FLOQUET STABILITY ANALYSIS OF THE WAKE OF A CIRCULAR CYLINDER IN LOW REYNOLDS NUMBER FLOW

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ABSTRACT

This study describes the Floquet stability analysis of the wake flow behind a mechanically oscillated circular cylinder placed in a low Reynolds-number flow. After defining the base flow, three-dimensional perturbation is introduced into the flow. The governing equations for the evolution of disturbance are derived. The integration of the linearised perturbation equations is carried out in the same way as for the base flow. The Floquet multiplier, which indicates whether the flow is stable or not, is determined using Krylov subspace iteration. The code developed for the Floquet analysis is still being tested.

INTRODUCTION

The incompressible flow past a stationary or oscillating cylinder is a classical bluff body problem in fluid mechanics. If flow oscillation reaches high enough levels, it can result in damage or collapse of structures, and thus it is important to investigate the stability of the flow. When perturbation is added to the flow, the perturbation may grow (indicating instability of the flow) or decay (showing stability). For periodic flow, linear Floquet stability analysis is the most popular method.

There are several studies concerning the linear stability analysis of the wakes behind bluff bodies. Barkley and Henderson [1], using Floquet stability analysis, determined critical Reynolds numbers belonging to the onset of three-dimensional instabilities (mode A and mode B) for the wake behind a stationary circular cylinder. A method for direct linear stability analysis of the Navier-Stokes equations in general geometry was developed in [2]. Goria et al. [3] carried out a Floquet stability analysis for a circular cylinder forced to oscillate transverse to the main stream for different Reynolds numbers and oscillation amplitude values.

The objective of this paper is to describe the mathematical background of the Floquet analysis to be applied to the wake flow behind a circular cylinder oscillated in-line with the main stream, or following a two-degree-of-freedom path. This paper is considered to be the first step of a comprehensive future study.

BASE FLOW

A non-inertial system fixed to the accelerating cylinder is used to compute two-dimensional (2D) low Reynolds number flow around a circular cylinder placed perpendicular to an otherwise uniform flow. The non-dimensional Navier-Stokes equations for incompressible constant-property Newtonian fluid, the equation of continuity and the Poisson equation for pressure can be written as [4]

$$\frac{\partial \mathbf{u}}{\partial t}+\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}=-\nabla p+\frac{1}{\mathrm{Re}}\nabla^2\mathbf{u}-\mathbf{a}_0\,, \tag{1}$$

$$\nabla\cdot\mathbf{u}=0\,, \tag{2}$$

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=2\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right]-\frac{\partial D}{\partial t}\,, \tag{3}$$

where $\mathbf{u}(\mathbf{x},t)=(u,v)(x,y,t)$ is the two-dimensional (2D) velocity field and $p(\mathbf{x},t)=p(x,y,t)$ is the pressure field, $\mathbf{a}_0$ is the cylinder acceleration and Re is the Reynolds number based on the cylinder diameter d, free stream velocity U and kinematic viscosity υ. On the cylinder surface no-slip boundary condition is used for the velocity and a Neumann type boundary condition is used for the pressure. At the far region potential flow is assumed. Boundary-fitted coordinates are used to impose boundary conditions accurately. The physical domain, which consists of two concentric circles, is mapped into a rectangular computational domain. The governing equations and boundary conditions are also transformed to the computational domain (not shown here) and solved by finite difference method. Space derivatives are approximated by fourth-order central differences, except for the convective terms, for which a third-order modified upwind scheme is used. The Poisson equation for pressure is solved by the successive over-relaxation (SOR) method. The Navier-Stokes equations are integrated using a second order Runge-Kutta method. The in-house code developed by the first author has been extensively tested against experimental and other computational results, [4]. Using the method described in detail in [4], after some transition periodic velocity and pressure distributions can be obtained, if lock-in is reached. This periodic flow will be referred to as base flow. The objective of this paper is to analyse the linear stability of the base flow ($\mathbf{u}, p$) when subjected to three-dimensional (3D) perturbations.

LINEARISED GOVERNING EQUATIONS

We would like to follow the evolution of infinitesimal 3D perturbations ($\mathbf{u'}, p'$) (i.e., $\mathbf{u'}$=$(u',v',w')(x,y,z,t)$ and p'=$p'(x,y,z,t)$) to the base flow ($\mathbf{u}, p$). The governing equations for these perturbations can be obtained by substituting

$$\mathbf{U}=\mathbf{u}+\varepsilon\mathbf{u'}\,,\ P=p+\varepsilon p'\,,\ \varepsilon<<1 \tag{4}$$

into equations (1) - (3) and keeping only the linear terms in ε, [1, 2]. The resulting governing equations are

$$\frac{\partial \mathbf{u'}}{\partial t}=-\nabla p'+\frac{1}{\mathrm{Re}}\nabla^2\mathbf{u'}-\left[\left(\mathbf{u'}\cdot\nabla\right)\mathbf{u}+\left(\mathbf{u}\cdot\nabla\right)\mathbf{u'}\right], \tag{5}$$

