ISENTROPES AND LYAPUNOV EXPONENTS

ZOLTÁN BUCZOLICH* AND GABRIELLA KESZTHELYI**

Abstract. We consider skew tent maps $T_{\alpha,\beta}(x)$ such that $(\alpha,\beta)\in[0,1]^2$ is the turning point of $T_{\alpha,\beta}$, that is, $T_{\alpha,\beta}=\frac{\beta}{\alpha}x$ for $0\leq x\leq\alpha$ and $T_{\alpha,\beta}(x)=\frac{\beta}{1-\alpha}(1-x)$ for $\alpha<x\leq 1$. We denote by $\underline{M}=K(\alpha,\beta)$ the kneading sequence of $T_{\alpha,\beta}$, by $h(\alpha,\beta)$ its topological entropy and $\Lambda=\Lambda_{\alpha,\beta}$ denotes its Lyapunov exponent. For a given kneading squence $\underline{M}$ we consider isentropes (or equi-topological entropy, or equi-kneading curves), $(\alpha,\Psi_{\underline{M}}(\alpha))$ such that $K(\alpha,\Psi_{\underline{M}}(\alpha))=\underline{M}$. On these curves the topological entropy $h(\alpha,\Psi_{\underline{M}}(\alpha))$ is constant.

We show that $\Psi'_{\underline{M}}(\alpha)$ exists and the Lyapunov exponent $\Lambda_{\alpha,\beta}$ can be expressed by using the slope of the tangent to the isentrope. Since this latter can be computed by considering partial derivatives of an auxiliary function $\Theta_{\underline{M}}$, a series depending on the kneading sequence which converges at an exponential rate, this provides an efficient new method of finding the value of the Lyapunov exponent of these maps.

1. Introduction

Consider a point (α,β) in the unit square $[0,1]^2$. Denote by $T_{\alpha,\beta}(x)$ the skew tent map.

$$(1)\qquad T_{\alpha,\beta}(x)=\begin{cases} L_{\alpha,\beta}(x)=\frac{\alpha}{\beta}x & \text{if}\quad 0\leq x\leq\alpha,\\ R_{\alpha,\beta}(x)=\frac{\beta}{1-\alpha}(1-x) & \text{if}\quad \alpha<x\leq 1.\end{cases}$$

To avoid trivial dynamics we suppose that $0.5<\beta\leq 1$ and $\alpha\in(1-\beta,\beta)$. We denote by U the region of $[0,1]^2$ consisting of these $[\alpha,\beta]$. We denote by $\underline{M}=K(\alpha,\beta)$ the kneading sequence of $T_{\alpha,\beta}$, by $h(\alpha,\beta)$ its topological entropy and by $\Lambda=\Lambda_{\alpha,\beta}$ denotes its Lyapunov exponent. The set of all possible kneading sequences is denoted by $\mathfrak{M}=\{K(\alpha,\beta):(\alpha,\beta)\in U\}$. For a given kneading squence $\underline{M}$ we consider isentropes (or equi-topological entropy, or equi-kneading curves) $(\alpha,\Psi_{\underline{M}}(\alpha))\in U$ such that $K(\alpha,\Psi_{\underline{M}}(\alpha))=\underline{M}$. On these curves the topological entropy $h(\alpha,\Psi_{\underline{M}}(\alpha))$ is constant. On Figure 1 on the left half $T_{.3,.8}$ is considered.

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On the bottom part of the figure one can see the first few entries of the kneading sequence. To visualize the isentrope the computer plotted in black some pixels which correspond to parameter values with similar initial segment of kneading sequence. To obtain a not too thick region the length of this initial segment depends on the parameter region. For example on the left half of Figure 2 there is a thicker region, which can be made thinner by considering longer initial segments. However if the initial segment is too long, the computer is not finding enough pixels from the given equi-kneading region, see for example the right half of Figure 1 where close to the upper left corner of the unit square the plot is too thin.

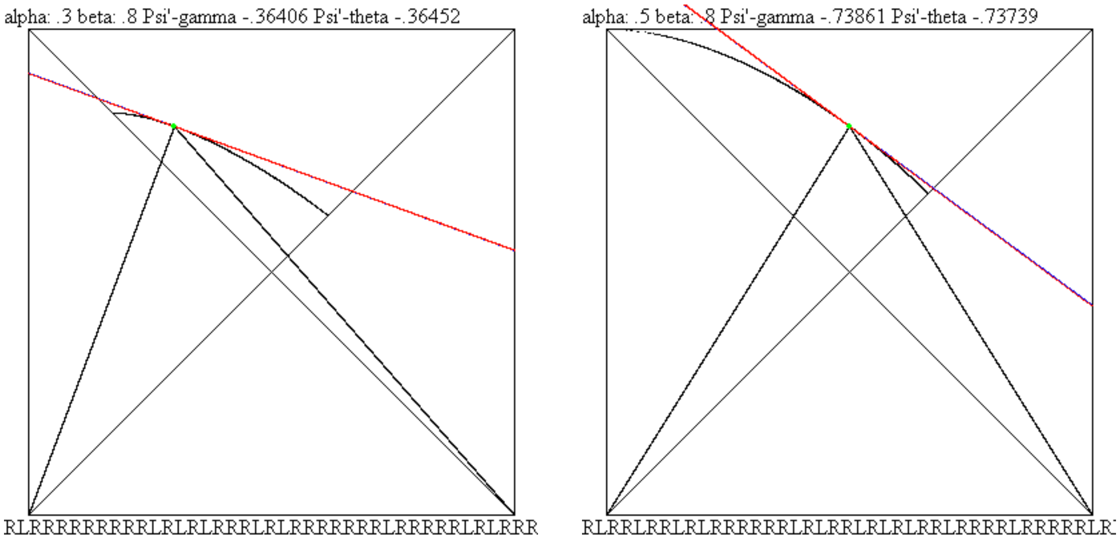

Figure 1. Tangents to isentropes computed from γ and from Θ

We will see in this paper that the isentropes $(\alpha, \Psi_{\underline{M}}(\alpha))$ are continuously differentiable curves. What we found really interesting that the derivatives of these curves can be used to compute the Lyapunov exponents of the skew tent maps $T_{\alpha,\beta}$.

To study equi-topological entropy, or equi-kneading curves in the region U in [4] we introduced the auxiliary functions $\Theta_{\underline{M}}$. Suppose that we have a given kneading-sequence $\underline{M}$ and

$$\underline{M}^{-} = R\underbrace{L\ldots L}_{m_1}R\underbrace{L\ldots L}_{m_2}R\underbrace{L\ldots L}_{m_3}R\ldots. \tag{2}$$

Here $\underline{M} = \underline{M}^{-}$ if the turning point is not periodic, that is $T^{k}_{\alpha,\beta}(\alpha) \neq \alpha$ for $k \in \mathbb{N}$. In this case there is no $C \in \underline{M}$. The set of such kneading sequences is denoted by $\mathfrak{M}_{\infty}$. The cases when the truning point is periodic, that is when C appears in $\underline{M}$ will play a very important role in this paper. The set of these kneading sequences is denoted by $\mathfrak{M}_{<\infty}$. These are the ones ending with C. In this case $\underline{M}^{-}$ can be defined in many ways. One such way was discussed in [4]. However, for our

α	β	γ	$\Psi'_{\underline{M}}$–γ	$\Psi'_{\underline{M}}$–Θ
.3	.8	.20444	-.36406	-.36452
.49	.56	.30996	-.40344	-.4244
.5	.7	.27034	-.64303	-.64064
.5	.8	.26918	-.73861	-.73739
.6	.75	.35597	-.76258	-.76132
.6	.9	.47736	-.4599	-.45991

Table 1. Tangents calculated from Θ and γ

definition of suitable $\Theta_{\underline{M}}$ functions any of the following definitions can be used. Concatenate $\underline{M}$ with itself infinitely many times. Then in the right infinite (right) periodic sequence replace the Cs in an arbitrary manner with Rs and Ls.

For example in our computer simulations each C was replaced by an L. This is due to the fact that if $T^k_{\alpha,\beta}(\alpha) = \alpha$ then $T^{k+1}_{\alpha,\beta}(\alpha) = \beta = L_{\alpha,\beta}(T^k_{\alpha,\beta}(\alpha)) = R_{\alpha,\beta}(T^k_{\alpha,\beta}(\alpha))$, that is both the left- and right- "half definitions" of $T^k_{\alpha,\beta}$ can be used in this case.

We put $\overline{m}_k = m_1 + m_2 + \cdots + m_k$ with m_i defined in (2) and

$$\Theta_{\underline{M}}(\alpha,\beta) = 1 - \beta + \sum_{k=1}^{\infty}(-1)^k \left(\frac{1-\alpha}{\beta}\right)^k \left(\frac{\alpha}{\beta}\right)^{\overline{m}_k}. \tag{3}$$

In [4] we showed that for $(\alpha,\beta) \in U$ it follows from $K(\alpha,\beta) = \underline{M}$ that $\Theta_{\underline{M}}(\alpha,\beta) = 0$. This means that the equi-topological entropy curve $\{(\alpha,\beta) \in U : K(\alpha,\beta) = \underline{M}\}$ is a subset of $\{(\alpha,\beta) \in U : \Theta_{\underline{M}}(\alpha,\beta) = 0\}$, the zero level set of $\Theta_{\underline{M}}$. This means that the isnetrope $(\alpha, \Psi_{\underline{M}}(\alpha))$ satisfies the implicit equation $\Theta_{\underline{M}}(\alpha, \Psi_{\underline{M}}(\alpha)) = 0$. By implicit differentiation

$$\Psi'_{\underline{M}}(\alpha) = -\frac{\partial_1\Theta_{\underline{M}}(\alpha, \Psi_{\underline{M}}(\alpha))}{\partial_2\Theta_{\underline{M}}(\alpha, \Psi_{\underline{M}}(\alpha))}, \tag{4}$$

provided that $\partial_2\Theta_{\underline{M}}(\alpha, \Psi_{\underline{M}}(\alpha)) \neq 0$. Since the series in (3) converges at an exponential rate if we consider the partial derivatives we also obtain an exponential convergence rate for the partial derivatives and hence it is very easy to compute/approximate $\Psi'_{\underline{M}}(\alpha)$ by using (4). On Figures 1, 2 and in Table 1 the entries `Psi'-theta` and $\Psi'_{\underline{M}} - \Theta$ were computed by using this implicit differentiation method by taking into consideration the first 200 elements of the kneading sequence.

The other approach is to estimate $\Psi'_{\underline{M}}(\alpha)$ via the Lyapunov exponents. For the skew tent map $T_{\alpha,\beta}$, $(\alpha,\beta) \in U$ there is a unique ergodic *acim* $\mu_{\alpha,\beta} = \mu$, that is a measure absolutely continuous with respect to the Lebesgue measure, λ. Its density f is an invariant function/fixed point of the Frobenius-Perron operator

$P_{\alpha,\beta}$, that is $P_{\alpha,\beta} f = f$. By Birkhoff's ergodic theorem the Lyapunov exponent

$$\Lambda_{\alpha,\beta} = \lim_{N\to\infty} \frac{1}{N} \log |(T^N_{\alpha,\beta})'(x)| = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log |T'_{\alpha,\beta}(T^n_{\alpha,\beta}(x))| \text{ for } \mu. \text{ a.e. } x. \tag{5}$$

In case of skew tent maps $|T'_{\alpha,\beta}(x)| = \beta/\alpha$ if $x < \alpha$ and $|T'_{\alpha,\beta}(x)| = \beta/(1-\alpha)$ if $x > \alpha$ hence if we let

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{[0,\alpha]}(T^n_{\alpha,\beta}(x)) \text{ then } \Lambda_{\alpha,\beta} = \gamma \log \frac{\beta}{\alpha} + (1-\gamma) \log \frac{\beta}{1-\alpha}, \tag{6}$$

for μ a.e. x.

