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DYNAMICS OF REDUCED ORDER MODELS OF THE FORCED KARMAN CYLINDER WAKE

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ABSTRACT

Employing symmetry-group equivariant bifurcation theory discrete reduced order models for the dynamics of the forced Karman wake have been developed. It turns out that the bifurcation behaviour of the complex forced wake flow can be approximately described by these simple models.

Although the symmetry approach is highly convenient when deriving the reduced order models, the general form of the resulting model requires the extra step of determination of the model coefficients for a given physical flow. In addition, an important assumption of the approach above is that symmetry holds exactly over the complete flow domain (and to infinity). In reality this is not the case – particularly for flow assumed to have translational symmetry.

The present paper addresses these two concerns. In a first attempt to obtain a physical interpretation of the reduced order model coefficients, the POD modes are employed in a Galerkin reduction of the complex Ginzburg-Landau (CGL) equation. The latter equation is a well known approximate model for spatio-temporal flow dynamics. To account for imperfect flow field symmetry, POD modes are derived directly from experimental PIV measurements.

An analysis of the CGL-based reduced order model yields the key bifurcations of the symmetrically forced Karman wake. Most importantly the model confirms the saddle-node bifurcation, supporting the findings based on the symmetry-based approach.

Keywords: bifurcation theory, Ginzburg-Landau equation, PIV, POD, Poincare map, saddle-node bifurcation, symmetry equivariance, von Karman wake.

NOMENCLATURE

a_i	$[-]$	CGL equation mode amplitude
C_L	$[-]$	lift coefficient
D	$[m]$	reference cylinder diameter
K_n	$[-]$	Karman mode amplitude
S_n	$[-]$	forcing mode amplitude
u	$[ms^{-1}]$	in-flow perturbation velocity
U	$[ms^{-1}]$	average flow velocity
v	$[ms^{-1}]$	transverse perturbation flow velocity
β	$[-]$	diffusion parameter in CGL equation
ϕ_n	$[-]$	POD mode
Γ_K	$[-]$	Karman mode symmetry
Γ_S	$[-]$	forcing mode symmetry
ν_1, ν_2	$[-]$	convection coefficients
σ	$[-]$	linear stability parameter in CGL equation

1. INTRODUCTION

The Karman wake is an ideal fluid system for the study of the dynamics, stability and control of fluid flow phenomena. Following the onset of vortex shedding at the critical velocity (near *Re*=45) via a Hopf bifurcation the Karman wake behaves, dynamically, as a well-defined spatio-temporal oscillator which can be described in terms of a finite number of orthogonal modes.

Recent work aimed at developing controllers for the Karman wake has taken the low order model approach. Low order modelling involves reduction of the effective number of degree-of-freedom of the fluid system to a small finite number while still retaining the essential features of the dynamical behaviour.

Low order models may be derived by direct numerical reduction of the NS equations (e.g. via Galerkin projections) or by a phenomenological approach based on physical and geometrical arguments. An example of the latter is the symmetry-equivariance based approach which exploits the 'known' symmetries of the dominant modes to derive low order models which are equivariant under the action of the chosen base flow symmetries. This method has led Mureithi et al. [1] and Rodriguez [2] to develop simple discrete models for the forced Karman wake.

A pair of simple low order equations describing the nonlinear interaction of the Karman and reflection-symmetric modes was derived in the Poincare space [1,3]. An analysis of the model showed that a number of standard bifurcations of the forced Karman mode could be obtained as the amplitude of the reflection-symmetric mode was varied. Possible changes in the wake, induced by increased forcing, such as period-doubling or symmetry breaking in the case of stream-wise harmonic forcing were correctly predicted.

One of the challenges in exploiting the symmetry-equivariance based model is the determination of model parameters and their correct (physical) interpretation. This is because the model is only limited by symmetry hence may represent any physical phenomenon having the correct symmetry – this being the universality property of the symmetry based approach. It is the model parameters (or coefficients) which map the model to a specific physical phenomenon.

An alternative approach to using symmetry-equivariance is derivation of reduced-order models using the Galerkin projection of either the full Navier-Stokes (NS) equations or simpler equations capturing the basic flow features. The advantage of the Galerkin projection is the fact that, in principle, no external parameters need to be determined. As a first step we propose to use the complex Ginzburg Landau (CGL) equation as a spatio-temporal model for the Karman wake flow in the subcritical and super critical regimes. The CGL equation has been successfully used to model key spatio-temporal instabilities and limit cycle behaviour in fluid systems (see, for instance, the excellent review by Aranson and Kramer [4] and references therein). The CGL equation has the advantage of being simpler than the NS equations making it easier to interpret the predicted dynamical phenomena.