$$\nabla\cdot\mathbf{u'}=D'=0\,, \tag{6}$$

$$\nabla^2 p' = \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2} = \frac{1}{\mathrm{Re}} \nabla^2 D' - \frac{\partial D'}{\partial t} - \mathrm{div}\left[(\mathbf{u'} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u'} \right], \tag{7}$$

where D' is the divergence of perturbation velocity $\mathbf{u'}$. It can be seen that cylinder acceleration does not occur in these equations hence these equations are identical for stationary and accelerating cylinder flows. The linearised Navier-Stokes equations (5) and continuity equation (6) in the physical plane can be written in detail as

$$\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{1}{\mathrm{Re}} \nabla^2 u' - \left[u\frac{\partial u'}{\partial x} + v\frac{\partial u'}{\partial y} + u'\frac{\partial u}{\partial x} + v'\frac{\partial u}{\partial y} \right], \tag{8}$$

$$\frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial y} + \frac{1}{\mathrm{Re}} \nabla^2 v' - \left[u\frac{\partial v'}{\partial x} + v\frac{\partial v'}{\partial y} + u'\frac{\partial v}{\partial x} + v'\frac{\partial v}{\partial y} \right], \tag{9}$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} + \frac{1}{\mathrm{Re}} \nabla^2 w' - \left[u\frac{\partial w'}{\partial x} + v\frac{\partial w'}{\partial y} \right], \tag{10}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = D' = 0. \tag{11}$$

The perturbed flow ($\mathbf{U}, P$) satisfies the same boundary conditions as the base flow ($\mathbf{u}, p$). Hence the perturbation flow ($\mathbf{u'}, p'$) on the cylinder surface satisfies homogeneous (zero) Dirichlet for the velocity. The boundary condition for pressure on the cylinder surface is very similar to that for the base pressure (Neumann type). In the far-field ($\mathbf{u'}, p'$) satisfies homogeneous Dirichlet boundary conditions.

FLOQUET STABILITY ANALYSIS

Floquet stability analysis is used for the time-periodic wake of a cylinder oscillating in-line or transverse to the main stream. With this, we seek the stability of the two-dimensional base flow $\mathbf{u}(\mathbf{x},t)$ of period *T*. Equations (5) and (6) define a linear evolution operator **L** [1, 2]:

$$\frac{\partial \mathbf{u'}}{\partial t} = \mathbf{L}(\mathbf{u'}). \tag{12}$$

As the base flow is *T*-periodic, the operator $\mathbf{L}(\mathbf{u'})$ is also *T*-periodic and therefore equation (12) is of Floquet type. The solution of the equation is assumed in the form of the sum of $\tilde{\mathbf{u}}(x,y,z,t)e^{\sigma t}$ components, where $\tilde{\mathbf{u}}(x,y,z,t)$ are also *T*-periodic functions, [3, 4]. These are the Floquet modes of the operator **L**. The complex numbers σ are the Floquet exponents, and $\mu \equiv e^{\sigma T}$ are called Floquet multipliers. Floquet multipliers inside the unit circle in the complex plane correspond to stable, exponentially decaying solutions, while Floquet multipliers outside the unit circle correspond to instable solutions. Floquet multipliers are computed by first creating the operator (based on equations (5), (6) or (12)) representing the linear evolution of the system over period *T*

$$\mathbf{u}'_{n+1} = \mathbf{A}(\mathbf{u}'_{n}), \tag{13}$$

where $\mathbf{u}'_{n} = \mathbf{u}'(\mathbf{x}, t_0 + nT)$ is the perturbation after n periods. In this way the operator $\mathbf{A}$ is equivalent to the linearised Poincaré map belonging to the base flow. The eigenvalues μ of $\mathbf{A}$ are the Floquet multipliers of $\mathbf{L}$.

The effect of $\mathbf{A}$ on the perturbation is obtained by integrating the linearised Navier-Stokes equations over the period T, using essentially the same method as for the base flow (see equation (1)). To obtain the eigenvalues of $\mathbf{A}$ subspace iterations are used. The subspace iteration is carried out on a Krylov subspace (Arnoldi iteration with Gram-Schmidt process) [5].

Because the system is homogenous in the spanwise direction z, general perturbations can be expressed as the Fourier-integral [6]

$$\mathbf{u}'(x,y,z,t) = \int_{-\infty}^{\infty} \hat{\mathbf{u}}(x,y,\beta,t) e^{i\beta z} d\beta . \tag{14}$$

Here β is the wavenumber defined by $\beta = 2\pi/\lambda$, where λ is the wavelength of perturbation. Since equations (5) and (6) are linear, different β values do not couple, thus the perturbations can be written in the following forms

$$\begin{aligned} \mathbf{u}'(x,y,z,t) &= \hat{\mathbf{u}}_1(x,y,t)\cos(\beta z) + \hat{\mathbf{u}}_2(x,y,t)\sin(\beta z) = \\ &= (\hat{u}_1, \hat{v}_1, \hat{w}_1)(x,y,t)\cos(\beta z) + (\hat{u}_2, \hat{v}_2, \hat{w}_2)(x,y,t)\sin(\beta z) \end{aligned} \tag{15}$$

$$p'(x,y,z,t) = \hat{p}_1(x,y,t)\cos(\beta z) + \hat{p}_2(x,y,t)\sin(\beta z). \tag{16}$$

