathbb{M}_n$ (i.e., $\|A\|$ is the square-root of the largest eigenvalue of the positive semi-definite matrix A^*A). If not specified otherwise, when we speak of metrical or topological properties related to $\mathbb{U}_n$ we always mean the metric induced by the norm $\|.\|$. In what follows I stands for the identity matrix, ^{tr} denotes the transpose of matrices, Tr is the usual trace functional, and $\overline{}$ refers to complex conjugate. Recall that a norm $N(.)$ on $\mathbb{M}_n$ is called unitarily invariant if $N(UAV) = N(A)$ holds for all $A \in \mathbb{M}_n$, $U, V \in \mathbb{U}_n$. In what follows we assume that $n \geq 2$ (in the case $n = 1$ the results below follow from classical mathematical analysis).

Our first main result which gives the complete description of Jordan triple endomorphisms of $\mathbb{U}_n$ reads as follows.

Theorem 1. *Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous map which is a Jordan triple endomorphism, i.e., assume that ϕ satisfies*

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad V, W \in \mathbb{U}_n.$$

Then there exist a unitary matrix $U \in \mathbb{U}_n$, an integer k, a number $c \in \{-1, 1\}$, and a set $\{P_1, \dots, P_n\}$ of mutually orthogonal rank-one projections in $\mathbb{M}_n$, a set $\{k_1, \dots, k_n\}$ of integers and a set $\{c_1, \dots, c_n\} \subset \{-1, 1\}$ such that ϕ is of one of the following forms:

(j1) $\phi(V) = c(\det V)^k UVU^{-1}, \quad V \in \mathbb{U}_n$;
(j2) $\phi(V) = c(\det V)^k UV^{-1}U^{-1}, \quad V \in \mathbb{U}_n$;
(j3) $\phi(V) = c(\det V)^k UV^{tr}U^{-1}, \quad V \in \mathbb{U}_n$;
(j4) $\phi(V) = c(\det V)^k U\overline{V}U^{-1}, \quad V \in \mathbb{U}_n$;
(j5) $\phi(V) = \sum_{j=1}^n c_j(\det V)^{k_j} P_j, \quad V \in \mathbb{U}_n$.

From the above result one can immediately deduce the structure of all continuous Jordan triple automorphisms of $\mathbb{U}_n$.

Corollary 2. *Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous Jordan triple automorphism, i.e., a continuous bijective map which satisfies*

$$\phi(VWV) = \phi(V)\phi(W)\phi(V), \quad V, W \in \mathbb{U}_n.$$

Then there exist a unitary matrix $U \in \mathbb{U}_n$ and a number $c \in \{-1, 1\}$ such that ϕ is of one of the following forms:

(a1) $\phi(V) = cUVU^{-1}, \quad V \in \mathbb{U}_n$;
(a2) $\phi(V) = cUV^{-1}U^{-1}, \quad V \in \mathbb{U}_n$;
(a3) $\phi(V) = cUV^{tr}U^{-1}, \quad V \in \mathbb{U}_n$;
(a4) $\phi(V) = cU\overline{V}U^{-1}, \quad V \in \mathbb{U}_n$.

As our second main aim in this paper, in the next theorem we determine the structure of all isometries of the unitary group $\mathbb{U}_n$ with respect to any unitarily invariant norm given on $\mathbb{M}_n$.

Theorem 3. *Let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n$. If $\phi : \mathbb{U}_n \to \mathbb{U}_n$ is an isometry, i.e., ϕ is a map which satisfies*

$$N(\phi(V) - \phi(W)) = N(V - W), \quad V, W \in \mathbb{U}_n,$$

then there exists a pair $U, U' \in \mathbb{U}_n$ of unitary matrices such that ϕ is of one of the following forms:

(i1) $\phi(V) = UVU', \quad V \in \mathbb{U}_n$;
(i2) $\phi(V) = UV^{-1}U', \quad V \in \mathbb{U}_n$;
(i3) $\phi(V) = UV^{tr}U', \quad V \in \mathbb{U}_n$;
(i4) $\phi(V) = U\overline{V}U', \quad V \in \mathbb{U}_n$.

In our fourth theorem we determine the isometries of $\mathbb{U}_n$ with respect to a recently defined class of interesting metrics on $\mathbb{U}_n$. Motivated by considerations in quantum information processing, in [5] Chau introduced a certain family of metrics on $\mathbb{U}_n$. In the paper [6] Chau, Li, Poon and Sze have extended this class significantly and presented a number of its interesting properties. It is a remarkable fact that starting from a very much different origin in [2], Antezana, Larotonda and Varela have been led practically to the same class of distances on $\mathbb{U}_n$.

As for the definition of the metrics in question, we first remark the following. To any $V \in \mathbb{U}_n$ there corresponds a unique self-adjoint matrix $H \in \mathbb{H}_n$ with spectrum in $]-\pi, \pi]$ such that $V = \exp(iH)$. Indeed, H can be obtained in the following way. Applying an appropriate unitary similarity transformation, V is transformed into a diagonal matrix. The diagonal elements of this matrix are complex numbers of modulus 1. For each such diagonal element take the corresponding unique angle that belongs to $]-\pi, \pi]$. From the so obtained angles form the corresponding diagonal matrix and finally transform it with the inverse of the previously mentioned unitary similarity transformation. What we get is just the self-adjoint matrix H that we have been looking for. For temporary use, in this paper we call this self-adjoint matrix H the angular matrix of V. Now, given a unitarily invariant norm $N(.)$ on $\mathbb{M}_n$, for any pair $V, W \in \mathbb{U}_n$ of unitary matrices pick the angular matrix H of VW^{-1} and define $d_N(V, W) = N(H)$. It has been proven in [6] that d_N is a metric on $\mathbb{U}_n$ and several interesting properties of d_N have been derived.

In our last result we determine the structure of the corresponding isometries of $\mathbb{U}_n$.