Galerkin projection of the CGL equations yields discrete model coefficients which can be directly related to the CGL parameters e.g. the convective velocity of spatial perturbations, and the diffusion parameter. Other parameters include the modal stability parameter and the limit cycle amplitude (nonlinear damping) parameter.

2. REDUCED ORDER MODELS

In this section a brief summary of the derivation of the low order models is presented. The key assumptions and derivations are outlined in order to clearly highlight the strengths and limitations of each approach. Reduced order models are based on the idea of the existence of fundamental coherent structures (modes) in the flow, [5]. The coherent structures may be described by their symmetries [6] which also govern their nonlinear dynamics. For the symmetry based approach, the detailed derivation may be found in Mureithi et al. [1]. An example of the CGL based model reduction may be found in Ilak et al. [7].

2.1. Symmetry based Model

Experimental measurements show that the Karman wake is dominated by two key modes. These modes correspond to the alternate shedding mode (***K***) and the symmetrical shedding mode (***S***). As detailed in Mureithi et al. [1,3], The spatial symmetries of these modes are, respectively, $\Gamma_K = Z_2(\kappa,\pi)$, and $\Gamma_S = D_2(\kappa,\pi)$. Starting with the symmetries Γ_K and Γ_S, Mureithi et al. [3] employed equivariant bifurcation theory, [8,9], to derive the general form of the amplitude equations governing the interactions between modes ***K*** and ***S***.

Representing the complex mode amplitude by *K* and *S,* respectively, the discrete form of the mode amplitude interaction equations to third order is (Mureithi et al. [1,3])

$$\begin{aligned} K_{n+1} &= \left(1+\alpha_0+\gamma_{11}\left|S_n\right|^2+\alpha_2\left|K_n\right|^2\right)K_n+\delta_{01}S_n^2\bar{K}_n \\ S_{n+1} &= \left(1+\beta_0+\beta_2\left|S_n\right|^2+\gamma_{21}\left|K_n\right|^2\right)S_n+\mu_{01}\bar{S}_nK_n^2 \end{aligned} \quad (1)$$

The map resulting when the mode ***S*** is considered "constant" (or the externally controlled parameter) has $Z_2(\kappa,\pi)$ symmetry. The map and its complex conjugate are:

$$\begin{aligned} K_{n+1} &= \left(\mu+\alpha_2\left|K_n\right|^2\right)K_n+\delta\bar{K}_n \\ \bar{K}_{n+1} &= \left(\bar{\mu}+\bar{\alpha}_2\left|K_n\right|^2\right)\bar{K}_n+\delta K_n \end{aligned} ; \quad (2)$$

$$\mu = 1+\alpha_0+\gamma_{11}\left|S\right|^2$$

$$\delta = \delta_{01}S^2$$

$\alpha_0, \gamma_{11}, \alpha_2, \beta_0, \delta_{01}$.... are empirical constants.

2.2. CGL based Model

The CGL equation is a convection-diffusion partial differential equation. To take into account amplitude limitation of unstable oscillations due to a Hopf bifurcation a nonlinear term of third order is incorporated which can be justified by a normal form analysis. The scaled CGL equation, which will be used in this study, has the form:

$$\partial_t u(x,y,t) = \begin{bmatrix} -\{\nu_1\partial_x + \nu_2\partial_y\} + \\ \beta\{\partial_{xx} + \partial_{yy}\} + (\sigma - c_1^2) \end{bmatrix} u(x,y,t) + g(u(x,y,t))$$

$$g(u(x,y,t)) = -\gamma |u(x,y,t)|^2 u(x,y,t) \quad (3)$$

The parameters $\nu_1 = U + 2ic_1$ and $\nu_2 = 2ic_1$ are the convection coefficients in the flow direction (i.e. convective velocity) and the transverse direction, respectively, U being the average inflow propagation velocity. The parameter β models the diffusion process while σ is the linear stability parameter for local perturbations. In equation (3) $u(x,y,t)$ is the (x-) direction velocity fluctuations about the mean velocity $\overline{u} = U$.