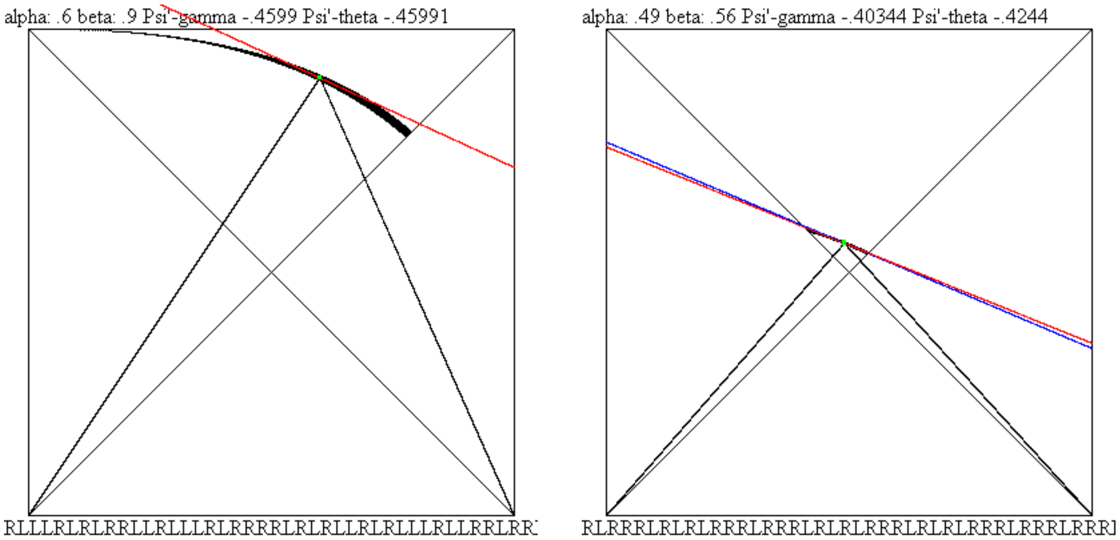

Figure 2. More tangents to isentropes computed from γ and from Θ

Hence to estimate the Lyapunov exponent we need to estimate γ. This is usually done by using a computer program. For a sufficiently large N and a "randomly" selected x one computes the sum in (6). Actually we have done this as well in our computer simulations. It has turned out that $N = 200000$ was sufficiently large to have a reasonably good estimate for γ. In Table 1 there is a column γ containing these estimates for the randomly selected parameter values. The main result of this paper is the fact that γ, and hence $\Lambda_{\alpha,\beta}$ can be expressed by using $\Psi'_{\underline{M}}(\alpha)$. We show in Proposition 10 and in Theorem 13 that

$$\gamma = \alpha(1-\alpha)\frac{\Psi'_{\underline{M}}(\alpha)}{\beta} + \alpha \text{ or, taking inverese } \Psi'_{\underline{M}}(\alpha) = \frac{(\gamma-\alpha)\beta}{\alpha(1-\alpha)}. \tag{7}$$

Since $\Psi'_{\underline{M}}(\alpha)$ can be calculated by (4) using (5), (6) and (7) we can calculate the Lyapunov exponent for any $T_{\alpha,\beta}$ with $(\alpha,\beta) \in U$. To illustrate the connection between $\Psi'_{\underline{M}}(\alpha)$ and $\Lambda_{\alpha,\beta}$, or γ in our computer simulations followed a reverse approach. This means that the computer program calculated an estimate of γ (and hence of Λ) and this estimate was used for calculating the slope of an approximate tangent to the isentrope. As the images show this method, based on (7) works, that is the approximate tangents really seem to be tangent to the isentrope.

In Table 1 there is a column labeled $\Psi'_{\underline{M}}$–Θ which contains the estimates we obtained for $\Psi'_{\underline{M}}(\alpha)$ by using the estimate for γ based on (6). As one can see that the estimates we obtained for $\Psi'_{\underline{M}}(\alpha)$ by using the $\Theta_{\underline{M}}$ function in (4) are quite close to the ones obtained by using γ. On Figures 1 and 2 we plotted both approximate tangents to the isentropes, the one calculated from γ and the one calculated from $\Theta_{\underline{M}}$. On the color pdf version of the paper the first approximate tangent is in red and the second is in blue. In case only one, the red tangent is visible then it means that the two approximate tangents are on top of each other. It is also visible that they are indeed "tangent" to the isentrope as well. On the right half of Figure 2 the two approximate tangents are not exactly on top of each other. This is due to the fact that for the parameter values $\alpha = 0.49$ and $\beta = 0.56$ both α/β and $(1-\alpha)/\beta$ are close to one and the convergence in the series giving the partial derivatives of $\Theta_{\underline{M}}$ is slower. To get a better estimate one needs to consider more than the first 200 entries of the kneading sequence. On this figure the tiny black region corresponding to the equi-kneading region is almost completely covered by the blue and red approximate tangents. We would like to emphasize that our new method based on $\Theta_{\underline{M}}$, even if the number of iterates is increased from 200 to a larger number requires still much less many iterates than the other method which needed 1000 times more iterates for about the same accuracy.

Finally, there is one more illustration showing that indeed there is a link between γ and $\Psi'_{\underline{M}}(\alpha)$. On Figure 3 the color of pixels in U was calculated based on the first 10 entries of the kneading sequence. Hence equi-kneading regions containing isentropes are of the same color (modulo screen/pixel resolution). We also plotted three skew tent maps with three different colors and the approximate tangent line computed by using γ from (6) substituted into (7).

As far as we know in the literature there were two ways to estimate/approximate Lyapunov exponents of skew tent maps. One method is based on computer programs approximating γ, or the acim, or its density as we also did in some calculations on our illustrations. In [2] for the Markov case a histogram of the distribution of the location in the Markov partition of the first 50000 iterates of a "generic" point is used to approximate the piecewise constant invariant density function of the acim. Here again a rather high number of iterates was used. In [7] a central limit theorem is discussed for the convergence in (6). The other method, discussed in [2] is based on the fact that if $K(\alpha, \beta) \in \mathfrak{M}_{<\infty}$, that is when the turning point is periodic for $T_{\alpha,\beta}$ then there is a Markov partition for $T_{\alpha,\beta}$. Based on the Markov partition one can obtain a system of linear equations and the solution of this system gives us the invariant density function $f_{\alpha,\beta}$ of the acim $\mu_{\alpha,\beta}$ of $T_{\alpha,\beta}$. Then $\gamma = \mu_{\alpha,\beta}([0, \alpha])$. (In [2] a different parametrization and notation was used, but we translated it to our notation.) The drawback of this calculation is that the number of equations is the number of elements in the Markov partition. If $K(\alpha,\beta)\in\mathfrak{M}_\infty$ then there is no Markov partition, but isentropes corresponding to skew tent maps with Markov partition are dense in U. It was remarked in [2] that in this case we can also approximate the invariant density by invariant densities of Markov skew tent maps. In this case the number of elements in the Markov partition of these appproximating maps tends to infinity, making it more and more difficult to solve the system of linear equations. It also seems for us that Theorem 10.3.2 from [3] was used in an incorrect way in [2]. By this we mean, that the way these Markov skew tent maps are approximating the non-Markov one is not satisfying the exact assumptions of Theorem 10.3.2 in [3]. Since in our paper we also need approximations of skew tent maps by other ones in Proposition 6 we clarify the way these approximations work. For some specific Markov parameter values in [10] a central limit behavior is discussed.

Properties of isentropes, especially connectedness in different families of dynamical systems were also studied for example in [1], [9] and [12].

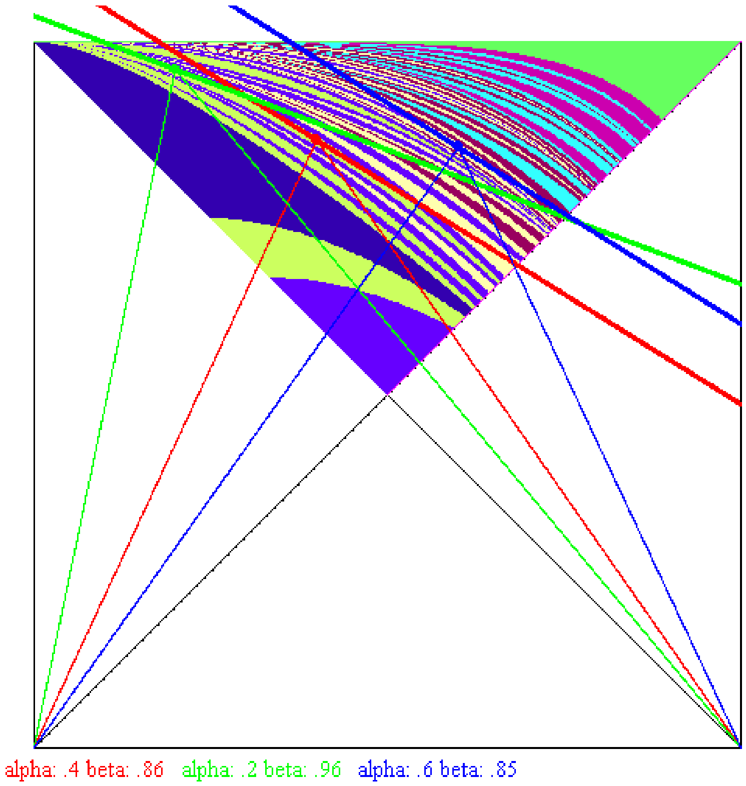

Figure 3. Isentropes and tangents computed from γ

This paper is organized the following way. In Section 2 we recall some definitions and results concerning skew tent maps and invariant densities. In Section 3 we continue to discuss some known results about absolutely continuous invariant measures and prove Proposition 6 which will be the key lemma about approximations of skew tent maps by other ones. This section concludes with some remarks about uniform Lipschitz properties of isentropes.

The most involved part of the paper is Section 4 in which we prove Proposition 10. This is a special version of the main result of the paper about the relationship between Lyapunov exponents and tangents to isentropes. In this proposition we suppose that the isentrope is differentiable at the point considered and we also suppose that we work with a Markov map. In later sections we aim towards Theorem 13 to use some approximation arguments to remove the assumptions about differentiability and Markovness.

In Section 5 by using Proposition 10 first we show that isentropes are continuously differentiable for Markov skew-tent maps. In this argument we use Proposition 6 and approximations of our skew tent map by other ones with the same topological entropy. Then by using another approximation argument based on Proposition 6 and approximation of non-Markov maps by Markov maps we generalize this result for arbitrary maps.

Finally, in Section 6 we prove Theorem 13 which is the main result of our paper. It is again an approximation argument of non-Markov maps by Markov maps. This way we obtain the general version of Proposition 10.

2. Preliminaries

Kneading theory was introduced by J. Milnor and W. Thurston in [8]. For symbolic itineraries and for the kneading sequences we follow the notation of [6].

Suppose $T = T_{\alpha,\beta}$ is fixed for an $(\alpha, \beta) \in U$ and $x \in [0, 1]$. The extended kneading sequence $K(\alpha, \beta) = \underline{M}_{ext} = (\mathfrak{m}_1, \mathfrak{m}_2, ...) \in \{L, R, C\}^{\mathbb{N}}$ is defined as follows. If $T^n_{\alpha,\beta}(\alpha) < \alpha$ then $\mathfrak{m}_n = L$, if $T^n_{\alpha,\beta}(\alpha) = \alpha$ then $\mathfrak{m}_n = C$, and if $T^n_{\alpha,\beta}(\alpha) > \alpha$ then $\mathfrak{m}_n = R$. If there is no C in $\underline{M}_{ext}$ then the kneading sequence $K(\alpha, \beta) = \underline{M} = \underline{M}_{ext}$. If there are Cs in $\underline{M}_{ext}$ then the kneading squence $K(\alpha, \beta) = \underline{M}$ is a finite string which is obtained by stopping at the first C and throwing away the rest of the infinite string $\underline{M}_{ext}$.

Following notation of [11] we denote by $\mathfrak{M}$ the class of kneading sequences $K(0.5, \beta)$, $\beta \in (0.5, 1]$, which is identical to all possible kneading sequences of the form $K(\alpha, \beta)$, $(\alpha, \beta) \in U$.