Theorem 4. *Let $N(.)$ be a unitarily invariant norm on $\mathbb{M}_n$. The structure of the isometries of $\mathbb{U}_n$ with respect to the metric d_N defined above is exactly the same as in Theorem 3.*

Remark 5. At this point let us remark the following. The results in the previous statements can all be reversed, meaning that all transformations of any of the forms which appear in the conclusions are in fact isometries, continuous Jordan triple automorphisms, and continuous Jordan triple endomorphisms, respectively. To verity these one needs to apply only simple observations.

2. Proofs

In this section, after verifying some auxiliary results, we present the proofs of our main theorems.

Our first lemma that follows states that the continuous Jordan triple endomorphisms of $\mathbb{U}_n$ are all Lipschitz functions. The result could also be derived following the argument given in [13] (p. 177, Satz 1) relating to group endomorphisms of linear groups. For the sake of completeness below we present a more direct and simple proof in the case of Jordan triple endomorphisms of $\mathbb{U}_n$.

Lemma 6. *Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous Jordan triple endomorphism. Assume $\phi(I) = I$. Then ϕ is a Lipschitz function.*

Proof. We begin with the following observation. For an arbitrary $V \in \mathbb{U}_n$ let H be the angular matrix of V and denote the operator norm $\|H\|$ of H by $m(V)$. Apparently, we have the inequalities

$$\|V - I\| \leq m(V) \leq 2\|V - I\|.$$

Moreover, if $m(V) < \pi$ and k is a positive integer such that $k\, m(V) < \pi$, then we have $m(V^k) = k\, m(V)$.

Turning to the proof of the lemma, we first assert that there exists a positive real number L such that $\|\phi(U) - I\| \leq L\|U - I\|$ holds for all $U \in \mathbb{U}_n$. Assume on the contrary that we have a sequence (U_k) in $\mathbb{U}_n$ and a sequence (c_k) of positive integers such that $c_k \to \infty$ and

$$\|\phi(U_k) - I\| > c_k\|U_k - I\| \tag{1}$$

holds for every $k \in \mathbb{N}$. Since $\mathbb{U}_n$ is a compact metric space, (U_k) has a convergent subsequence. Without serious loss of generality we may and do assume that already the original sequence (U_k) is convergent. If its limit were different from I, by (1) we would have $\|\phi(U_k) - I\| \to \infty$ which contradicts $\|\phi(U_k) - I\| \leq 2$. Therefore, $U_k \to I$ and $\phi(U_k) \to I$ as $k \to \infty$. We can also assume that

$$\|\phi(U_k) - I\| = \epsilon_k, \quad \epsilon_k < 1/2$$

holds for all $k \in \mathbb{N}$. Choose positive integers l_k such that

$$1/(l_k + 1) \leq \epsilon_k < 1/l_k.$$

Clearly, $l_k \geq 2$. Since $\epsilon_k > c_k\|U_k - I\|$, we have $\|U_k - I\| < \epsilon_k/c_k$ and this implies that

$$m(U_k) \leq 2\|U_k - I\| < (2\epsilon_k)/c_k.$$

On the other hand, we have

$$\frac{2\epsilon_k l_k}{c_k} < \frac{2}{c_k} < \pi.$$

Therefore, we infer $m(U_k^{l_k}) = l_k m(U_k) < 2/c_k$ which implies

$$\|U_k^{l_k} - I\| \leq m(U_k^{l_k}) < 2/c_k.$$

Consequently, $U_k^{l_k} \to I$ and since $U_k \to I$ also holds, we have $U_k^{l_k+1} \to I$ as $k \to \infty$.

We continue with the inequalities

$$m(\phi(U_k)) \leq 2\|\phi(U_k) - I\| = 2\epsilon_k$$

and

$$2\epsilon_k(l_k + 1) < 2(l_k + 1)/l_k < \pi,$$

where in the last inequality we have used $l_k \geq 2$. These imply that

$$m(\phi(U_k)^{l_k+1}) = (l_k + 1)m(\phi(U_k)).$$

Hence we compute

$$1 = (1/\epsilon_k)\|\phi(U_k) - I\| \leq (l_k+1)\|\phi(U_k) - I\|$$
$$\leq (l_k+1)m(\phi(U_k)) = m(\phi(U_k)^{l_k+1}) \leq 2\|\phi(U_k^{l_k+1}) - I\|.$$

Consequently, $\phi(U_k^{l_k+1}) \not\to I$ and this contradicts $U_k^{l_k+1} \to I$. Therefore, we have a positive real number L such that $\|\phi(U) - I\| \leq L\|U - I\|$ holds for every $U \in \mathbb{U}_n$.

To complete the proof pick arbitrary unitaries $W, W' \in \mathbb{U}_n$. We can choose $V \in \mathbb{U}_n$ such that $V^2 = W'$ and then find $U \in \mathbb{U}_n$ such that $VUV = W$. We infer

$$\|\phi(W) - \phi(W')\| = \|\phi(VUV) - \phi(V^2)\| = \|\phi(V)\phi(U)\phi(V) - \phi(V)I\phi(V)\|$$
$$= \|\phi(U) - I\| \leq L\|U - I\| = L\|VUV - V^2\| = L\|W - W'\|.$$

This proves that ϕ is a Lipschitz function. □

In the next auxiliary result we show that every continuous Jordan triple endomorphism of $\mathbb{U}_n$ which is unital (i.e., maps I to I) gives rise to a linear transformation on $\mathbb{H}_n$. The use of one-parameter groups in the proof that originates from [12] has already been exploited in the papers [11] and [1].

Lemma 7. *Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous Jordan triple endomorphism with $\phi(I) = I$. Then there exists a linear transformation $f : \mathbb{H}_n \to \mathbb{H}_n$ such that*

$$\phi(e^{itA}) = e^{itf(A)}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n.$$

Moreover, f satisfies

$$f(VAV) = \phi(V)f(A)\phi(V)$$

for every $A \in \mathbb{H}_n$ and symmetry $V \in \mathbb{U}_n$.