The CGL equation (3) is discretized via Galerkin projection using POD modes of the flow velocity field $u(x,y,t)$. The velocity field is decomposed as

$$u(x,y,t) = \sum_{n=1}^{N} \phi_n(x,y) a_n(t), \quad (4)$$

where $\phi_n(x,y)$ are the POD modes of the wake flow. Substituting (4) into (3) and performing a two mode (N=2) Galerkin projection we obtain the following pair of differential equations for the complex mode amplitudes

$$\dot{a}_1(t) = \left[\sigma_1 + \gamma_{12}|a_2(t)|^2\right] a_1(t) + \gamma_{11}|a_1(t)|^2 a_1(t) + ..$$

$$\dot{a}_2(t) = \left[\sigma_2 + \gamma_{21}|a_1(t)|^2\right] a_2(t) + \gamma_{22}|a_2(t)|^2 a_2(t) + .. \quad (5)$$

In equation (5) the coefficients $\sigma_i, \gamma_{ij}, ...$ must be determined from experimental or numerical data. These coefficients represent integrals of the POD modes and their spatial derivatives, e.g.,

$$\sigma_i = \int\left[-\{\nu_1\phi_{ix} + \nu_2\phi_{iy}\} + \beta\{\phi_{ixx} + \phi_{iyy}\} + \sigma\right]\phi_i dxdy$$

$$\gamma_{ij} = -\gamma \iint \phi_i^2 \phi_j dxdy. \quad (6)$$

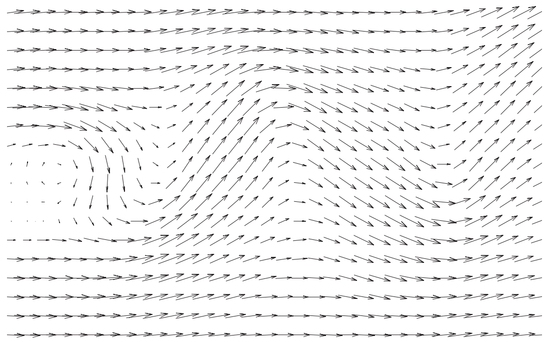

Figure 1. Velocity field in the wake of the cylinder for *Re*=200

3. EXPERIMENTAL MEASUREMENTS AND POD MODE COMPUTATION

To determine the POD wake modes, experimental tests have been conducted for a Reynolds number *Re*=200. The tests were conducted in a low-speed wind tunnel of test section dimensions 60 *cm* x 60 *cm*. The temperature controlled recirculating wind tunnel has a maximum empty test section speed of 90 *m/s* and turbulence intensity below 0.5%. The 76.2 *mm* diameter test cylinders spanned the test section eliminating unwanted 3D end effects. The flow field in the cylinder wake was measured using particle image velocimetry (PIV) and the resulting data analyzed using Dantec Dynamics's flow manager software. An in-house code was then used to determine the spatial (x,y) and spatial-temporal (x,t) POD modes of the measured wake flow. Further experimental details may be found in [10].

Figure 1 shows an image of the measured wake flow field for *Re*=200. The measurement area extends over a distance of approximately 5*D* (*D* being the cylinder diameter) downstream of the test cylinder. From this flow field the spatio-temporal POD modes are computed by using the transverse velocity data $v(x=D,y,t)$ at a location 1*D* downstream of the cylinder. Figure 2(a) shows a contour representation of the time varying transverse velocity profile $v(x=D,y,t)$; transverse position is given in cylinder diameters, *y/D.* At this location immediately downstream of the cylinder the flow is highly symmetrical as confirmed by the time invariance of the contours. The first two spatio-temporal modes are shown in Figs. 2(b-c). The first two modes are clearly highly structured and temporally 'stable'. The third mode (not shown) showed significant time variation (or