When substituting these solutions into equations (8) - (11) we found (as did [7]) that identical equations are obtained for functions with subscripts 1 and 2, so the perturbations are simplified to

$$\mathbf{u}'(x,y,z,t) = \left[\hat{u}(x,y,t)\cos(\beta z), \hat{v}(x,y,t)\cos(\beta z), \hat{w}(x,y,t)\sin(\beta z)\right], \tag{17}$$

$$p'(x,y,z,t) = \hat{p}(x,y,t)\cos(\beta z). \tag{18}$$

Substituting these perturbations into equations (8) - (11) leads eventually to

$$\frac{\partial \hat{u}}{\partial t} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} - \beta^2 \hat{u}\right) - \left(u\frac{\partial \hat{u}}{\partial x} + v\frac{\partial \hat{u}}{\partial y} + \hat{u}\frac{\partial u}{\partial x} + \hat{v}\frac{\partial u}{\partial y}\right), \tag{19}$$

$$\frac{\partial \hat{v}}{\partial t} = -\frac{\partial \hat{p}}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{\partial^2 \hat{v}}{\partial y^2} - \beta^2 \hat{v}\right) - \left(u\frac{\partial \hat{v}}{\partial x} + v\frac{\partial \hat{v}}{\partial y} + \hat{u}\frac{\partial v}{\partial x} + \hat{v}\frac{\partial v}{\partial y}\right), \tag{20}$$

$$\frac{\partial \hat{\mathrm{w}}}{\partial t} = \beta \hat{p} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \hat{w}}{\partial x^2} + \frac{\partial^2 \hat{w}}{\partial y^2} - \beta^2 \hat{w}\right) - \left(u\frac{\partial \hat{w}}{\partial x} + v\frac{\partial \hat{w}}{\partial y}\right), \tag{21}$$

$$\frac{\partial \hat{u}}{\partial x}+\frac{\partial \hat{v}}{\partial y}+\beta \hat{w}=\hat{D}=0\,. \tag{22}$$

Substituting (17) and (18) into the linearised pressure Poisson equation (7), after some complicated algebra and neglecting the convective derivatives of $\hat{D}$, yields the equation for $\hat{p}$

$$\frac{\partial^2 \hat{p}}{\partial x^2}+\frac{\partial^2 \hat{p}}{\partial y^2}-\beta^2 \hat{p}=-\frac{\partial \hat{D}}{\partial t}-2\left[\frac{\partial u}{\partial x}\cdot\frac{\partial \hat{u}}{\partial x}+\frac{\partial v}{\partial x}\cdot\frac{\partial \hat{u}}{\partial y}+\frac{\partial \hat{v}}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial \hat{v}}{\partial y}\cdot\frac{\partial v}{\partial y}\right]. \tag{23}$$

The next step is the transformation of the governing equations (19-23) and boundary conditions to the computational plane in the same way as was done for the base flow. For the sake of brevity, this is not shown here; see [4].

Following transformation, equations (19-23) are integrated from a starting time t_0 until t_0+T, then from t_0+T until t_0+2T, etc. To determine the stability of the perturbed flow, the Floquet multipliers are needed, which are the eigenvalues of matrix **A** (see equation (13)). There is no need to create and store the huge matrix **A** (corresponding to operator **A**); a matrix-free method can be used. Only the effect of **A** is taken into account while integrating the perturbation equations over periods of T. Computational results for the evolution of the perturbation velocities $\left(\hat{u}, \hat{v}, \hat{w}\right)$ are stored as the column vectors, where one integration over period T gives one column of a matrix. After every period the new column is made orthonormal to the previous columns using the Gram-Schmidt process. The scalar products of these column vectors constitute the elements of a so-called Hessenberg matrix whose largest eigenvalues are identical to those of matrix **A**, i.e., the Floquest multipliers. The integration is carried out over 30-40 periods and the results are stored in the columns of a matrix, from which the Hessenberg matrix is created. This number of repetitions typically ensures the determination of the eigenvalues with high accuracy [1]. The eigenvalues of the Hessenberg matrix are obtained using Arnoldi iteration [5].

Computations should be repeated for several wavenumbers β while Reynolds number, oscillation amplitude A and frequency f are fixed. The absolute value of the Floquet multiplier μ plotted against β reveals the instability domains (where $|\mu|\geq 1$). Then computations should be repeated for a different set of (Re, A, f) while systematically changing β. In this way instability domains in different parameters can be determined. Naturally, this process is computationally very expensive in the three-parameter system needed for a cylinder oscillating with one degree of freedom (for two degrees of freedom 4-5 parameters are needed).

The code was implemented for both CPU and GPGPU hardware (NVIDIA® CUDA™) for two different temporal discretisations: Euler method and 2nd order Runge-Kutta method [8] within a new in-house code called FlowCFD developed at the Department of Fluid and Heat Engineering at the University of Miskolc. However, extensive testing is still in progress, and further validation is required.

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