Proof. Since ϕ is a unital Jordan triple endomorphism, it is easy to check that $\phi(V^k) = \phi(V)^k$ holds for every $V \in \mathbb{U}_n$ and positive integer k. We show that ϕ preserves the inverse operation. To prove this, let $W \in \mathbb{U}_n$ be such that $W^2 = V$. We compute

$$\phi(W)\phi(V^{-1})\phi(W) = \phi(WV^{-1}W) = \phi(I) = I$$

which implies that

$$\phi(V^{-1}) = \phi(W)^{-2} = \phi(W^2)^{-1} = \phi(V)^{-1}.$$

Therefore, we obtain that $\phi(V^k) = \phi(V)^k$ holds for every integer k and $V \in \mathbb{U}_n$. In the rest of the paper we shall use several times that, in particular, ϕ maps symmetries to symmetries.

In the next step, following an argument similar to the proof of Theorem 7 in [11] we show that ϕ maps one-parameter unitary groups to one-parameter unitary groups. Pick an arbitrary self-adjoint matrix $T \in \mathbb{H}_n$ and define $S_T : \mathbb{R} \to \mathbb{U}_n$ by

$$S_T(t) = \phi(e^{itT}), \quad t \in \mathbb{R}.$$

We assert that S_T is a continuous one-parameter unitary group in $\mathbb{M}_n$. Since ϕ is continuous, we only need to prove that $S_T(t+t') = S_T(t)S_T(t')$ holds for every pair t, t' of real numbers. First select rational numbers r and r' such that $r = \frac{k}{m}$ and $r' = \frac{k'}{m'}$ with integers k, k', m, m'. We compute

$$\begin{aligned} S_T(r+r') &= \phi(e^{i\frac{km'+k'm}{mm'}T}) = \phi(e^{i\frac{1}{mm'}T})^{km'+k'm} \\ &= \phi(e^{i\frac{1}{mm'}T})^{km'}\phi(e^{i\frac{1}{mm'}T})^{k'm} = S_T(r)S_T(r'). \end{aligned}$$

Since ϕ is continuous, we deduce that $S_T(t+t') = S_T(t)S_T(t')$ holds for every pair t, t' of real numbers. By Stone's theorem we obtain that there exists a unique self-adjoint matrix $f(T) \in \mathbb{H}_n$ (the generator of the one-parameter unitary group S_T) such that

$$\phi(e^{itT}) = S_T(t) = e^{itf(T)}, \quad t \in \mathbb{R}.$$

We next prove that $f : \mathbb{H}_n \to \mathbb{H}_n$ is in fact a linear transformation. Pick $A, B, C \in \mathbb{H}_n$. We compute

$$\begin{aligned} &\frac{e^{i(t/2)A}e^{itB}e^{i(t/2)A} - e^{itC}}{it} \\ = &\frac{(e^{i(t/2)A} - I)e^{itB}e^{i(t/2)A} + (e^{itB} - I)e^{i(t/2)A} + (e^{i(t/2)A} - I) - (e^{itC} - I)}{it} \\ &\to A/2 + B + A/2 - C = A + B - C \end{aligned}$$

as $t \to 0$. It follows that

$$\lim_{t\to 0} \frac{e^{i(t/2)A}e^{itB}e^{i(t/2)A} - e^{itC}}{it} = 0 \Longleftrightarrow C = A + B.$$

If $C = A + B$, then using the Lipschitz property of ϕ proven in Lemma 6 we have

$$\begin{aligned} &\frac{e^{i(t/2)f(A)}e^{itf(B)}e^{i(t/2)f(A)} - e^{itf(C)}}{it} \\ &= \frac{\phi(e^{i(t/2)A})\phi(e^{itB})\phi(e^{i(t/2)A}) - \phi(e^{itC})}{it} \\ &= \frac{\phi(e^{i(t/2)A}e^{itB}e^{i(t/2)A}) - \phi(e^{itC})}{it} \to 0 \end{aligned}$$

as $t \to 0$. On the other hand, just as above we deduce

$$\frac{e^{i(t/2)f(A)}e^{itf(B)}e^{i(t/2)f(A)} - e^{itf(C)}}{it} \to f(A) + f(B) - f(C).$$

This gives us that $f(A) + f(B) - f(A+B) = 0$, i.e., f is additive. The homogeneity of f is trivial to see. Indeed, we have

$$e^{it\lambda f(A)} = \phi(e^{it\lambda A}) = e^{itf(\lambda A)}$$

for every $t, \lambda \in \mathbb{R}$ which implies $\lambda f(A) = f(\lambda A)$. Consequently, f is a linear transformation on $\mathbb{H}_n$.

To obtain the last statement of the result we compute

$$e^{it\phi(V)f(A)\phi(V)} = \phi(V)e^{itf(A)}\phi(V)$$
$$= \phi(V)\phi(e^{itA})\phi(V) = \phi(Ve^{itA}V) = e^{itf(VAV)}.$$

Since this holds for every $t \in \mathbb{R}$ we easily get the desired equality $f(VAV) = \phi(V)f(A)\phi(V)$ for every $A \in \mathbb{H}_n$ and symmetry $V \in \mathbb{U}_n$. □

In what follows $\mathbb{T}$ denotes the circle group which is just the unitary group in the one-dimensional case. The next auxiliary result describes the structure of continuous Jordan triple functionals on $\mathbb{U}_n$. It can be viewed also as a characterization of the determinant function on the unitary group.