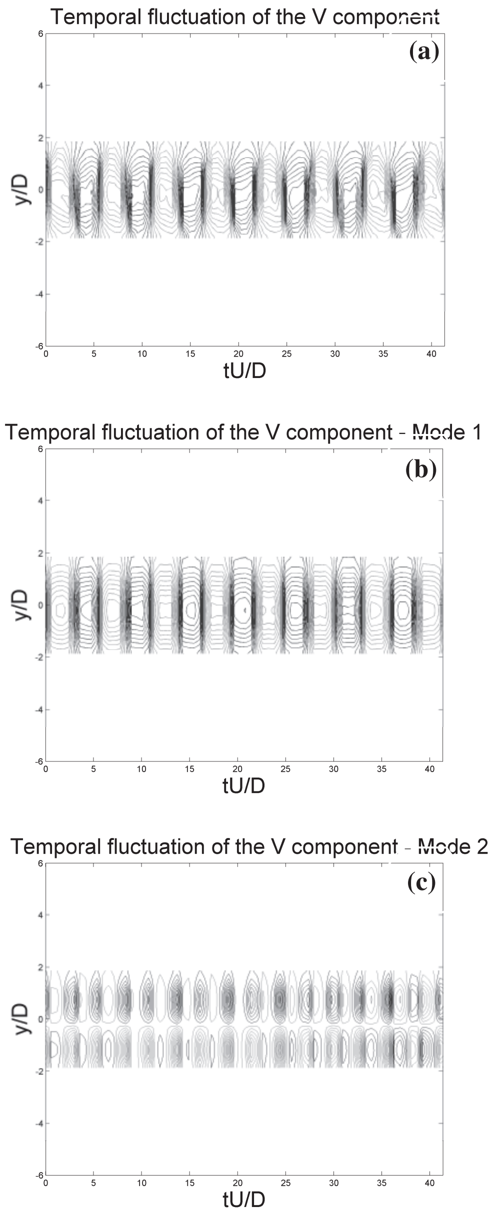

Figure 2. (a) Transverse velocity (*v*) contours, (b) POD mode 1, (c) POD mode 2 based on the velocity (*v*) at an *x*-location 1D downstream of the cylinder

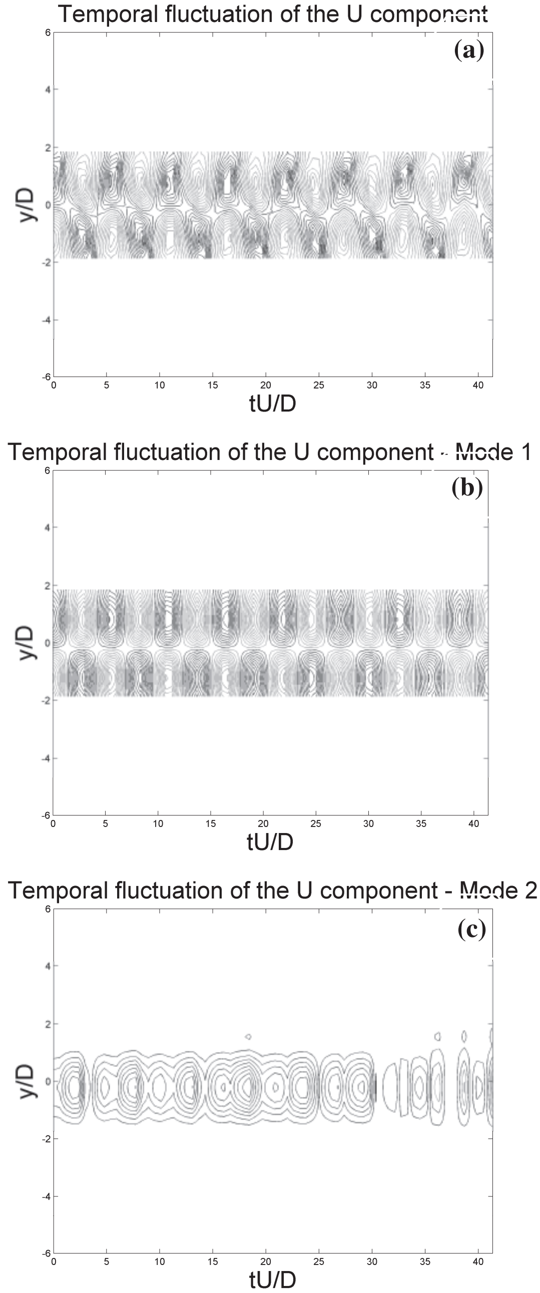

Figure 3. (a) Inflow velocity (*u*) contours, (b) POD mode 1, (c) POD mode 2

'defects') suggesting the appearance of significant 3D effects. POD modes are obtained in Matlab.

The inflow velocity profile $u(x,y,t)$ and corresponding spatio-temporal modes (1 and 2) are presented in Figs. 3(a-c). Once again the strongly symmetrical nature of the first two modes is clearly evident. For reduced order modelling it is important to determine the minimal number of modes that can be expected to capture the dynamics of the infinite-dimensional system. An estimate can be obtained by looking at the energy distribution between the modes. In the POD analysis the singular values provide a measure of the energy contained in a given mode.