In [11] a different parametrization of skew tent maps was used. The functions

$$F_{\lambda,\mu}(x) = \begin{cases} 1 + \lambda x & \text{if} \quad x \leq 0 \\ 1 - \mu x & \text{if} \quad x \geq 0 \end{cases}$$

were considered on $\mathbb{R}$.

A simple calculation shows that if $(\alpha, \beta) \in U$ then $(\lambda(\alpha, \beta), \mu(\alpha, \beta)) = (\frac{\beta}{\alpha}, \frac{\beta}{1-\alpha})$ belongs to the region $D' = \{(\lambda, \mu) : \lambda > 1, \mu > 1, \frac{1}{\lambda} + \frac{1}{\mu} \geq 1\}$ this, apart from a boundary segment, coincides with the parameter region $D = \{(\lambda, \mu) : \lambda \geq$

$1, \mu > 1, \frac{1}{\lambda} + \frac{1}{\mu} \geq 1\}$ considered in [11]. In [4] we gave the explicit formula for the linear homeomorphism showing that $T_{\alpha,\beta}$ and $F_{\lambda(\alpha,\beta),\mu(\alpha,\beta)}$ are topologically conjugate. We use the notation $\mathcal{K}(\lambda,\mu)$ for the kneading sequence of $F_{\lambda,\mu}$. In this parametrization $\mathfrak{M}$ corresponds to the kneading sequences of functions $F_{\mu,\mu}$ with $1 < \mu \leq 2$.

We denote by $\prec$ the parity lexicographical ordering of kneading sequences, symbolic itineraries, for the details see [6].

Without discussing too much details of renormalization we need to say a few words about it. The interested reader is refered to more details in [6] or [11]. For $j = 0, 1, ...$ we denote by $\mathfrak{M}^j$ the set of those kneading sequences $\underline{M}$ for which there exists $\beta \in ((\sqrt{2})^{j+1}, \sqrt{2}^j]$ such that $\underline{M} = K(\frac{1}{2}, \beta)$. The kneading sequences in $\mathfrak{M}^0$ correspond to the non-renormalizable case. We denoute by U^j the set of those $(\alpha,\beta) \in U$ for which $K(\alpha,\beta) \in \mathfrak{M}^j$. In [11], D_0 denotes the region of those $(\lambda,\mu) \in D$ for which $\lambda > \frac{\mu}{\mu^2-1}$. This is the non-renormalizable region in the $\lambda - \mu$-parametrization. In [2] and [11] mainly the non-renormalizable region is considered. In Section 5 of [11] renormalization, and the way of extension the result obtained for the non-renormalizable case is discussed. It turns out that if $\mathcal{K}(\lambda,\mu) \in \mathfrak{M}^j$ with $j \geq 1$ then $F^2_{\lambda,\mu}$ can be restricted onto a suitable interval mapped into itself by this map. This restriction is topologically conjugate to $F_{\mu^2,\lambda\mu}$ and $\mathcal{K}(\mu^2, \lambda\mu) \in \mathfrak{M}^{j-1}$. In our parametrization if $K(\alpha,\beta) \in \mathfrak{M}^j$ with $j \geq 1$ then $T^2_{\alpha,\beta}$ restricted onto a suitable interval is topologically conjugate to $T_{1-\alpha,\beta^2/(1-\alpha)}$ and $K(1-\alpha, \beta^2/(1-\alpha)) \in \mathfrak{M}^{j-1}$. In this paper we only use that the density of Markov maps in U^1, shown in [2] implies via renormalization density of Markov maps in U.

We recall a corollary of Theorem C of [11] adapted to our $\alpha-\beta$-parametrization.

Theorem 1. *For each* $\underline{M} \in \mathfrak{M}$ *there exist two numbers* $\alpha_1(\underline{M}) < \alpha_2(\underline{M})$ *and a continuous function* $\Psi_{\underline{M}} : (\alpha_1(\underline{M}), \alpha_2(\underline{M})) \to U$ *such that for* $(\alpha,\beta) \in U$ *we have* $K(\alpha,\beta) = \underline{M}$ *if and only if* $\beta = \Psi_{\underline{M}}(\alpha)$*. The graphs of the functions* $\Psi_{\underline{M}}$ *fill up the whole set* U*. Moreover,* $\lim_{\alpha\to\alpha_1(\underline{M})+} \Psi_{\underline{M}}(\alpha) = 1$ *if* $\underline{M} \succeq RLR^\infty$*. If* $\underline{M} \prec RLR^\infty$ *then the curve* $(\alpha, \Psi_{\underline{M}}(\alpha))$ *converges to a point on the line segment* $\{(\alpha, 1-\alpha) : 0 < \alpha < \frac{1}{2}\}$ *as* $\alpha \to \alpha_1(\underline{M})+$*. If* $\underline{M} = RL^\infty$ *then* $\alpha_1(\underline{M}) = 0$*,* $\alpha_2(\underline{M}) = 1$ *and* $\Psi_{\underline{M}}(\alpha) = 1$ *for all* $\alpha \in (0,1)$*.*

For the skew tent map $T_{\alpha,\beta}$, $(\alpha,\beta) \in U$ we define the Frobenius-Perron operator $P_{\alpha,\beta} : L^1[0,1] \to L^1[0,1]$ by

$$P_{\alpha,\beta}f(x) = \sum_{z\in\{T^{-1}_{\alpha,\beta}(x)\}} \frac{f(z)}{|T'_{\alpha,\beta}(z)|},$$

which in a more explicit form is

$$P_{\alpha,\beta}f(x) = \frac{\alpha}{\beta} f\left(\frac{\alpha x}{\beta}\right) + \frac{1-\alpha}{\beta} f\left(1 - \frac{1-\alpha}{\beta}x\right) \text{ if } 0 \leq x \leq \beta, \tag{8}$$

and $P_{\alpha,\beta} f(x) = 0$ if $x > \beta$.
We recall for example from Proposition 4.2.4 of [3] the contraction property of Frobenius-Perron operator

$$\|P_{\alpha,\beta} f\|_1 \leq \|f\|_1. \tag{9}$$

We also remind to the definition of the variation of a real function $f : [a, b] \to \mathbb{R}$.

$$Vf = V_{[a,b]} f = \sup_{\mathcal{P}} \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}$$

where sup is taken for all partitions $\mathcal{P} = \{[x_0, x_1], [x_1, x_2], \dots [x_{n-1}, x_n]\}$ of $[a, b]$. If $V_{[a,b]} f < +\infty$ then f is of bounded variation, BV on $[a, b]$.

Definition 2. Suppose $I = [a, b]$, $T : I \to I$. A partition

$$\mathcal{P} = \{[a_0, a_1], [a_1, a_2], \dots, [a_{n-1}, a_n]\}$$

of $[a, b]$ is Markov for T if for any $i = 1, \dots, n$ the transformation $T|_{(a_{i-1}, a_i)}$ is a homeomorphism onto the interior of the connected union of some elements of $\mathcal{P}$, that is onto an interval $(a_{j(i)}, a_{k(i)})$.

Observe that if $T^n_{\alpha,\beta}(\beta) = \alpha$, that is C appears in $K(\alpha, \beta) \in \mathfrak{M}_{<\infty}$ then the partition determined by the points $\{0, \alpha, \beta, T_{\alpha,\beta}(\beta), \dots, T^{n-1}_{\alpha,\beta}(\beta), 1\}$ provides a Markov partition.

3. Absolutely continuous invariant measures and densities for skew tent maps

We recall some definitions and results from [3] p. 96. We denote by $\mathcal{T}(I)$ the set of those transformations $T : I \to I$ which satisfy the next two properties:

I. T is piecewise expanding, that is there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, \dots, n\}$ of I such that $T|_{I_i}$ is C^1 and $|T'(x)| \geq \alpha > 1$ for any i and for all $x \in (a_{i-1}, a_i)$.
II. $g(x) = \frac{1}{|T'(x)|}$ is a function of bounded variation, where $T'(x)$ is an appropriately calculated one-sided derivative at the endpoints of $\mathcal{P}$.

For every $n \geq 1$ we define $\mathcal{P}^{(n)}$ as

$$\mathcal{P}^{(n)} = \bigvee_{k=0}^{n-1} T^{-k}(\mathcal{P}) = \{I_{i_0} \cap T^{-1}(I_{i_1}) \cap \dots \cap T^{-n+1}(I_{i_{n-1}}) : I_{i_j} \in \mathcal{P}, j = 0, \dots, n-1\}.$$

One can easily see that if $T \in \mathcal{T}(I)$ then T^n is piecewise expanding on $\mathcal{P}^{(n)}$.

Since $|T'_{\alpha,\beta}(x)| = \frac{\beta}{\alpha}$ on $[0, \alpha]$ and $|T'_{\alpha,\beta}(x)| = \frac{\beta}{1-\alpha}$ on $[\alpha, 1]$, for $(\alpha, \beta) \in U$ we obtain that $T_{\alpha,\beta} \in \mathcal{T}([0, 1])$ with $\mathcal{P} = \{[0, \alpha], [\alpha, 1]\}$.

The next theorem is about the existence of absolutely continuous invariant measures, acims and it is Theorem 5.2.1. from [3].

Theorem 3. *If $T \in \mathcal{T}(I)$ then it admits an absolutely continuous invariant measure, acim whose density is of bounded variation.*

In case of skew tent maps this acim is unique. Theorem 8.2.1 of [3] gives an upper bound on the number of distinct ergodic acims for a $T \in \mathcal{T}(I)$.

Theorem 4. *Let $T \in \mathcal{T}(I)$ be defined on a partition $\mathcal{P}$. Then the number of distinct ergodic acims for T is at most $\#\mathcal{P} - 1$.*

In our case when $(\alpha, \beta) \in U$ and $I_0 = [0,1]$ then $\mathcal{P} = \{[0,\alpha], [\alpha, 1]\}$. Since $\#\mathcal{P} = 2$ we obtain that for $T_{\alpha,\beta}$ there is only one ergodic acim. Using this and the results about the spectral decomposition of the Frobenius-Perron operator in Chapter 7 of [3] one can see that invariant densities are linear combinations of densities of ergodic acims. Hence in case of our skew tent maps the following Lemma holds:

Lemma 5. *For every $(\alpha, \beta) \in U$ there is a unique invariant density for $T_{\alpha,\beta}$, and it is the density of the unique ergodic acim.*

As Theorems 10.2.1 and 10.3.2 are proved in [3] we show the next proposition.

Proposition 6. *Suppose $(\alpha_n, \beta_n) \in U$ for $n = 0, 1, \dots,$ $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$ and $\mathcal{P}_n = \{[0, \alpha_n], [\alpha_n, 1]\}$. Suppose that*

$$\forall\, m \geq 1, \exists\, \delta_m > 0 \text{ such that if}$$

$$\mathcal{P}_n^{(m)} = \bigvee_{j=0}^{m-1} T_{\alpha_n,\beta_n}^{-j}(\mathcal{P}_n) \text{ then } \min_{I \in \mathcal{P}_n^{(m)}} \lambda(I) \geq \delta_m > 0. \tag{10}$$

Then:
(A) For any density f of bounded variation there exists a constant M such that for any n and $k = 1, 2, \dots$

$$V P_{\alpha_n,\beta_n}^k f \leq M.$$

This implies that for any n there is an invariant density f_n of T_{α_n,β_n} and the set $\{f_n\}$ is a precompact set in $L^1([0,1], \lambda)$.
(B) Moreover, if $f_{n_k} \to f_0$ in L^1 then f_0 is an invariant density for T_{α_0,β_0}.

In a similar situation in [2] there is a direct reference to Theorem 10.3.2 of [3] but it seems that after a careful check, this reference is not applicable in the situation of the Markov approximations in [2], neither in our case.

Next we discuss what the problem is with the direct application of Theorem 10.3.2 then by using the ideas of the proofs of Theorems 10.2.1 and 10.3.1 of [3] we prove our Proposition 6.