Lemma 8. *Let $\varphi : \mathbb{U}_n \to \mathbb{T}$ be a continuous Jordan triple functional, i.e., assume that φ is continuous and satisfies*

$$\varphi(VWV) = \varphi(V)\varphi(W)\varphi(V), \quad V, W \in \mathbb{U}_n.$$

Then there is an integer k and a number $c \in \{-1, 1\}$ such that

$$\varphi(V) = c(\det V)^k, \quad V \in \mathbb{U}_n.$$

Proof. Clearly, $\varphi(I)^3 = \varphi(I)$ implying that $\varphi(I) = \pm 1$. There is no loss of generality in assuming that $\varphi(I) = 1$. Since, by the transformation $\lambda \mapsto \operatorname{diag}(\lambda, 1, \dots, 1)$, the group $\mathbb{T}$ embeds trivially into $\mathbb{U}_n$, the functional $\varphi : \mathbb{U}_n \to \mathbb{T}$ gives rise to a continuous unital Jordan triple endomorphism of $\mathbb{U}_n$. Applying Lemma 7 to this transformation one can easily check that there is a linear functional $l : \mathbb{H}_n \to \mathbb{R}$ such that $\varphi(e^{itA}) = e^{itl(A)}$, $t \in \mathbb{R}, A \in \mathbb{H}_n$. By the second statement in Lemma 7 we further have that $l(VAV) = l(A)$ holds for all $A \in \mathbb{H}_n$ and symmetry $V \in \mathbb{U}_n$.

Proceeding further, since l is a linear functional on the real Hilbert space $\mathbb{H}_n$, by Riesz representation theorem there is an element $H \in \mathbb{H}_n$ such that $l(A) = \operatorname{Tr}(AH)$, $A \in \mathbb{H}_n$. Then

$$\operatorname{Tr}(AH) = l(A) = l(VAV) = \operatorname{Tr}(VAVH) = \operatorname{Tr}(AVHV)$$

holds for every $A \in \mathbb{H}_n$ which implies that $H = VHV$ for every symmetry $V \in \mathbb{H}_n$. Multiplying by V, this gives us that H commutes with all symmetries in $\mathbb{U}_n$. Since any symmetry V is of the form $V = 2P - I$ with some projection, it follows that H commutes with every projection and we obtain that H is necessarily a scalar multiple of the identity. Let $h \in \mathbb{R}$ be such that $H = hI$. We have

$$\varphi(e^{itA}) = e^{ith \operatorname{Tr}(A)}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n.$$

Pick a rank-one projection $P \in \mathbb{M}_n$. Since $\exp i\pi P$ is a symmetry, it follows that its image under φ is a number that has square equal to 1. Therefore, $\exp(i\pi h \operatorname{Tr}(P)) = \exp(i\pi h)$ equals ± 1 which yields that h is an integer. Denote it by k. We compute

$$\varphi(e^{itA}) = e^{itk \operatorname{Tr}(A)} = (e^{\operatorname{Tr}(itA)})^k = (\det(e^{itA}))^k$$

which shows that $\varphi(V) = (\det V)^k$ holds for every $V \in \mathbb{U}_n$.

In the case where $\varphi(I) = -1$ we apply the above argument to the Jordan triple functional $-\varphi$. □

After these preliminaries we are now in a position to prove our first main theorem. The basic idea of the proof is the use of a structural result concerning commutativity preserving linear transformations of $\mathbb{H}_n$. That result holds only when $n \geq 3$. The two-dimensional case requires some special considerations.

Proof of Theorem 1. Let $\phi : \mathbb{U}_n \to \mathbb{U}_n$ be a continuous Jordan triple endomorphism.

Clearly, we have $\phi(I) = \phi(I)^3$ which implies that $I = \phi(I)^2$, i.e., $\phi(I)$ is a symmetry.

We have $\phi(V) = \phi(IVI) = \phi(I)\phi(V)\phi(I)$ for every $V \in \mathbb{U}_n$. Multiplying by $\phi(I)$ from either side we obtain that $\phi(I)$ commutes with every element $\phi(V)$ of the range of ϕ. Defining $\psi(V) = \phi(I)\phi(V)$, the transformation $\psi : \mathbb{U}_n \to \mathbb{U}_n$ is a continuous map which is easily seen to be a Jordan triple endomorphism of $\mathbb{U}_n$ which sends I to I. In what follows we assume that already our original map ϕ fixes the identity I.

By Lemma 7 we have a linear transformation $f : \mathbb{H}_n \to \mathbb{H}_n$ such that

$$\phi(e^{itA}) = e^{itf(A)}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n. \tag{2}$$

We prove that f preserves commutativity meaning that if $A, B \in \mathbb{H}_n$ are such that $AB = BA$, then $f(A)f(B) = f(B)f(A)$ holds, too. Pick commuting matrices $A, B \in \mathbb{H}_n$. Then for every $t, s \in \mathbb{R}$ we have

$$e^{itA}e^{i2sB}e^{itA} = e^{isB}e^{i2tA}e^{isB}$$

implying

$$\phi(e^{itA})\phi(e^{i2sB})\phi(e^{itA}) = \phi(e^{isB})\phi(e^{i2tA})\phi(e^{isB})$$

and hence

$$e^{itf(A)}e^{i2sf(B)}e^{itf(A)} = e^{isf(B)}e^{i2tf(A)}e^{isf(B)}.$$

Fixing the real variable s and putting the complex variable z in the place of it we have that the equality

$$e^{zf(A)}e^{i2sf(B)}e^{zf(A)} = e^{isf(B)}e^{2zf(A)}e^{isf(B)}$$

between matrix valued holomorphic (entire) functions of the variable z holds along the y-axis. By the uniqueness theorem of holomorphic functions we infer that the above equality holds necessarily on the whole complex plane. Next, fixing z and inserting the complex variable w in the place of is, the same reasoning leads to that the equality

$$e^{zf(A)}e^{2wf(B)}e^{zf(A)} = e^{wf(B)}e^{2zf(A)}e^{wf(B)}$$

holds for all values of the variables $z, w \in \mathbb{C}$. In particular, for arbitrary real numbers t, s setting $z = t/2, w = s/2$ we have

$$\sqrt{e^{tf(A)}}e^{sf(B)}\sqrt{e^{tf(A)}} = \sqrt{e^{sf(B)}}e^{tf(A)}\sqrt{e^{sf(B)}}. \tag{3}$$