Figure 4 shows the energy distribution between the POD modes for v-velocity modes (Fig. 4(a)) and the u-velocity modes in Fig. 4(b). In both cases it is found that the first mode contains close to 85% of the flow energy. The second mode, on the other hand carries roughly 9% while the third mode carries 3% of the energy. The first mode is therefore clearly predominant as expected for this low Reynolds number and can therefore be expected to provide reasonable single mode approximation of the wake flow. The first two modes contain over 90% of the flow 'energy' and are therefore

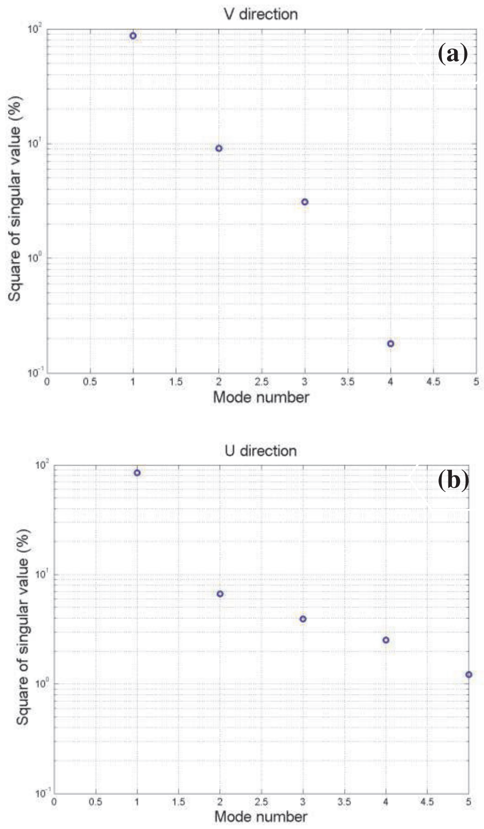

Figure 4. POD mode energy distributions for (a) transverse velocity (*v*) modes and (b) inflow velocity (*u*) modes

sufficient for a low order representation of the flow with a higher accuracy than the single mode case.

The spatio-temporal POD modes presented above contain the implicit assumption of perfect translation symmetry of the flow (along the flow direction -*x*). Indeed it is this assumption that makes it possible to derive the symmetry-equivariance based low order model in equations (1-2). Figure 5(a) presents an example of a contour representation of the instantaneous transverse velocity field (*v*) in the cylinder wake. The first two modes obtained from a POD analysis of this flow velocity field are presented in Figs. 5(b-c). Due to the spatial development of the flow as it propagates downstream, it is evident that perfect translation symmetry cannot be conserved. However, the flow does remain nearly or approximately translation symmetric. This is an encouraging result for the symmetry-equivariance based model. The first spatial and spatio-temporal modes have very similar features as confirmed by comparing Fig. 2(b) and Fig. 5(b). The comparison is less clear for modes 2 and 3. For the present measurements the mode order is reversed relative to the spatio-temporal case in Fig. 3. The singular values show that the second and third spatial modes have nearly equal energy levels