The main problem of the direct application in [2] of the theorems from [3] to the case of approximations by skew tent maps is the following. In the assumptions of these theorems given a piecewise expanding transformation $T : I \to I$, a family $\{T_n\}_{n \geq 1}$ of approximating Markov transformations associated with T is considered.

Assume $\mathcal{Q}^{(0)}$ denotes the endpoints of intervals belonging to $\mathcal{P}^{(0)}$, where $\mathcal{P}^{(0)}$ is a partition such that T is C^1 and expanding on the partition intervals of $\mathcal{P}^{(0)}$.

If one checks in Section 10.3, p. 217 of [3] the definition of the approximating Markov transformations associated with T one can see that there is a sequence of partitions $\mathcal{P}^{(n)}$. It is supposed that the transformations T_n are piecewise expanding and Markov transformations with respect to $\mathcal{P}^{(n)}$.

Moreover, in assumption (a) on p. 217 of [3] it is stated that if $J = [c, d] \in \mathcal{P}^n$ and $J \cap \mathcal{Q}^{(0)} = \emptyset$ then $T_n|_J$ is a C^1 monotonic function such that

$$T_n(c) = T(c), \quad T_n(d) = T(d) \tag{11}$$

Assumption (11) is clearly not satisfied if $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$, $(\alpha_n, \beta_n) \neq (\alpha_0, \beta_0)$, $T_n = T_{\alpha_n,\beta_n}$, $T = T_{\alpha_0,\beta_0}$ and $\mathcal{P}^{(n)}$ has subintervals $[c, d]$ which do not contain 0, α_0 or 1. This means that contrary to what is claimed by the authors of [2] Theorem 10.3.2 of [3], cannot be applied directly to the case of Markov approximations they want to use. Our Proposition 6 can be used in their case as well. Moreover, it is also an advantage of our Proposition 6 that we do not assume that the approximating skew tent maps are Markov.

Proof of Proposition 6. First we check that assumptions of Theorem 10.2.1 in [3] are satisfied by T_{α_n,β_n} and T_{α_0,β_0} given in Proposition 6. First observe that by $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$ we can choose $\boldsymbol{\gamma} > 1$ such that $|T'_{\alpha_n,\beta_n}(x)| \geq \boldsymbol{\gamma}$ for any x where the derivative exists for any n, this implies condition (1) of Theorem 10.2.1 of [3]. Since $\frac{1}{|T'_{\alpha_n,\beta_n}|}$ is constant on $(0, \alpha_n)$ and $(\alpha_n, 1)$, from $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$ it clearly follows that there exists $W > 0$ such that $V\left(\frac{1}{T'_{\alpha_n,\beta_n}}\right) \leq W$ for any $n \in \mathbb{N}$. This shows that condition (2) of Theorem 10.2.1 of [3] is also satisfied. Observe that by $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$ the partitions $\mathcal{P}_n$ have the property that we can choose $\delta > 0$ such that if $I \in \mathcal{P}_n$ then $T_{\alpha_n,\beta_n}|_I$ is one-to-one, $T_{\alpha_n,\beta_n}(I)$ is an interval and $\min_{I \in \mathcal{P}_n} \lambda(I) > \delta$. This is condition (3) of Theorem 10.2.1 of [3].

Finally, (10) is assumption (4) of Theorem 10.2.1. Therefore this theorem is applicable to the sequence T_{α_n,β_n}. This yields that conclusion (A) of our Proposition 6 holds true. The only thing which needs extra proof that in conclusion (B) the function f_0, which is the L^1 limit of the $P0_{\alpha_{n_k},\beta_{n_k}}$ invariant densities f_{n_k}, is P_{α_0,β_0} invariant. Since $f_{n_k} \to f_0$ in L^1 and $\int f_{n_k} = 1$ for all k, it is clear that $|\int f_0 - \int f_{n_k}| \leq \int |f_0 - f_{n_k}| \to 0$ and hence $\int f_0 = 1$.

For the invariance of f_0 we need to show that $P_{\alpha_0,\beta_0} f = f_0$ a.e.. As on page 220 of [3] it is sufficient to show that $\|P_{\alpha_0,\beta_0} f_0 - f_0\|_1 = 0$, which will be verified by the following estimates:

$$\|P_{\alpha_0,\beta_0} f_0 - f_0\|_1 \leq \|P_{\alpha_0,\beta_0} f_0 - P_{\alpha_{n_k},\beta_{n_k}} f_0\|_1 + \|P_{\alpha_{n_k},\beta_{n_k}} f_0 - P_{\alpha_{n_k},\beta_{n_k}} f_{n_k}\|_1$$

$$+\|P_{\alpha_{n_k},\beta_{n_k}} f_{n_k} - f_{n_k}\|_1 + \|f_{n_k} - f_0\|_1 = A_{1,n_k} + A_{2,n_k} + A_{3,n_k} + A_{4,n_k}.$$

By (9), $A_{2,n_k} \leq \|P_{\alpha_{n_k},\beta_{n_k}}\|_1 \cdot \|f_0 - f_{n_k}\|_1 \leq \|f_0 - f_{n_k}\|_1 \to 0$.
Since f_{n_k} is an invariant density of $T_{\alpha_{n_k},\beta_{n_k}}$ we have $A_{3,n_k} = 0$ for any k. It is also clear that $A_{4,n_k} \to 0$ as $k \to \infty$.

The only non-trivial part is the estimation of A_{1,n_k}. Suppose $\varepsilon > 0$ is given and choose an $f_\varepsilon \in C^1[0,1]$ such that $\|f_0 - f_\varepsilon\|_1 < \varepsilon$. Put

$$M_\varepsilon = \max\{|f_\varepsilon(x)| + |f'_\varepsilon(x)| : x \in [0,1]\}. \tag{12}$$

We suppose that K_0 is chosen in a way that

$$\begin{gathered}\left|\frac{\alpha_0}{\beta_0} - \frac{\alpha_{n_k}}{\beta_{n_k}}\right| < \frac{\varepsilon}{M_\varepsilon}, \ \left|\frac{1-\alpha_0}{\beta_0} - \frac{1-\alpha_{n_k}}{\beta_{n_k}}\right| < \frac{\varepsilon}{M_\varepsilon}, \\ \text{and } \frac{|\beta_k - \beta_0|}{\beta_k} < \frac{\varepsilon}{M_\varepsilon} \text{ hold for } k \geq K_0.\end{gathered} \tag{13}$$

We suppose that $\beta_{n_k} \geq \beta_0$, the case $\beta_{n_k} < \beta_0$ is similar and is left to the reader. For ease of notation we denote n_k by k in the sequel. We have for $k \geq K_0$

$$A_{1,k} = \|P_{\alpha_0,\beta_0} f_0 - P_{\alpha_k,\beta_k} f_0\|_1 \leq \|P_{\alpha_0,\beta_0} f_\varepsilon - P_{\alpha_k,\beta_k} f_\varepsilon\|_1 + \|P_{\alpha_0,\beta_0}\| \cdot \|f - f_\varepsilon\|$$
$$+ \|P_{\alpha_k,\beta_k}\| \cdot \|f_0 - f_\varepsilon\|$$

(using (9))

$$\leq \|P_{\alpha_0,\beta_0} f_\varepsilon - P_{\alpha_k,\beta_k} f_\varepsilon\| + 2\varepsilon \leq \int_0^{\beta_0} \left|\frac{\alpha_0}{\beta_0} f_\varepsilon\left(\frac{\alpha_0}{\beta_0}x\right) - \frac{\alpha_k}{\beta_k} f_\varepsilon\left(\frac{\alpha_k}{\beta_k}x\right)\right| dx$$
$$+ \int_0^{\beta_0} \left|\frac{1-\alpha_0}{\beta_0} f_\varepsilon\left(1 - \frac{1-\alpha_0}{\beta_0}x\right) - \frac{1-\alpha_k}{\beta_k} f_\varepsilon\left(1 - \frac{1-\alpha_k}{\beta_k}x\right)\right| dx$$
$$+ \int_{\beta_0}^{\beta_k} \left|\frac{\alpha_k}{\beta_k} f_\varepsilon\left(\frac{\alpha_k}{\beta_k}x\right)\right| + \left|\frac{1-\alpha_k}{\beta_k} f_\varepsilon\left(1 - \frac{1-\alpha_k}{\beta_k}x\right)\right| dx + 2\varepsilon$$
$$\leq \int_0^{\beta_0} \frac{\alpha_0}{\beta_0} \left|f_\varepsilon\left(\frac{\alpha_0}{\beta_0}x\right) - f_\varepsilon\left(\frac{\alpha_k}{\beta_k}x\right)\right| + \left|\frac{\alpha_0}{\beta_0} - \frac{\alpha_k}{\beta_k}\right| f_\varepsilon\left(\frac{\alpha_k}{\beta_k}x\right) dx$$
$$+ \int_0^{\beta_0} \frac{1-\alpha_0}{\beta_0} \left|f_\varepsilon\left(1 - \frac{1-\alpha_0}{\beta_0}x\right) - f_\varepsilon\left(1 - \frac{1-\alpha_k}{\beta_k}x\right)\right|$$
$$+ \left|\frac{1-\alpha_0}{\beta_0} - \frac{1-\alpha_k}{\beta_k}\right| f_\varepsilon\left(1 - \frac{1-\alpha_k}{\beta_k}x\right) dx + |\beta_k - \beta_0| \left(\frac{\alpha_k}{\beta_k} M_\varepsilon + \frac{1-\alpha_k}{\beta_k} M_\varepsilon\right) + 2\varepsilon$$

(using (12) and (13))

$$< \frac{\alpha_0}{\beta_0} M_\varepsilon \left|\frac{\alpha_0}{\beta_0} - \frac{\alpha_k}{\beta_k}\right| \int_0^{\beta_0} |x| dx + \beta_0 \left|\frac{\alpha_0}{\beta_0} - \frac{\alpha_k}{\beta_k}\right| \cdot M_\varepsilon$$
$$+ \frac{1-\alpha_0}{\beta_0} \left|\frac{1-\alpha_0}{\beta_0} - \frac{1-\alpha_k}{\beta_k}\right| \cdot M_\varepsilon \cdot \int_0^{\beta_0} |x| dx + \beta_0 \left|\frac{1-\alpha_0}{\beta_0} - \frac{1-\alpha_k}{\beta_k}\right| \cdot M_\varepsilon + \varepsilon + 2\varepsilon$$

(using again (13))

$$< \frac{\alpha_0}{\beta_0}\cdot\varepsilon+\beta_0\cdot\varepsilon+\frac{1-\alpha_0}{\beta_0}\varepsilon+\beta_0\cdot\varepsilon+3\varepsilon<\left(\frac{1}{\beta_0}+2\beta_0+3\right)\varepsilon.$$

Thus $\|P_{\alpha_0,\beta_0}f-P_{\alpha_k,\beta_k}f\|_1\to 0$ as $k\to\infty$ and hence $A_{1,k}\to 0$ as $k\to\infty$ and completes the proof of the Proposition. □

The next lemma shows that if T_{α_0,β_0} is non-Markov, that is $K(\alpha_0,\beta_0)\in\mathfrak{M}_\infty$ then (10) is satisfied.