This is an important equality. Given positive semi-definite matrices or Hilbert space operators D, F, the operation defined by $\sqrt{D}F\sqrt{D}$ is called the sequential product of D and F, see [8]. It was proved in [8, Corollary 3] that commutativity of positive semi-definite operators with respect to that product is equivalent to the commutativity relative to the usual product. Using that result we infer from (3) that

$$e^{tf(A)}e^{sf(B)} = e^{sf(B)}e^{tf(A)}$$

holds for all $t, s \in \mathbb{R}$. This easily implies $f(A)f(B) = f(B)f(A)$ which verifies that f indeed preserves commutativity.

Assume $n \geq 3$. In that case the structure of commutativity preserving linear maps on $\mathbb{H}_n$ is known and was described in [7]. The result [7, Theorem 2] tells us that we have the following possibilities for f:

(a) either the range of f is commutative;
(b) or there exist a unitary matrix $U \in \mathbb{U}_n$, a linear functional $l : \mathbb{H}_n \to \mathbb{R}$ and a nonzero scalar $c \in \mathbb{R}$ such that f is of the form

$$f(A) = cUAU^* + l(A)I, \quad A \in \mathbb{H}_n \tag{4}$$

or of the form

$$f(A) = cUA^{tr}U^* + l(A)I, \quad A \in \mathbb{H}_n. \tag{5}$$

First consider the case (b) where f is of the form (4). We have

$$\phi(e^{itA}) = e^{it(cUAU^*+l(A)I)}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n.$$

Apparently, it follows that

$$U^*\phi(e^{itA})U = e^{it(cA+l(A)I)}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n.$$

Clearly, $U^*\phi(.)U$ is a continuous Jordan triple endomorphism of $\mathbb{U}_n$ which maps I to I. Hence, in the remaining steps of the proof we may and do assume without serious loss of generality that

$$\phi(e^{itA}) = e^{it(cA+l(A)I)} = e^{itl(A)}e^{it(cA)} \tag{6}$$

holds for all $t \in \mathbb{R}, A \in \mathbb{H}_n$. If f is of the form (5), then considering the map $(U^*\phi(.)U)^{tr}$ we can again assume that (6) holds.

It follows from (6) that $\phi(\exp(itA))$ is a scalar multiple of $\exp(it(cA))$ for every $t \in \mathbb{R}$ and $A \in \mathbb{H}_n$. We claim that $c = \pm 1$. To verify this, we first recall that ϕ sends symmetries to symmetries (this follows easily from the fact that I is mapped into I). Let P be a rank-one projection in $\mathbb{M}_n$. Then $\exp(i\pi P)$ is a symmetry and it follows that the symmetry $\phi(\exp(i\pi P))$ is a scalar multiple of $\exp(i\pi cP)$. It is easy to see that this scalar multiplier is necessarily ± 1 and then we obtain that the number $\exp(i\pi c)$ also equals ± 1. From this we infer that c is an integer, say $c = m$. Since every element of $\mathbb{U}_n$ is of the form $\exp(iA)$ with some $A \in \mathbb{H}_n$, we thus obtain from (6) that $\phi(V)$ is a scalar multiple of V^m for every $V \in \mathbb{U}_n$. By the Jordan triple multiplicativity of ϕ this gives us that m is a nonzero integer for which $(VWV)^m$ and $V^mW^mV^m$ are scalar multiples of each other whenever

$V, W \in \mathbb{U}_n$. Taking inverses here if necessary, we can obviously assume that the integer m is positive. Since $(VWV)^m V^{-m} W^{-m} V^{-m}$ is a scalar multiple of the identity, it follows that it commutes with every matrix $T \in \mathbb{M}_n$. Inserting one-parameter groups of unitaries in the places of V and W we deduce that

$$(e^{itH} e^{isJ} e^{itH})^m e^{-imtH} e^{-imsJ} e^{-imtH} T$$
$$= T(e^{itH} e^{isJ} e^{itH})^m e^{-imtH} e^{-imsJ} e^{-imtH}$$

holds for all $t, s \in \mathbb{R}$, $H, J \in \mathbb{H}_n$ and $T \in \mathbb{M}_n$. We now apply the method that we have used to verify the commutativity preserving property of f to replace the real variables t, s in the above displayed formula by general complex variables z, w. Namely, fixing the real variable s and putting the complex variable z in the place of it we obtain that the equality

$$(e^{zH} e^{isJ} e^{zH})^m e^{-mzH} e^{-imsJ} e^{-mzH} T$$
$$= T(e^{zH} e^{isJ} e^{zH})^m e^{-mzH} e^{-imsJ} e^{-mzH}$$

between matrix valued holomorphic (entire) functions of the complex variable z holds along the y-axis. It follows that the equality must hold for every complex value of z, too. Next, fixing z and inserting the complex variable w in the place of is, the same reasoning yields that

$$(e^{zH} e^{wJ} e^{zH})^m e^{-mzH} e^{-mwJ} e^{-mzH} T = T(e^{zH} e^{wJ} e^{zH})^m e^{-mzH} e^{-mwJ} e^{-mzH}$$

holds for all z, w in $\mathbb{C}$. In particular, we obtain that for all $t, s \in \mathbb{R}$, $H, J \in \mathbb{H}_n$ and $T \in \mathbb{M}_n$ we have

$$(e^{tH} e^{sJ} e^{tH})^m e^{-mtH} e^{-msJ} e^{-mtH} T = T(e^{tH} e^{sJ} e^{tH})^m e^{-mtH} e^{-msJ} e^{-mtH}.$$

Since every positive definite matrix in $\mathbb{M}_n$ is the exponential of an element of $\mathbb{H}_n$, we deduce that $(ABA)^m A^{-m} B^{-m} A^{-m}$ commutes with all elements of $\mathbb{M}_n$ and hence it is a scalar multiple of the identity whenever $A, B \in \mathbb{M}_n$ are positive definite. Therefore, $(ABA)^m$ and $A^m B^m A^m$ are scalar multiples of each other. Letting B converge to an arbitrary rank-one projection P we obtain that $(APA)^m$ and $A^m P A^m$ are linearly dependent for every positive definite A and rank-one projection P in $\mathbb{M}_n$. It requires only elementary considerations to see that this can happen only if $m = 1$. Referring back to the general case where m is not necessarily positive, we obtain that $m = \pm 1$.