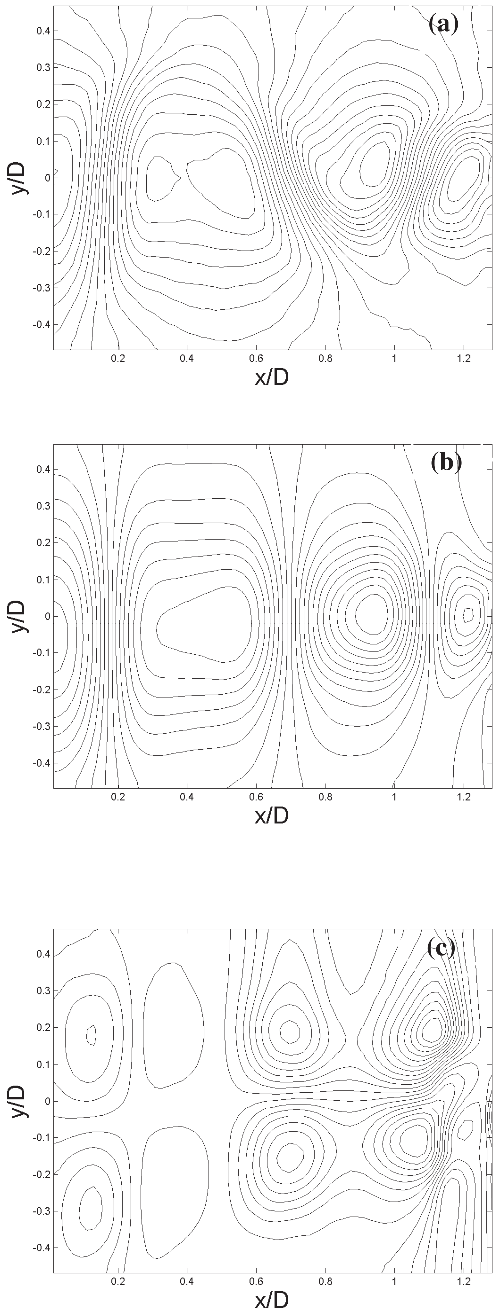

Figure 5. (a) Spatial transverse velocity (*v*) contours and (b) POD spatial mode 1, and (c) POD spatial mode 2, based on transverse velocity *v*

with the result that a mode order switching occurs.

It is interesting to note that the experimental spatial modes are closely similar to spatial modes obtained from a 2D CFD simulation for *Re*=200. Figure 6 shows the transverse velocity contours and the corresponding first POD mode. Comparison with Figs. 5(a, b) shows that the 2D simulation does capture the key modal features. This result is particularly relevant for the symmetry based reduced order model.

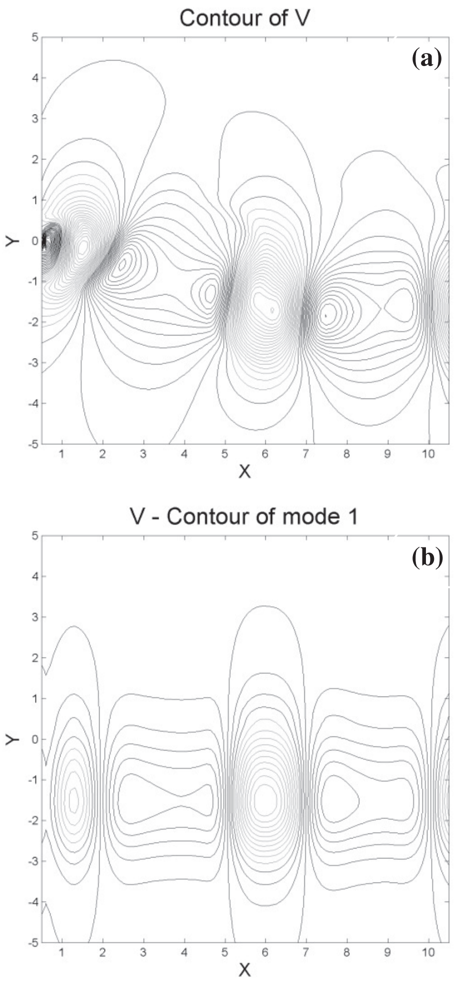

Figure 6. 2D CFD computation based (a) spatial transverse velocity (v) contours and (b) POD spatial mode 1

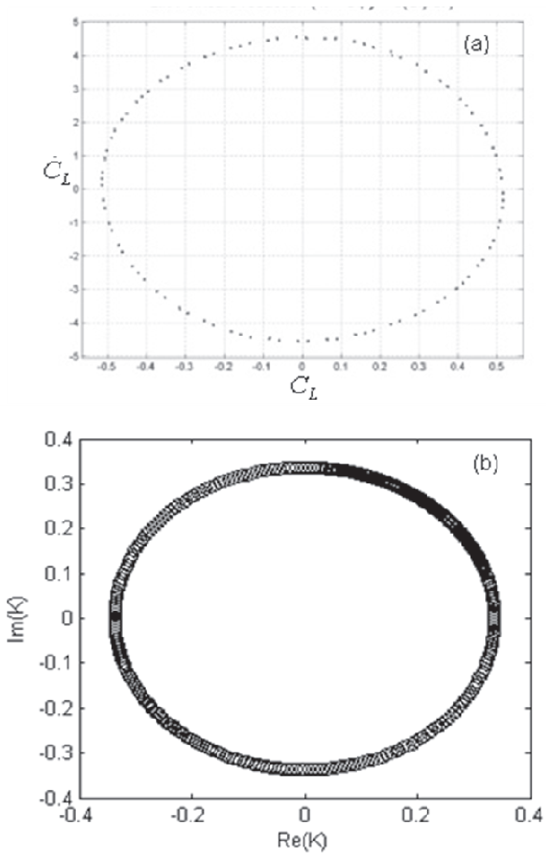

Figure 7. Low order model prediction relative to CFD simulations showing (a) computed and (b) predicted limit-cycles