Lemma 7. *Suppose* $(\alpha_0,\beta_0)\in U$, $K(\alpha_0,\beta_0)=\underline{M}\in\mathfrak{M}_\infty$. *The sequence* $(\alpha_n,\beta_n)\to(\alpha_0,\beta_0)$, $(\alpha_n,\beta_n)\in U$, $\mathcal{P}_n=\{[0,\alpha_n],[\alpha_n,1]\}$, $n=0,1,\dots$ *then* (10) *is satisfied.*

Proof. Since $\underline{M}\in\mathfrak{M}_\infty$ we have $T^{k+1}_{\alpha_0,\beta_0}(\alpha_0)=T^{k}_{\alpha_0,\beta_0}(\beta_0)\neq\alpha_0$ for $k=0,1,\dots$. This implies that

$$T^{k+1}_{\alpha_0,\beta_0}(\alpha_0)\neq T^{k'}_{\alpha_0,\beta_0}(\alpha_0)\text{ if }k'>k\geq 0. \tag{14}$$

Observe that the division points of $\mathcal{P}^{(m)}_n$, $(n=0,1,\dots)$ are $0,1$ and points of the form $T^{-j}_{\alpha_n,\beta_n}(\alpha_n)$ with $0\leq j\leq m-1$. Denote the set of division points of $\mathcal{P}^{(m)}_n$ by $\mathcal{Q}^{(m)}_n$. By (14) we have

$$\alpha_0\notin T^{-j}_{\alpha_0,\beta_0}(\alpha_0)\text{ for any }j=1,2,\dots \tag{15}$$

and in general

$$T^{-j'}_{\alpha_0,\beta_0}(\alpha_0)\text{ and }T^{-j}_{\alpha_0,\beta_0}(\alpha_0)\text{ are disjoint finite sets for }j'\neq j. \tag{16}$$

Indeed, if we had for a $j'>j\geq 1$, $x\in T^{-j'}_{\alpha_0,\beta_0}(\alpha_0)\cap T^{-j}_{\alpha_0,\beta_0}(\alpha_0)$ then

$$T^{j'-j}_{\alpha_0,\beta_0}(T^{j}_{\alpha_0,\beta_0}(x))=\alpha_0=T^{j}_{\alpha_0,\beta_0}(\alpha_0)$$

and hence $T^{j'-j}_{\alpha_0,\beta_0}(\alpha_0)=\alpha_0$, which contradicts (14).

Denote by $\delta_{0,m}$ the length of the shortest interval in $\mathcal{P}^{(m)}_0$. By using $\alpha_n\to\alpha_0$, $\beta_n\to\beta_0$, (15) and (16) we can select N_m such that

$$\mathrm{dist}_{\mathrm{Hau}}(\mathcal{Q}^{(m)}_n,\mathcal{Q}^{(m)}_0)<\delta_{0,m}/3\text{ holds for }n\geq N_m. \tag{17}$$

This implies that for $\min_{I\in\mathcal{P}^{(m)}_n}\lambda(I)\geq\delta_{0,m}/3>0$ holds for $n\geq N_m$. Since $\min\{\lambda(I):I\in\mathcal{P}^{(m)}_n,\ n\leq N_m\}>0$ we obtain that (10) is satisfied. □

Finally, in this section we make a few remarks about the Lipschitz property of the isentropes. By Theorem A of [11] if $\mu'>\mu$ and $\lambda'>\lambda$ then the topological entropy of $F_{\lambda',\mu'}$ is larger than that of $F_{\lambda,\mu}$. Recalling that $\lambda=\frac{\beta}{\alpha}$ and $\mu=\frac{\beta}{1-\alpha}$ we obtain that if the isentrope $\{(\alpha,\Psi_{\underline{M}}(\alpha)):\alpha\in(\alpha_1(\underline{M}),\alpha_2(\underline{M}))\}$ is passing through the point $(\alpha_0,\beta_0)=(\alpha_0,\Psi_{\underline{M}}(\alpha_0))$ then

$$\frac{\Psi_{\underline{M}}(\alpha)-\Psi_{\underline{M}}(\alpha_0)}{\alpha-\alpha_0}\leq\frac{\beta_0}{\alpha_0}\text{ for }\alpha>\alpha_0 \tag{18}$$

and

$$\frac{\Psi_{\underline{M}}(\alpha)-\Psi_{\underline{M}}(\alpha_0)}{\alpha-\alpha_0} \geq -\frac{\beta_0}{1-\alpha_0} \text{ for } \alpha < \alpha_0. \tag{19}$$

Now suppose that we selected an interval $[\overline{\alpha}_1, \overline{\alpha}_2] \subset (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. Then we can choose a constant $\overline{B} > 0$ for which

$$\frac{\Psi_{\underline{M}}(\alpha)-\Psi_{\underline{M}}(\alpha_0)}{\alpha-\alpha_0} \leq \overline{B} \text{ if } \alpha > \alpha_0,\ \alpha, \alpha_0 \in [\overline{\alpha}_1, \overline{\alpha}_2],$$

and

$$\frac{\Psi_{\underline{M}}(\alpha)-\Psi_{\underline{M}}(\alpha_0)}{\alpha-\alpha_0} \geq -\overline{B} \text{ if } \alpha < \alpha_0,\ \alpha, \alpha_0 \in [\overline{\alpha}_1, \overline{\alpha}_2].$$

This implies that we proved the following:

Proposition 8. *Suppose* $\underline{M} \in \mathfrak{M}$ *and* $[\overline{\alpha}_1, \overline{\alpha}_2] \subset (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. *Then there exists a* $\overline{B}$ *such that*

$$\left| \frac{\Psi_{\underline{M}}(\alpha_1)-\Psi_{\underline{M}}(\alpha_2)}{\alpha_1-\alpha_2} \right| \leq \overline{B} \tag{20}$$

if $\alpha_1, \alpha_2 \in [\overline{\alpha}_1, \overline{\alpha}_2]$, *that is* $\Psi_{\underline{M}}$ *is Lipschitz on* $[\overline{\alpha}_1, \overline{\alpha}_2]$ *and hence is absolutely continuous on* $[\overline{\alpha}_1, \overline{\alpha}_2]$, $\Psi'_{\underline{M}}$ *exists almost everywhere on* $[\overline{\alpha}_1, \overline{\alpha}_2]$ *and for any* $\alpha_1, \alpha_2 \in [\overline{\alpha}_1, \overline{\alpha}_2]$, $\alpha_1 < \alpha_2$ *we have* $\Psi_{\underline{M}}(\alpha_2) - \Psi_{\underline{M}}(\alpha_1) = \int_{\alpha_1}^{\alpha_2} \Psi'_{\underline{M}}(\alpha) d\alpha$.

Remark 9. From (18) and (19) it is also clear that we have a locally uniform Lipschitz property of the isentropes. This means that if $(\alpha_0, \beta_0) \in U$ and $[\alpha_0 - \delta, \alpha_0 + \delta] \times [\beta_0 - \delta, \beta_0 + \delta] \subset U$ then one can choose $\overline{B}$ such that for any $\alpha_1, \alpha_2 \in U$ if $\Psi_{\underline{M}}(\alpha_1), \Psi_{\underline{M}}(\alpha_2) \in [\alpha_0 - \delta, \alpha_0 + \delta] \times [\beta_0 - \delta, \beta_0 + \delta]$ then we have (20).

4. ISENTROPES AND LYAPUNOV EXPONENTS, THE MARKOV CASE

Proposition 10. *Suppose* $(\alpha_0, \beta_0) \in U$, $\underline{M} = K(\alpha_0, \beta_0) \in \mathfrak{M}_{<\infty}$, *that is there exists a minimal* $n_{\underline{M}} > 1$ *such that* $T^{n_{\underline{M}}}_{\alpha_0,\beta_0}(\beta_0) = \alpha_0$. *Assume that* $\Lambda = \Lambda_{\alpha_0,\beta_0}$ *denotes the Lyapunov exponent of* T_{α_0,β_0} *and* $(\alpha, \Psi_{\underline{M}}(\alpha))$ *is the isentrope satisfying* $\beta_0 = \Psi_{\underline{M}}(\alpha_0)$. *We also suppose that* $\Psi'_{\underline{M}}(\alpha_0)$ *exists, that is the isentrope is differentiable at* α_0. *Then we have the following formula*

$$\Lambda_{\alpha_0,\beta_0} = \Lambda = \gamma \log \frac{\beta_0}{\alpha_0} + (1-\gamma) \log \frac{\beta_0}{1-\alpha_0}, \text{ where } \gamma \text{ satisfies} \tag{21}$$

$$\gamma = \frac{\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0} + \frac{1}{1-\alpha_0}}{\frac{1}{\alpha_0} + \frac{1}{1-\alpha_0}} = \alpha_0(1-\alpha_0)\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0} + \alpha_0. \tag{22}$$

Moreover, if μ *denotes the acim of* T_{α_0,β_0} *then*

$$\gamma = \mu([0, \alpha_0]). \tag{23}$$

Proof. Since $\underline{M} \in \mathfrak{M}_{<\infty}$ we know that $\{T^n_{\alpha_0,\beta_0}(\alpha_0) : n \in \mathbb{N}\}$ is a finite set which has $k = n_{\underline{M}} + 1$ many elements. We denote this finite set by $c_1 < c_2 < \dots < c_k$. Then $T^k_{\alpha_0,\beta_0}(\alpha_0) = \alpha_0$, $c_1 = T_{\alpha_0,\beta_0}(\beta_0)$, $c_k = \beta_0$ and $[c_1, c_k]$ is the dynamical core of the dynamical system $([0,1], T_{\alpha_0,\beta_0})$. The orbit of any $x \in (0,1)$ enters $[c_1, c_k]$ and then for higher iterates $T^n_{\alpha_0,\beta_0}(x)$ stays in this interval.
Moreover, since $T_{\alpha_0,\beta_0}([c_1, c_k]) = [c_1, c_k]$ we can study the restriction of T_{α_0,β_0} onto $[c_1, c_k]$, which for ease of notation is still denoted by T_{α_0,β_0}.
Since μ can be obtained as the weak limit of a subsequence of the measures $\frac{1}{N}\sum_{n=0}^{N-1} \boldsymbol{\delta}_{T^n_{\alpha_0,\beta_0}(x)}$ for μ almost every x, it is clear that the support of μ is a subset of $[c_1, c_k]$. (Recall that $\boldsymbol{\delta}_x$ is the Dirac measure centred on x.) By Proposition 5, μ is unique and ergodic. By (6), γ in (21) satisfies (23) and by Birkhoff's ergodic theorem

$$\gamma = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_{[0,\alpha_0]}(T^n_{\alpha_0,\beta_0}(x)) \tag{24}$$

holds for μ almost every x. Since μ is absolutely continuous with respect to the Lebesgue measure the set S_γ which consist of those x for which (24) holds is of positive Lebesgue measure. It is also well-known, and is easy to check, that the partition $\mathcal{P}_{\alpha_0} = \{[c_1, c_2], \dots, [c_{k-1}, c_k]\}$ is a Markov partition of the dynamical core $[c_1, c_k]$.
We select $\overline{\alpha}_1 < \overline{\alpha}_2$ such that $\alpha_0 \in (\overline{\alpha}_1, \overline{\alpha}_2) \subset [\overline{\alpha}_1, \overline{\alpha}_2] \subset (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. Since $\Psi_{\underline{M}}$ is an isentrope, the maps $T_{\alpha,\Psi_{\underline{M}}(\alpha)}$ are topologically conjugate,

$$T^k_{\alpha,\Psi_{\underline{M}}(\alpha)}(\alpha) = \alpha \text{ holds for } \alpha \in [\overline{\alpha}_1, \overline{\alpha}_2], \tag{25}$$

and the dynamical systems $T_{\alpha,\Psi_{\underline{M}}(\alpha)}$ are also Markov with Markov partitions $\mathcal{P}_\alpha = \{[c_1(\alpha), c_2(\alpha)], \dots, [c_{k-1}(\alpha), c_k(\alpha)]\}$ where $c_i(\alpha) = T^{n_i}_{\alpha,\Psi_{\underline{M}}(\alpha)}(\alpha)$ with $n_i < k$ not depending on α. By Proposition 8 and by topological conjugacy of the maps $T_{\alpha,\Psi_{\underline{M}}(\alpha)}$, $\alpha \in [\overline{\alpha}_1, \overline{\alpha}_2]$ the functions $c_i(\alpha)$, $i = 1, \dots, k$ are Lipschitz on $[\overline{\alpha}_1, \overline{\alpha}_2]$. Moreover, we can choose $M_c > 0$ such that

$$|c_i(\alpha_1) - c_i(\alpha_2)| \le M_c|\alpha_1 - \alpha_2| \text{ for } \alpha_1, \alpha_2 \in [\overline{\alpha}_1, \overline{\alpha}_2] \text{ and } i = 1, \dots, k. \tag{26}$$