By (6) it follows that one of the following two possibilities must hold:

$$\phi(e^{itA}) = e^{itl(A)} e^{itA}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n;$$
$$\phi(e^{itA}) = e^{itl(A)} (e^{itA})^{-1}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n.$$

Apparently, this shows that ϕ can be decomposed as $\phi(V) = \varphi(V)V$, $V \in \mathbb{U}_n$ or as $\phi(V) = \varphi(V)V^{-1}$, $V \in \mathbb{U}_n$ where $\varphi : U \to \mathbb{T}$ is a continuous Jordan triple functional. Hence Lemma 8 applies and we obtain that there is an integer k such that ϕ is of one of the following forms:

$$\phi(V) = (\det V)^k V, \quad V \in \mathbb{U}_n;$$
$$\phi(V) = (\det V)^k V^{-1}, \quad V \in \mathbb{U}_n.$$

Remember now the reductions what we may have applied to get (6). First we may have multiplied the original transformation by a symmetry (the image of the identity) that is in the commutant of the range to obtain a unital continuous Jordan triple endomorphism and then we may have composed it with an inner automorphism $U^*(.)U$ and/or with the transpose operation. Having these in mind we deduce the following. In the present case what we are considering ($n \geq 3$, and the possibility (b) holds for f) the above mentioned commutant is trivial implying that the symmetry in question equals ± 1 times the identity. Consequently, dropping all reductions, for the original transformation ϕ we have that it is of one of the forms (j1)-(j4) which appear in the formulation of Theorem 1.

Keeping further the assumptions $n \geq 3$ and $\phi(I) = I$, let us now examine the case where the range of the linear transformation $f : \mathbb{H}_n \to \mathbb{H}_n$ satisfying

$$\phi(e^{itA}) = e^{itf(A)}, \quad t \in \mathbb{R}, A \in \mathbb{H}_n$$

(see (2)) is commutative. Clearly, the same holds for the range of ϕ, too. Therefore, the elements of that range can be diagonalized simultaneously and hence we have a commuting set $\{P_1, \ldots, P_n\}$ of rank-one projections in $\mathbb{M}_n$ and a collection $\varphi_1, \ldots, \varphi_n : \mathbb{U}_n \to \mathbb{T}$ of continuous Jordan triple functionals such that

$$\phi(V) = \sum_{j=1}^n \varphi_j(V) P_j, \quad V \in \mathbb{U}_n.$$

Applying Lemma 8 we obtain that there are integers $k_1, \ldots, k_n$ such that

$$\phi(V) = \sum_{j=1}^n (\det V)^{k_j} P_j, \quad V \in \mathbb{U}_n.$$

Again remember that in the beginning of the course of the proof we may have multiplied the original transformation by a symmetry from the commutant of its range to get a unital Jordan triple endomorphism. In the present case the commutant in question can be described as follows. From the set $\{k_1, \ldots, k_n\}$ of integers and the set $\{P_1, \ldots, P_n\}$ of mutually orthogonal rank-one projections form the new set $\{k'_1, \ldots, k'_d\}$ of pairwise different integers such that

$$\{k'_1, \ldots, k'_d\} = \{k_1, \ldots, k_n\}$$

and then define the set $\{P'_1, \ldots, P'_d\}$ of mutually orthogonal projections with sum I by

$$P'_l = \sum \{P_j \,:\, k_j = k'_l\}, \quad l = 1, \ldots, d.$$

The commutant we are about to describe consists precisely of the unitaries which commute with all projections $P'_1, \ldots, P'_d$. Regarding the symmetry in question which belongs to that commutant it follows that the projections P'_l all split into two mutually orthogonal projections $P'_l = P'_{l_1} + P'_{l_2}$ and we have scalars $c_{l_1}, c_{l_2} \in \{-1, 1\}$ such that the symmetry equals

$$\sum_{l=1}^{d} (c_{l_1} P'_{l_1} + c_{l_2} P'_{l_2}).$$

We easily conclude that the original transformation is of the form (j5) in Theorem 1. This completes the proof if $n \geq 3$.

In the remaining part we treat the case where $n = 2$. Applying the reduction to get a unital Jordan triple endomorphism again, according to (2) we have a linear transformation $f : \mathbb{H}_2 \to \mathbb{H}_2$ such that

$$\phi(e^{itA}) = e^{itf(A)}$$

holds for all $t \in \mathbb{R}$ and $A \in \mathbb{H}_2$. We also know that f preserves commutativity.

Now, we have the following possibilities concerning the range of f: It is either commutative or not commutative. In the former case the proof can be completed as in the corresponding higher dimensional case above (we obtain the possibility (j5)). So we need to consider the case where the range of f is not commutative. The commutativity preserving property of f implies that the range of f commutes with $f(I)$. If $f(I)$ is not a scalar multiple of the identity, then, considering matrices as linear operators, its spectral distribution consists of two mutually orthogonal rank-one projections. Since all elements of the range of f commute with $f(I)$, their spectral distributions must commute with the previously mentioned two rank-one projections. This easily implies that every element of the range of f is a linear combination of those spectral projections and hence this range is necessarily commutative, a contradiction. Therefore, we obtain that $f(I)$ is a scalar multiple of the identity, $f(I) = sI$ holds for some real number s. This yields $\phi(e^{it}I) = e^{its}I$ for every $t \in \mathbb{R}$. Inserting $t = \pi$, we have that $\phi(-I) = \phi(e^{i\pi}I) = e^{i\pi s}I$ is a symmetry which implies that s is an integer. It follows that $f(I) = mI$ holds with some integer m.