4. REDUCED ORDER MODEL DYNAMICS

We investigate next the dynamic interaction between the reflection-symmetric mode and the Karman mode. This corresponds to equation (2) for the symmetry based model with mode 2 amplitude (S) considered as a 'controlled' forcing parameter. For the CGL based model, the amplitude $a_2(t)$ is the controlled forcing parameter. The dynamic behaviour of mode 1 when forced by mode 2 is investigated. Experimentally, or in CFD simulations, mode 2 is 'generated' at a chosen frequency (here the vortex shedding frequency) by periodically forcing the cylinder in the flow direction.

Figure 7(a) shows a Poincare map of the lift coefficient obtained from CFD simulations (using CFX, [1]) for the fixed cylinder and *Re*=200. The symmetry based reduced order model prediction is shown in Fig. 7(b).

The model (as expected) correctly reproduces the limit cycle in Fig. 7 (see [1] for details). For a higher forcing amplitude *A*/*D*=0.35 CFD simulations show that wake undergoes a saddle-node bifurcation. The Poincare map shows a series of saddle-node pairs (S, N) pairs as indicated in Fig. 8(a). As discussed in [1] the nodes appear to be unstable likely due to the effects of higher (unmodelled) modes. The symmetry based reduced order model correctly predicts the saddle-node bifurcation as shown in Fig. 8(b). Here two saddle-node pairs are predicted due to the low order of the model.

Simulations carried out with the CGL based model yield particularly interesting results. The spatial POD based model attempts to make the link between the observed bifurcations and the model parameters, particularly the stability parameter σ and the dissipation γ. Figure 9 shows the complex amplitude variation for increasing values of the effective stability parameter $\bar{\mu}$ containing the effect of mode 2 on mode 1, ($\bar{\mu} = \alpha\mu_0 - c_i^2$) where α is the parameter adjusted to study the dynamics (by simulating increasing mode 2 excitation). For

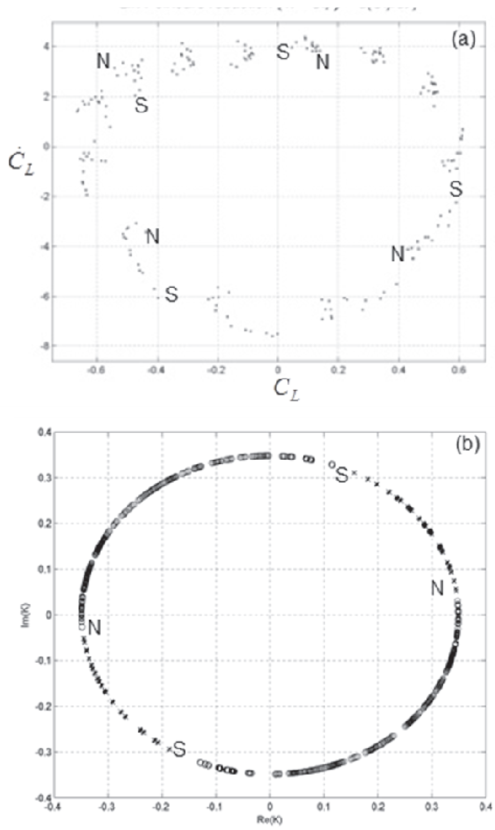

Figure 8. Low order model prediction relative to CFD simulations showing (a) computed and (b) predicted saddle-node bifurcation

low $\bar{\mu}(\approx 0.017)$ the zero amplitude state is stable, (see Fig. 9(a)) corresponding to laminar flow with no vortex shedding; (to justify the validity of employing the POD modes in this state, we note the observation of Baranyi [11] showing the Karman mode, for instance, appears when the steady flow is perturbed below the vortex-shedding critical Reynolds number). As $\bar{\mu}$ is increased a Hopf bifurcation occurs resulting in limit cycle oscillations, Fig. 9(b). The limit cycle amplitude increase with $\bar{\mu}$ (Fig. 10(a)) while, more importantly, it deforms revealing four preferred points which become increasingly locally stable (at the same time limit cycle period becomes increasingly long, approaching infinity at the bifurcation point). For $\bar{\mu} \approx 0.078$, the limit cycle bifurcates into four saddle-node pairs as shown in Fig. 10(b). The latter figure shows the mode 1 amplitude evolution for different initial conditions. The four stable nodes are clearly identifiable by the convergence of orbits. The saddle points can be identified by observing the approach along the stable manifolds followed by sudden divergence at the saddle point. The results confirm the suspected saddle-nodes in the numerical computations of Mureithi et al. [1], Fig. 8(a).