We denote by $\boldsymbol{\Delta}_c$ the minimum distance among the points $c_i = c_i(\alpha_0)$, $i = 1, \dots, k$ that is

$$\boldsymbol{\Delta}_c = \min\{c_{i+1} - c_i : i = 1, \dots k - 1\}. \tag{27}$$

Next, proceeding towards a contradiction we suppose that γ defined in (24) does not satisfy (22). By Proposition 8, $\Psi_{\underline{M}}$ is a Lipschitz function on $[\overline{\alpha}_1, \overline{\alpha}_2]$. Hence $\Psi'_{\underline{M}}(\alpha)$ exists almost everywhere on $[\overline{\alpha}_1, \overline{\alpha}_2]$ and we can put

$$\widehat{\gamma}(\alpha) = \alpha(1-\alpha)\frac{\Psi'_{\underline{M}}(\alpha)}{\Psi_{\underline{M}}(\alpha)} + \alpha \text{ for } \lambda \text{ a.e. } \alpha \in [\overline{\alpha}_1, \overline{\alpha}_2]. \tag{28}$$

Since $\Psi_{\underline{M}}(\alpha_0)=\beta_0$ our assumption that γ does not satisfy (22) can be written in the form $\widehat{\gamma}(\alpha_0)\neq\gamma$. Recall that we supposed that $\Psi'_{\underline{M}}(\alpha_0)$ exists and hence $\widehat{\gamma}(\alpha_0)$ is well defined. Moreover $\Psi_{\underline{M}}(\alpha_0)=\beta_0$ and (28) imply

$$\frac{\widehat{\gamma}(\alpha_0)-\alpha_0}{\alpha_0(1-\alpha_0)}=\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0},\ \text{that is } 0=\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0}-\frac{\widehat{\gamma}(\alpha_0)}{\alpha_0}+\frac{1-\widehat{\gamma}(\alpha_0)}{1-\alpha_0}, \tag{29}$$

since

$$\begin{aligned}&\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0}-\frac{\widehat{\gamma}(\alpha_0)}{\alpha_0}+\frac{1-\widehat{\gamma}(\alpha_0)}{1-\alpha_0}\\ &=\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0}+\frac{(1-\widehat{\gamma}(\alpha_0))\alpha_0-\widehat{\gamma}(\alpha_0)(1-\alpha_0)}{\alpha_0(1-\alpha_0)}\\ &=\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0}+\frac{\alpha_0-\widehat{\gamma}(\alpha_0)}{\alpha_0(1-\alpha_0)}=0.\end{aligned}$$

Put $s(\alpha,t)=\Psi_{\underline{M}}(\alpha)\left(\frac{1}{\alpha}\right)^t\left(\frac{1}{1-\alpha}\right)^{1-t}$. Then $\partial_1 s(\alpha,t)$ exists at α_0 and for fixed t, $s(\alpha,t)$ is Lipschitz in α on $[\overline{\alpha}_1,\overline{\alpha}_2]$. Using (29) we obtain

$$\begin{aligned}\partial_1 s(\alpha_0,t)&=\left(\frac{\Psi'_{\underline{M}}(\alpha_0)}{\Psi_{\underline{M}}(\alpha_0)}-\frac{t}{\alpha_0}+\frac{1-t}{1-\alpha_0}\right)s(\alpha_0,t)\\ &=\left(\frac{\Psi'_{\underline{M}}(\alpha_0)}{\beta_0}-\frac{\hat{\gamma}(\alpha_0)}{\alpha_0}+\frac{1-\widehat{\gamma}(\alpha_0)}{1-\alpha_0}+\frac{\widehat{\gamma}(\alpha_0)-t}{\alpha_0}-\frac{t-\widehat{\gamma}(\alpha_0)}{1-\alpha_0}\right)s(\alpha_0,t) \qquad (30)\\ &=s(\alpha_0,t)(\widehat{\gamma}(\alpha_0)-t)\left(\frac{1}{\alpha_0}+\frac{1}{1-\alpha_0}\right).\end{aligned}$$

Since $\widehat{\gamma}(\alpha_0)-\gamma\neq 0$ we have $\partial_1 s(\alpha_0,\gamma)\neq 0$. Select and fix $\delta_0>0$ such that for $|\Delta\alpha|<\delta_0$

$$|s(\alpha_0+\Delta\alpha,\gamma)-s(\alpha_0,\gamma)-\Delta\alpha\cdot\partial_1 s(\alpha_0,\gamma)|<\frac{1}{2}|\Delta\alpha|\cdot|\partial_1 s(\alpha_0,\gamma)|. \tag{31}$$

Since $s(\alpha_0,\gamma)>0$, by (30), $\operatorname{sgn}(\partial_1 s(\alpha_0,\gamma))=\operatorname{sgn}(\widehat{\gamma}(\alpha_0)-\gamma)$. Choose $\Delta\alpha$ with $|\Delta\alpha|<\delta_0$ such that

$$\alpha_0+\Delta\alpha\in[\overline{\alpha}_1,\overline{\alpha}_2],\ \Delta\alpha\cdot\partial_1 s(\alpha_0,\gamma)<0,\ \text{and } |\Delta\alpha|<\frac{\boldsymbol{\Delta}_c}{4M_c}. \tag{32}$$

By (31)

$$s(\alpha_0+\Delta\alpha,\gamma)<s(\alpha_0,\gamma)+\frac{1}{2}\Delta\alpha\cdot\partial_1 s(\alpha_0,\gamma)<s(\alpha_0,\gamma). \tag{33}$$

Since $s(\alpha_0,t)$ and $\partial_1 s(\alpha_0,t)$ are continuous in t, choose $\delta_1>0$ such that if $|t-\gamma|<\delta_1$ then

$$s(\alpha_0+\Delta\alpha,t)<s(\alpha_0,t)+\frac{1}{2}\Delta\alpha\cdot\partial_1 s(\alpha_0,t),\ \text{and } |\widehat{\gamma}(\alpha_0)-t|>\frac{|\widehat{\gamma}(\alpha_0)-\gamma|}{2}. \tag{34}$$

Put

$$\gamma_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_{[0,\alpha_0]}(T^n_{\alpha_0,\beta_0}(x)).$$

By Lemma 5, μ is ergodic and hence $\gamma_N(x) \to \gamma = \mu([0,\alpha_0])$ for μ a.e. x and there exists $\widehat{S}_\gamma \subset S_\gamma$ and $N_0 \in \mathbb{N}$ such that $\lambda(\widehat{S}_\gamma) > \lambda(S_\gamma)/2 > 0$ and we have

$$|\gamma_N(x) - \gamma| < \delta_1 \text{ for any } N \geq N_0 \text{ and } x \in \widehat{S}_\gamma. \tag{35}$$

We will fix an $N \geq N_0$ later. Suppose N is given and fixed. We can select a system of intervals $I_l = [d_l, e_l]$ such that $T^N_{\alpha_0,\beta_0}$ is linear and non-constant on I_l but is non-linear on any larger interval containing I_l, moreover

$$(d_l, e_l) \cap \widehat{S}_\gamma \neq \emptyset \text{ and } \widehat{S}_\gamma \subset \bigcup_l I_l. \tag{36}$$

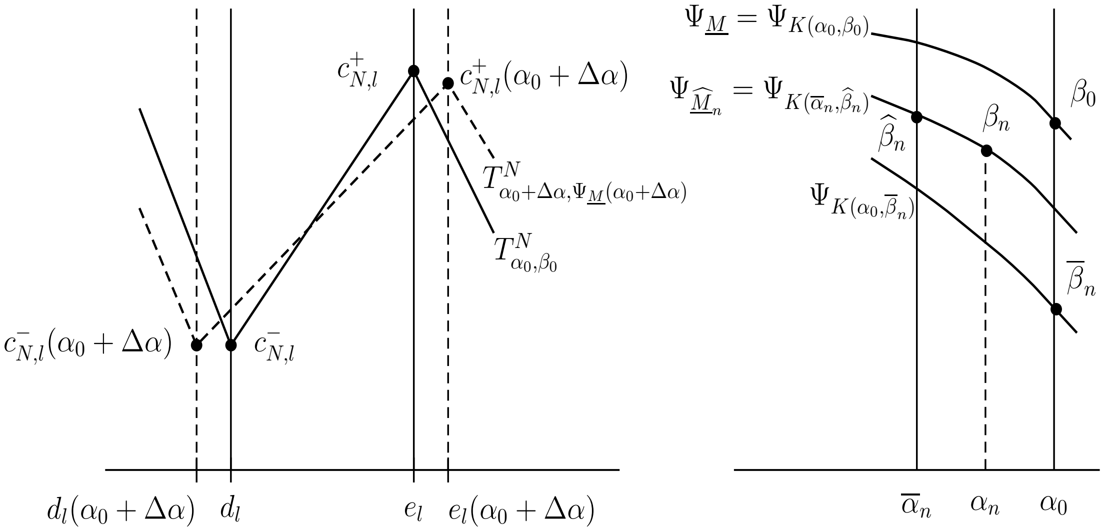

Figure 4. Illustration for the proofs of Proposition 10 and Theorem 13

The maximality of the intervals I_l implies that

$$T^N_{\alpha_0,\beta_0}(d_l), T^N_{\alpha_0,\beta_0}(e_l) \in \{c_i : i = 1,\dots,k\} \text{ and } T^N_{\alpha_0,\beta_0}(d_l) \neq T^N_{\alpha_0,\beta_0}(e_l). \tag{37}$$

From (36) it follows that

$$\sum_l \lambda(I_l) \geq \lambda(\widehat{S}_\gamma). \tag{38}$$

By using (37) we introduce the notation

$$c^-_{N,l} = T^N_{\alpha_0,\beta_0}(d_l) \text{ and } c^+_{N,l} = T^N_{\alpha_0,\beta_0}(e_l). \tag{39}$$

From (27) and (37) it follows that

$$|c^+_{N,l} - c^-_{N,l}| \geq \boldsymbol{\Delta}_c. \tag{40}$$

An elementary calculation shows that

$$\left|\frac{d}{dx}(T^N_{\alpha_0,\beta_0}(x))\right| = \left|(T^N_{\alpha_0,\beta_0})'(x)\right| = \left(\left(\frac{\beta_0}{\alpha_0}\right)^{\gamma_N(x)}\left(\frac{\beta_0}{1-\alpha_0}\right)^{1-\gamma_N(x)}\right)^N \tag{41}$$

for any $x \in (d_l, e_l)$.