Pick a rank-one projection $P \in \mathbb{H}_2$. Since $\exp(i\pi P)$ is a symmetry, so is $\phi(\exp(i\pi P)) = \exp(i\pi f(P))$. It follows that the elements of the spectrum of $f(P)$ are integers. By the continuity of the spectrum on $\mathbb{H}_2$ and using the connectedness of the set of rank-one projections in $\mathbb{H}_2$, it follows that the spectrum of $f(P)$ does not depend on the choice of the rank-one projection P, instead, it is independent of P. Let j, k be the elements of that uniquely determined spectrum. If $j = k$, then it follows that the image of any rank-one projection under f is a scalar multiple of the identity. This implies that every element of the range of f is also a scalar multiple of the identity which contradicts the assumption that this range is not commutative. Therefore, we have $j \neq k$. Now set $Q = I - P$. Then Q is a rank-one projection and hence the spectra of $f(P)$ and $f(Q)$ are equal. Simultaneously diagonalizing the commuting matrices $f(P), f(Q)$, we can conclude that $f(I) = f(P) + f(Q) = (j+k)I$. Since $f(I) = mI$, we obtain $j + k = m$.

Now, consider the linear transformation $g : \mathbb{H}_2 \to \mathbb{H}_2$ defined by

$$g(A) = \frac{f(A) - (k\,\mathrm{Tr}(A))I}{j-k}, \quad A \in \mathbb{H}_2.$$

It is easy to see that for any rank-one projection R, the spectrum of $g(R)$ equals $\{0, 1\}$ and hence $g(R)$ is a projection. Moreover, we have $g(I) = I$ and, trivially, $g(0) = 0$. Therefore, $g : \mathbb{H}_2 \to \mathbb{H}_2$ sends projections to projections. It is well known (cf., [14], Appendix) that any linear transformation of $\mathbb{H}_n$ (for any integer $n \geq 2$) which sends each projection to a projection is necessarily either zero, or a Jordan *-automorphism of $\mathbb{H}_n$. It follows that there exists a unitary matrix U such that either we have

$$g(A) = UAU^*, \quad A \in \mathbb{H}_2$$

or we have

$$g(A) = UA^{tr}U^*, \quad A \in \mathbb{H}_2.$$

As for f, this means that either

$$f(A) = (j-k)UAU^* + k\,\mathrm{Tr}(A)I, \quad A \in \mathbb{H}_2$$

or

$$f(A) = (j-k)UA^{tr}U^* + k\,\mathrm{Tr}(A)I, \quad A \in \mathbb{H}_2$$

holds true. The same analysis as the one presented above after the formulas (4), (5) in the case where $n \geq 3$ shows that $j - k$ is ± 1 and we can conclude the proof obtaining one of the possible forms (j1)-(j4) for the original transformation ϕ.

The proof of the theorem is complete. □

Having the above result, the proof of Corollary 2 is just trivial, therefore we omit it.

For the proof of Theorems 3 and 4 we need to recall the following assertion from the paper [9] (see Corollary 3.9 there) which states that under certain conditions the surjective isometries between metric groups with translation and inverse invariant metrics locally preserve the inverted Jordan triple product.

Proposition 9. *Suppose that G_1, G_2 are metric groups with translation and inverse invariant metrics (i.e., we assume that the two-sided multiplication operators and the inverse operation are all isometries). Let $\phi : G_1 \to G_2$ be a surjective isometry. Suppose that for a given pair $a, b \in G_1$ there exists a constant $K > 1$ such that*

$$d(bx^{-1}b, x) \geq Kd(b, x)$$

holds for all $x \in L_{a,b} = \{x \in X : d(x, a) = d(ba^{-1}b, x) = d(a, b)\}$. Then we have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

We mention that in [9] (and also in [10]) some even more general results have been presented leading to the same conclusion.

In the light of the above statement it is no wonder that we also need the following proposition which describes the structure of all continuous bijections of $\mathbb{U}_n$ that preserve the inverted Jordan triple product. Observe that the trivial idea of the proof together with the use of Theorem 1 could result in the description of all continuous self-maps (not necessarily bijective) of $\mathbb{U}_n$ that preserve the inverted Jordan triple product.

Proposition 10. *Assume $\phi : \mathbb{U}_n \to \mathbb{U}_n$ is a continuous bijective map which preserves the inverted Jordan triple product, i.e., ϕ satisfies*

$$\phi(VW^{-1}V) = \phi(V)\phi(W)^{-1}\phi(V), \quad V, W \in \mathbb{U}_n.$$

Then there exists a pair $U, U' \in \mathbb{U}_n$ of unitary matrices such that ϕ is of one of the following forms:

(t1) $\phi(V) = UVU', \quad V \in \mathbb{U}_n$;
(t2) $\phi(V) = UV^{-1}U', \quad V \in \mathbb{U}_n$;
(t3) $\phi(V) = UV^{tr}U', \quad V \in \mathbb{U}_n$;
(t4) $\phi(V) = U\overline{V}U', \quad V \in \mathbb{U}_n$.

Proof. Just as in the proof of Theorem 2.1 in [15] observe that $\phi(I)^{-1}\phi(.)$ is a continuous Jordan triple automorphism of $\mathbb{U}_n$. The result follows from Corollary 2. □

We can now present the proof of Theorem 3.