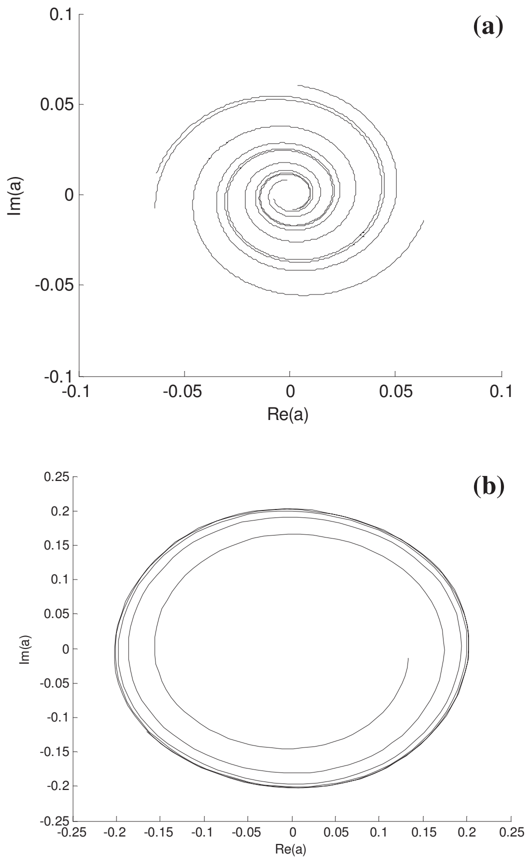

Figure 9. (a) stable fixed point and (b) limit cycle following a Hopf bifurcation

5. CONCLUSIONS

The CGL equation has been employed to derive a reduced order model for the forced wake dynamics behind a circular cylinder. The model was found to predict the most important bifurcations including the Hopf bifurcation leading to vortex shedding and, more importantly, the saddle-node bifurcation of the Karman wake observed in CFD computations for *Re*=200.

The CGL based reduced order model provides a useful and simple tool for the study of the dynamics of the Karman wake which retains the key aspects of spatial-temporal energy propagation and dissipation effects. Most importantly the model provides an alternative confirmation of the validity of the symmetry-based reduced order models. The CGL model has the particularly important advantage that model coefficients can be related directly to physical quantities in the flow.

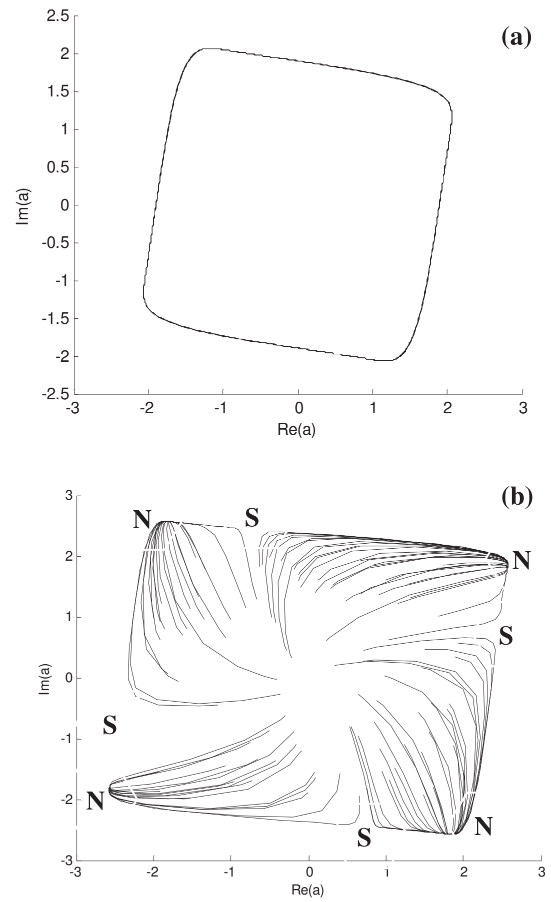

Figure 10. CGL low order model (a) limit cycle near bifurcation point ($\bar{\mu} \approx 0.055$) and (b) stationary states following a saddle-node bifurcation, $\bar{\mu} \approx 0.078$

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