During the rest of the proof the reader might find useful to look every so often at the left half of Figure 4. Also observe that the value of $\gamma_N(x)$ is constant on (d_l, e_l). Denote this constant by g_l. Using (35) and (36) we obtain

$$|g_l - \gamma| < \delta_1. \tag{42}$$

By topological conjugacy of $T_{\alpha,\Psi_{\underline{M}}(\alpha)}$ and T_{α_0,β_0} if we change $\alpha \in [\overline{\alpha}_1, \overline{\alpha}_2]$ then the system of maximal intervals of monotonicity of $T_{\alpha,\Psi_{\underline{M}}(\alpha)}$ is not changing in number and only endpoints of these intervals vary in a Lipschitz continuous way. This means that we can consider the intervals $[d_l(\alpha), e_l(\alpha)]$, $\alpha \in [\overline{\alpha}_1, \overline{\alpha}_2]$ and the absolute value of the slope of $T^N_{\alpha,\Psi_{\underline{M}}(\alpha)}$ on these intervals will be for any $x \in (d_l(\alpha), e_l(\alpha))$

$$\begin{aligned}\left|\frac{d}{dx}(T^N_{\alpha,\Psi_{\underline{M}}(\alpha)}(x))\right| &= \left(\left(\frac{\Psi_{\underline{M}}(\alpha)}{\alpha}\right)^{g_l}\cdot\left(\frac{\Psi_{\underline{M}}(\alpha)}{1-\alpha}\right)^{1-g_l}\right)^N \\ &= (\Psi_{\underline{M}}(\alpha))^N\cdot\left(\left(\frac{1}{\alpha}\right)^{g_l}\cdot\left(\frac{1}{1-\alpha}\right)^{1-g_l}\right)^N = (s(\alpha, g_l))^N.\end{aligned} \tag{43}$$

By (34) and (35) we have

$$s(\alpha_0 + \Delta\alpha, g_l) < s(\alpha_0, g_l) + \frac{1}{2}\Delta\alpha\partial_1 s(\alpha_0, g_l), \tag{44}$$

that is

$$\begin{aligned}\frac{s(\alpha_0, g_l)}{s(\alpha_0+\Delta\alpha, g_l)} &> \frac{1}{1+\frac{1}{2}\Delta\alpha\frac{\partial_1 s(\alpha_0,g_l)}{s(\alpha_0,g_l)}} \\ &= \frac{1}{1-\frac{1}{2}\left|\frac{\partial_1 s(\alpha_0,g_l)}{s(\alpha_0,g_l)}\right||\Delta\alpha|} > 1+\frac{1}{2}\left|\frac{\partial_1 s(\alpha_0,g_l)}{s(\alpha_0,g_l)}\right|\cdot|\Delta\alpha|.\end{aligned} \tag{45}$$

Using $\frac{1}{\alpha}+\frac{1}{1-\alpha} \geq 2$, (30), (34), (45) and Bernoulli's inequality

$$\begin{aligned}(s(\alpha_0+\Delta\alpha, g_l))^N &< \frac{(s(\alpha_0,g_l))^N}{\left(1+\frac{1}{2}\left|\frac{\partial_1 s(\alpha_0,g_l)}{s(\alpha_0,g_l)}\right||\Delta\alpha|\right)^N} \\ &< \frac{(s(\alpha_0,g_l))^N}{1+N\cdot\frac{1}{4}|\widehat{\gamma}(\alpha_0)-\gamma|(\frac{1}{\alpha}+\frac{1}{1-\alpha})|\Delta\alpha|} < \frac{(s(\alpha_0,g_l))^N}{1+\frac{N}{2}|\widehat{\gamma}(\alpha_0)-\gamma||\Delta\alpha|}.\end{aligned} \tag{46}$$

Since the choice of $\Delta\alpha$ did not depend on N we can suppose that N is so large that

$$1+\frac{N}{2}|\widehat{\gamma}(\alpha_0)-\gamma|\cdot|\Delta\alpha|>\frac{20}{\lambda(\widehat{S}_\gamma)}. \tag{47}$$

By (41) and (43) we know that

$$\lambda(I_l)=e_l-d_l=\frac{|c^+_{N,l}-c^-_{N,l}|}{(s(\alpha_0,g_l))^N}=\frac{|c^+_{N,l}(\alpha_0)-c^-_{N,l}(\alpha_0)|}{(s(\alpha_0,g_l))^N}. \tag{48}$$

We want to obtain an estimate of $e_l(\alpha_0+\Delta\alpha)-d_l(\alpha_0+\Delta\alpha)$. By (26)

$$|c^\pm_{N,l}(\alpha_0+\Delta\alpha)-c^\pm_{N,l}(\alpha_0)|\le M_c\cdot|\Delta\alpha|,$$

and hence using (27) and (32)

$$\begin{aligned}&|c^+_{N,l}(\alpha_0+\Delta\alpha)-c^-_{N,l}(\alpha_0+\Delta\alpha)|\\&>|c^+_{N,l}(\alpha_0)-c^-_{N,l}(\alpha_0)|-2M_c|\Delta\alpha|>\frac{1}{2}|c^+_{N,l}(\alpha_0)-c^-_{N,l}(\alpha_0)|.\end{aligned} \tag{49}$$

By (43), (46), (47), (48) and (49) we obtain

$$\begin{aligned}e_l(\alpha_0+\Delta\alpha)-d_l(\alpha_0+\Delta\alpha)&=\frac{|c^+_{N,l}(\alpha_0+\Delta\alpha)-c^-_{N,l}(\alpha_0+\Delta\alpha)|}{|s(\alpha_0+\Delta\alpha,g_l)|^N}\\&>\frac{\frac{1}{2}|c^+_{N,l}(\alpha_0)-c^-_{N,l}(\alpha_0)|}{|s(\alpha_0,g_l)|^N}\cdot\Big(1+\frac{N}{2}|\widehat{\gamma}(\alpha_0)-\gamma|\cdot|\Delta\alpha|\Big)\\&>\frac{|c^+_{N,l}(\alpha_0)-c^-_{N,l}(\alpha_0)|}{|s(\alpha_0,g_l)|^N}\cdot\frac{10}{\lambda(\widehat{S}_\gamma)}=\lambda(I_l)\cdot\frac{10}{\lambda(\widehat{S}_\gamma)}.\end{aligned} \tag{50}$$

By topological conjugacy of $T_{\alpha_0+\Delta\alpha,\Psi_{\underline{M}}(\alpha_0+\Delta\alpha)}$ and T_{α_0,β_0} the intervals $I_l(\alpha_0+\Delta\alpha)=[d_l(\alpha_0+\Delta\alpha),e_l(\alpha_0+\Delta\alpha)]$ are non-overlapping for fixed $\Delta\alpha$ and are in $[0,1]$. This contradicts (38) since we have

$$1\ge\sum_l\lambda(I_l(\alpha_0+\Delta\alpha))>\sum_l\lambda(I_l)\cdot\frac{10}{\lambda(\widehat{S}_\gamma)}\ge 10.$$

Hence γ satisfies (22) and Proposition 10 is proved. □

5. Differentiability of the isentropes (ergodic theory approach)

In this section we prove that isentropes are continuously differentiable curves. We have already seen that results of [11] imply that they are (locally uniformly) Lipschitz. There are two possible ways to verify that they are differentiable. One way, the one which we call analytic method, is to use the auxiliary function $\Theta_{\underline{M}}$, (4) and implicit differentiation. If one can verify that for $(\alpha,\beta)\in U$, $\underline{M}=K(\alpha,\beta)$ we have $\partial_2\Theta_{\underline{M}}(\alpha,\beta)\neq 0$ then this argument works. Unfortunately, to deal with partial derivatives of $\Theta_{\underline{M}}$ is a quite unpleasant and technical task. We have a manuscript in prepartion, [5] which discusses this other approach. In this paper we use a much more elegant and less technical argument which we called the ergodic theory approach and is based on Proposition 10 which says that the slope of the tangent of isentropes wherever it exists can be expressed by γ, which depends on the unique acim of the skew tent map considered. Then by using approximations, Proposition 6 and uniqueness of the acim first we verify in Lemma 11 continuous differentiability of the isentrope in the Markov case. Then by another approximation argument we prove the general case in Theorem 12.

Lemma 11. *If* $\underline{M} \in \mathfrak{M}_{<\infty}$ *then* $\Psi'_{\underline{M}}$ *exists and is continuous on* $(\alpha_1(\underline{M}), \alpha_2(\underline{M}))$.

Proof. Choose $\alpha_0 \in (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. We know that $\Psi'_{\underline{M}}(\alpha)$ exists for almost every $\alpha \in (\alpha_1(\underline{M})), \alpha_2(\underline{M}))$. Denote by $D_{\underline{M}}$ the set of those αs where $\Psi'_{\underline{M}}$ exists. Suppose that there exists $d_1 \neq d_2 \in [-\infty, \infty]$ and $\alpha_{i,n} \to \alpha_0$, $(i = 1, 2)$ such that $\alpha_{i,n} \in D_{\underline{M}}$, and $\Psi'_{\underline{M}}(\alpha_{i,n}) \to d_i$, $(i = 1, 2)$. Put $\beta_{i,n} = \Psi_{\underline{M}}(\alpha_{i,n})$, $i = 1, 2$. Then $(\alpha_{i,n}, \beta_{i,n}) \to (\alpha_0, \beta_0) = (\alpha_0, \Psi_{\underline{M}}(\alpha_0))$, for $i = 1, 2$ as $n \to \infty$. Since $\Psi_{\underline{M}}$ is an isentrope we know that the maps $T_{\alpha_{i,n},\beta_{i,n}}$ are all topologically conjugate to T_{α_0,β_0}. It is not difficult to check that the assumptions of Proposition 6 are satisfied. Hence if we denote by $f_{i,n}$ the invariant densities of $T_{\alpha_{i,n},\beta_{i,n}}$ which appear in Proposition 6 then there are subsequences $n_{k,i}$ such that $f_{i,n_{k,i}} \to f_{i,0}$ in L^1, and $f_{i,0}$, $(i = 1, 2)$ are both invariant densities of T_{α_0,β_0}. By Proposition 5, T_{α_0,β_0} has a unique invariant density and hence $f_{1,0} = f_{2,0} = f_0$ almost everywhere. Denote by $\mu_{i,n}$ and μ_0 the acims with densities $f_{i,n}$ and f_0, respectively. For $i = 1, 2$ we have

$$\gamma_{i,n_{k,i}} = \alpha_{i,n_{k,i}}(1 - \alpha_{i,n_{k,i}}) \frac{\Psi'_{\underline{M}}(\alpha_{i,n_{k,i}})}{\Psi_{\underline{M}}(\alpha_{i,n_{k,i}})} + \alpha_{i,n_{k,i}} \to \gamma_i = \alpha_0(1 - \alpha_0)\frac{d_i}{\beta_0} + \alpha_0. \tag{51}$$

From $d_1 \neq d_2$ it follows that $\gamma_1 \neq \gamma_2$. By Proposition 10

$$\gamma_{i,n_{k,i}} = \mu_{n_{k,i}}([0, \alpha_{i,n_{k,i}}]) = \int_{[0,\alpha_{i,n}]} f_{i,n_{k,i}} d\lambda, \ i = 1, 2. \tag{52}$$

Set $\gamma_0 = \mu_0([0, \alpha_0]) = \int_{[0,\alpha_0]} f_0 d\lambda$. We denote by $I_{k,i}$ the interval with endpoints α_0 and $\alpha_{i,n_{k,i}}$. We know that

$$\int_{[0,1]} |f_{i,n_{k,i}} - f_0| d\lambda \to 0 \text{ as } k \to +\infty, \text{ for } i = 1, 2. \tag{53}$$

Hence

$$\begin{aligned} |\gamma_0 - \gamma_{i,n}| &= \left| \int_{[0,\alpha_0]} f_0 d\lambda - \int_{[0,\alpha_{i,n_{k,i}}]} f_{i,n_{k,i}} d\lambda \right| \\ &\leq \left| \int_{I_{k,i}} f_0 d\lambda \right| + \int_{[0,\alpha_{i,n_{k,i}}]} |f_0 - f_{i,n_{k,i}}| d\lambda \\ &\leq \left| \int_{I_{k,i}} f_0 d\lambda \right| + \|f - f_{i,n_{k,i}}\|_1 \to 0 \text{ as } k \to \infty. \end{aligned} \tag{54}$$

Since $\gamma_{i,n_{k,i}} \to \gamma_i$, $i = 1,2$ and $\gamma_1 \neq \gamma_2$, it is impossible that $\gamma_{i,n_{k,i}} \to \gamma_0$, $i = 1,2$. Hence $\Psi'_{\underline{M}}|_{D_{\underline{M}}}$ has a limit at every $\alpha_0 \in (\alpha_1(\underline{M}), \alpha_2\underline{M})$. Since $\Psi_{\underline{M}}$ is locally Lipschitz and $D_{\underline{M}}$ is of full measure in $(\alpha_1(\underline{M}), \alpha_2(\underline{M}))$ we obtained that $\Psi'_{\underline{M}}(\alpha_0)$ exists and continuous at any $\alpha_0 \in (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. $\square$

Next we consider the general case.