Proof of Theorem 3. First we recall that every unitarily invariant norm on $\mathbb{M}_n$ is symmetric (e.g., see Proposition IV.2.4. in [3]) and hence $N(.)$ satisfies $N(ABC) \leq \|A\| N(B) \|C\|$ for all $A, B, C \in \mathbb{M}_n$. Since $\mathbb{M}_n$ is finite dimensional, the norms $N(.)$ and $\|.\|$ are equivalent. Therefore, $\mathbb{U}_n$ is compact also in the metric induced by $N(.)$. Next we refer to the fact that any isometry of a compact metric space is automatically surjective, see Exercise 2.4.1. in [4]. Therefore, the given isometry ϕ of $\mathbb{U}_n$ relative to the metric induced by $N(.)$ is also surjective. One can easily check step by step that the argument employed in the proof of Theorem 2.3 in [15] (that heavily rests on the application of Proposition 9 above) can be followed literally to verify that ϕ is a continuous bijective map on $\mathbb{U}_n$ which satisfies

$$\phi(VW^{-1}V) = \phi(V)\phi(W)^{-1}\phi(V), \quad V, W \in \mathbb{U}_n.$$

An apparent application of Proposition 10 completes the proof. □

The proof of our last theorem is as follows.

Proof of Theorem 4. First observe that d_N and the metric induced by the usual operator norm generate the same topology on $\mathbb{U}_n$. Indeed, since the norm $N(.)$ is equivalent to $\|.\|$, we see that U, V are close enough relative to d_N if and only if the norm $\|H\|$ of the angular matrix H of UV^{-1} is small enough which is the case if and only if $\|UV^{-1} - I\| = \|U - V\|$ is small. Therefore, $\mathbb{U}_n$ is compact also relative to the metric d_N and hence any isometry $\phi : \mathbb{U}_n \to \mathbb{U}_n$ with respect to the metric d_N is necessarily surjective.

Let us check that the conditions in Proposition 9 are satisfied. We first show that d_N is translation and inverse invariant. Indeed,

$$d_N(UW, VW) = d_N(U, V), \quad U, V, W \in \mathbb{U}_n$$

holds trivially and from the equality

$$V^{-1}U = V^{-1}(UV^{-1})V$$

we deduce that $d_N(V^{-1}, U^{-1}) = d_N(U, V)$. Therefore, d_N is inverse invariant and, since it is right translation invariant, we obtain that it is left translation invariant, too.

We have noted in the proof of the previous theorem that $N(.)$ is a symmetric norm equivalent to $\|.\|$. Hence we have a positive scalar c such that $c\|.\| \leq N(.) \leq N(I)\|.\|$.

Set $\alpha = \pi c/4$. Pick $V, W \in \mathbb{U}_n$ with $d_N(V, W) < \alpha$. Let $X \in \mathbb{U}_n$ be such that $d_N(X, V) = d_N(X, WV^{-1}W) = d_N(V, W)$. Then we have

$$d_N(X, W) \leq d_N(X, V) + d_N(V, W) = 2d_N(V, W) < 2\alpha = \pi c/2.$$

It follows that the angular matrix H of $WX^{-1} = e^{iH}$ satisfies $N(H) < \pi c/2$, which implies that $\|H\| < \pi/2$. From this we infer that the angular matrix of $(WX^{-1}W)X^{-1} = (WX^{-1})^2$ is just $2H$. This yields

$$d_N(WX^{-1}W, X) = 2d_N(W, X).$$

These show that the conditions in Proposition 9 are fulfilled (with constant $K = 2$) and we conclude that

$$\phi(VW^{-1}V) = \phi(V)\phi(W)^{-1}\phi(V)$$

holds for any $V, W \in \mathbb{U}_n$ with $d_N(V, W) < \alpha$. Next, choose a positive number β such that $\beta < \alpha/(2N(I))$. Assume $\|V - W\| < \beta$. Then $\|VW^{-1} - I\| < \beta$ and it easily follows that the angular matrix H of VW^{-1} satisfies $\|H\| < 2\beta$. Hence $d_N(V, W) = N(H) \leq N(I)\|H\| < N(I)2\beta < \alpha$ holds which further implies the equality

$$\phi(VW^{-1}V) = \phi(V)\phi(W)^{-1}\phi(V). \tag{7}$$

Consequently, for any pair $V, W \in \mathbb{U}_n$ with $\|V - W\| < \beta$ (i.e., for elements close enough relative to the usual metric) we have the above equality.

The argument given in the first part of the proof of Theorem 8 in [10] is about showing that the above property which tells us that (7) holds locally in $\mathbb{U}_n$ in fact implies that it holds also globally. We can employ that argument here too and obtain that ϕ satisfies (7) for all pairs $V, W \in \mathbb{U}_n$. Since ϕ is an isometry with respect to the metric d_N which induces the same topology as $\|.\|$, it follows that ϕ is continuous in the operator norm. Applying Proposition 10 we obtain the desired conclusion. □

Remark 11. We finish with the following open problem and remarks.

In Theorem 1 we have described the Jordan triple endomorphisms of $\mathbb{U}_n$. A natural problem arises which asks for the structure of all continuous Jordan triple endomorphisms from $\mathbb{U}_n$ into another unitary group $\mathbb{U}_m$. Of course, if $m \leq n$, our result can be applied ($\mathbb{U}_m$ embeds into $\mathbb{U}_n$), but what happens if $m > n$? Recall that in our proof above we have heavily used the structure of commutativity preserving linear maps of $\mathbb{H}_n$ which statement is no longer valid between different spaces.

We remark that the methods what we applied in [15] to determine the Jordan triple automorphisms of the unitary group of a complex separable infinite dimensional Hilbert space can most probably be modified to the finite dimensional setting where $n \geq 3$. However, because of the essential use of the structure of commutativity preserving non-linear maps in [15], the two-dimensional case would certainly remain uncovered. Observe that the approach we have followed in the present paper has provided result also in that low-dimensional case.

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