Theorem 12. *If $\underline{M} \in \mathfrak{M}$ then $\Psi'_{\underline{M}}$ exists and is continuous on $(\alpha_1(\underline{M}), \alpha_2(\underline{M}))$.*

Proof. The Markov case $\underline{M} \in \mathfrak{M}_{<\infty}$ is Lemma 11. In [2] there are some considerations showing that the curves $\{(\alpha, \Psi_{\underline{M}}(\alpha)) : \underline{M} \in \mathfrak{M}^0_{<\infty}\}$ are dense in U^0. By renormalization, or by using directly the argument from [2] one can see that the curves $\{(\alpha, \Psi_{\underline{M}}(\alpha)) : \underline{M} \in \mathfrak{M}_{<\infty}\}$ are dense in U. Suppose that $\underline{M} \in \mathfrak{M}_\infty$ is fixed $\beta_0 = \Psi_{\underline{M}}(\alpha_0)$, $(\alpha_0, \beta_0) \in U$, $K(\alpha_0, \beta_0) = \underline{M}$. Then there are no Cs in $\underline{M}$ and $T^{k+1}_{\alpha_0,\beta_0}(\alpha_0) = T^k_{\alpha_0,\beta_0}(\beta_0) \neq \alpha_0$ for any $k \geq 0$. This also implies that

$$T^k_{\alpha_0,\beta_0}(\alpha_0) \neq T^{k'}_{\alpha_0,\beta_0}(\alpha_0) \text{ if } k' > k \geq 0. \tag{55}$$

Choose $[\overline{\alpha}_1, \overline{\alpha}_2] \subset (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. By Proposition 8 , $\Psi_{\underline{M}}$ is Lipschitz on $[\overline{\alpha}_1, \overline{\alpha}_2]$ and $\Psi'_{\underline{M}}$ exists and is bounded almost everywhere on $[\overline{\alpha}_1, \overline{\alpha}_2]$. Suppose that $\Psi_{\underline{M}}$ is not differentiable at $\alpha_0 \in (\overline{\alpha}_1, \overline{\alpha}_2)$. This means that there is $d_1 \neq d_2$ such that we can select $\alpha_{i,n} \to \alpha_0$, $i = 1,2$, such that

$$\frac{\Psi_{\underline{M}}(\alpha_{i,n}) - \Psi_{\underline{M}}(\alpha_0)}{\alpha_{i,n} - \alpha_0} \to d_i,\ i = 1,2. \tag{56}$$

Since the Markov isentropes are dense in U we can choose $\underline{M}_n \in \mathfrak{M}_{<\infty}$ such that

$$\begin{aligned} &\frac{\Psi_{\underline{M}_n}(\alpha_{i,n}) - \Psi_{\underline{M}_n}(\alpha_0)}{\alpha_{i,n} - \alpha_0} \to d_i, \Psi_{\underline{M}_n}(\alpha_{i,n}) \to \beta_0, \\ &\text{and } \Psi_{\underline{M}_n}(\alpha_0) \to \beta_0, \text{ as } n \to \infty, i = 1,2. \end{aligned} \tag{57}$$

By Lemma 11 and by the Mean Value Theorem we can choose $\overline{\alpha}_{i,n} \to \alpha_0$ such that

$$\Psi_{\underline{M}_n}(\overline{\alpha}_{i,n}) = \overline{\beta}_{i,n} \to \beta_0 \text{ and } \Psi'_{\underline{M}_n}(\overline{\alpha}_{i,n}) \to d_i \text{ for } i = 1,2. \tag{58}$$

We denote by $\mu_{i,n}$ the acim of $T_{\overline{\alpha}_{i,n},\overline{\beta}_{i,n}}$, $i = 1,2$ and $f_{i,n}$ denotes the corresponding invariant density. By Lemma 7 assumption (10) is satisfied for $(\overline{\alpha}_{i,n}, \overline{\beta}_{i,n}) \to (\alpha_0, \beta_0)$ for $i = 1,2$. Then we can apply Proposition 6 in this case as well and we conclude that for suitable subsequences $f_{i,n_{k,i}} \to f_0$ as $k \to +\infty$ where f_0 is the unique invariant density of T_{α_0,β_0}. Now by using $\overline{\alpha}_{i,n_{k,i}}$ instead of $\alpha_{i,n_{k,i}}$ one can argue as we did in the end of the proof of Lemma 11 to obtain (51), (52), (53) and (54). This way we can obtain a contradiction as in Lemma (11). $\square$

6. ISENTROPES AND LYAPUNOV EXPONENTS, THE GENERAL CASE

Next we state the main result of our paper. Its special Markov case, assuming differentiability of the isentrope at the point considered was discussed in Section 4.

Theorem 13. *Suppose* $(\alpha_0, \beta_0) \in U$, $\Lambda = \Lambda_{\alpha_0,\beta_0}$ *denotes the Lyapunov exponent of* T_{α_0,β_0} *and* $(\alpha, \Psi_{\underline{M}}(\alpha))$ *is the isentrope satisfying* $\beta_0 = \Psi_{\underline{M}}(\alpha_0)$. *Then* $\Psi'_{\underline{M}}(\alpha_0)$ *exists, moreover* (21) *and* (22) *are satisfied.*

Proof. The case $K(\alpha_0, \beta_0) = \underline{M} \in \mathfrak{M}_{<\infty}$ was proved in Proposition 10. By Theorem 12 we know that $\Psi'_{\underline{M}}(\alpha)$ exists for any $\underline{M} \in \mathfrak{M}$ and $\alpha \in (\alpha_1(\underline{M}), \alpha_2(\underline{M}))$. Next we suppose that $K(\alpha_0, \beta_0) \in \mathfrak{M}_\infty$, that is there is no C in $K(\alpha_0, \beta_0)$. We use again the fact that isentropes corresponding to Markov systems are dense in U. We will select a suitable $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$ such that $K(\alpha_n, \beta_n) = \underline{M}_n \in \mathfrak{M}_{<\infty}$. Again we choose $\overline{\alpha}_1 < \overline{\alpha}_2$ such that $\alpha_0 \in (\overline{\alpha}_1, \overline{\alpha}_2) \subset [\overline{\alpha}_1, \overline{\alpha}_2] \subset (\overline{\alpha}_1(\underline{M}), \overline{\alpha}_2(\underline{M}))$. Suppose $n \in \mathbb{N}$ is given. Choose $\overline{\alpha}_n < \alpha_0$ such that

$$|\overline{\alpha}_n - \alpha_0| < \frac{1}{n} \text{ and } \left| \frac{\Psi_{\underline{M}}(\overline{\alpha}_n) - \Psi_{\underline{M}}(\alpha_0)}{\overline{\alpha}_n - \alpha_0} - \Psi'_{\underline{M}}(\alpha_0) \right| < \frac{1}{2n}. \tag{59}$$

Select $\overline{\beta}_n$ such that

$$0 < \beta_0 - \overline{\beta}_n = \Psi_{\underline{M}}(\alpha_0) - \overline{\beta}_n < \frac{1}{4n} |\overline{\alpha}_n - \alpha_0|. \tag{60}$$

The right half of Figure 4 might turn out to be useful to help to understand the rest of the proof.

Since isentropes do not cross $\Psi_{K(\alpha_0,\overline{\beta}_n)} < \Psi_{K(\alpha_0,\beta_0)} = \Psi_{\underline{M}}$ at points where they are both defined. By choosing $\overline{\beta}_n$ sufficiently close to β_0 we can ensure that they are both defined on $[\overline{\alpha}_n, \alpha_0]$.

Select $\widehat{\beta}_n$ such that

$$\begin{aligned} 0 < \Psi_{\underline{M}}(\overline{\alpha}_n) - \widehat{\beta}_n < \frac{1}{4n} |\overline{\alpha}_n - \alpha_0|, \quad \Psi_{K(\alpha_0,\overline{\beta}_n)}(\overline{\alpha}_n) < \widehat{\beta}_n \\ \text{and } K(\overline{\alpha}_n, \widehat{\beta}_n) = \widehat{\underline{M}}_n \in \mathfrak{M}_{<\infty}. \end{aligned} \tag{61}$$

Since isentropes do not cross we have

$$\overline{\beta}_n < \Psi_{\widehat{\underline{M}}_n}(\alpha_0) < \Psi_{\underline{M}_n}(\alpha_0) = \beta_0. \tag{62}$$

Recalling that $\Psi_{\widehat{\underline{M}}_n}(\overline{\alpha}_n) = \widehat{\beta}_n$ by (59), (60), (61) and (62) we obtain that

$$\left| \frac{\Psi_{\widehat{\underline{M}}_n}(\overline{\alpha}_n) - \Psi_{\widehat{\underline{M}}_n}(\alpha_0)}{\overline{\alpha}_n - \alpha_0} - \Psi'_{\underline{M}}(\alpha_0) \right| < \frac{1}{n}. \tag{63}$$

Since $\Psi_{\widehat{\underline{M}}_n}$ is differentiable on $[\overline{\alpha}_n, \alpha_0]$ by the Mean Value Theorem we can choose $\alpha_n \in (\overline{\alpha}_n, \alpha_0)$ such that

$$\Psi'_{\widehat{\underline{M}}_n}(\alpha_n) = \frac{\Psi_{\widehat{\underline{M}}_n}(\overline{\alpha}_n) - \Psi_{\widehat{\underline{M}}_n}(\alpha_0)}{\overline{\alpha}_n - \alpha_0}.$$

From (63) it follows that

$$|\Psi'_{\widehat{\underline{M}}_n}(\alpha_n) - \Psi'_{\underline{M}}(\alpha_0)| < \frac{1}{n}. \tag{64}$$

Set $\beta_n = \Psi_{\widehat{\underline{M}}_n}(\overline{\alpha}_n)$. By the local uniform Lischitz property of the isentropes mentioned in Remark 9 it is clear that $(\alpha_n, \beta_n) \to (\alpha_0, \beta_0)$. Since $\widehat{\underline{M}}_n \in \mathfrak{M}_{<\infty}$ we can apply Proposition 10 at the point (α_n, β_n) to the isentrope $\Psi_{\widehat{\underline{M}}_n}$.

By Lemma 7 assumption (10) is satisfied. Hence if μ_n and f_n denote the acim and its density for T_{α_n,β_n}, $n = 0, 1, \dots$ then by Proposition 6 for a suitable subsequence n_k the sequence $f_{n_k} \to f_0$ in L^1. Now

$$\gamma_{n_k} = \mu_{n_k}([0, \alpha_{n_k}]) = \int_0^{\alpha_{n_k}} f_{n_k} d\lambda$$

$$= \int_0^{\alpha_0} f_0 \, d\lambda + \int_0^{\alpha_{n_k}} f_{n_k} \, d\lambda - \int_0^{\alpha_0} f_0 \, d\lambda = \mu_0([0, \alpha_0]) + A_k = \gamma_0 + A_k.$$

We have

$$|A_k| = \left| \int_0^{\alpha_{n_k}} f_{n_k} d\lambda - \int_0^{\alpha_0} f_0 \, d\lambda \right| \le \int_0^{\alpha_{n_k}} |f_{n_k} - f_0| d\lambda + \int_{\alpha_{n_k}}^{\alpha_0} f_0 \, d\lambda$$

$$\le \|f_{n_k} - f_0\|_1 + \int_{\alpha_{n_k}}^{\alpha_0} f_0 \, d\lambda \to 0.$$

Hence $\gamma_{n_k} \to \gamma_0$.

By Proposition 10 we have

$$\Lambda_{\alpha_n,\beta_n} = \Lambda_n = \gamma_n \log \frac{\beta_n}{\alpha_n} + (1 - \gamma_n) \log \frac{\beta_n}{1 - \alpha_n} \text{ where } \gamma_n \text{ satisfies} \tag{65}$$

$$\gamma_n = \mu_n([0, \alpha_n]) = \alpha_n(1 - \alpha_n) \frac{\Psi'_{\widehat{\underline{M}}_n}(\alpha_n)}{\beta_n} + \alpha_n. \tag{66}$$

Using (64) and $\gamma_{n_k} \to \gamma_0$ by taking limit as $k \to \infty$ we obtain that (21) and (22) hold for T_{α_0,β_0}. $\square$

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* DEPARTMENT OF ANALYSIS
ELTE EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C, 1117 BUDAPEST, HUNGARY

E-mail address: `buczo@cs.elte.hu`

** DEPARTMENT OF ANALYSIS
ELTE EÖTVÖS LORÁND UNIVERSITY
PÁZMÁNY PÉTER SÉTÁNY 1/C, 1117 BUDAPEST, HUNGARY

E-mail address: `keszthelyig@gmail.com`