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SPECTRA AND FINE SPECTRA OF THE UPPER TRIANGULAR BAND MATRIX $U(a;0;b)$ OVER THE SEQUENCE SPACE c_0

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Abstract. The aim of this paper is to obtain the spectrum, fine spectrum, approximate point spectrum, defect spectrum and compression spectrum of the operator

$$U(a;0;b)=\begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & 0 & b_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & b_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \quad (b_0, b_1, b_2 \neq 0)$$

on the sequence space c_0 where the non-zero diagonals are the entries of an oscillatory sequence.

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1. INTRODUCTION

We can band matrices which occur finite element or finite difference problems in numerical analysis. We define the relationship between the problem variables helping these matrices. The bandedness is confirmed with variables which are not conjugate in arbitrarily large distances. We can furthermore divide these matrices. For example there are banded matrices with every element in the band is nonzero. We generally encounter to these matrices separating one-dimensional problems.

Also, there are band matrices in problems with higher dimensions. Herein the bands are more thin. For example, the matrix which its bandwidth is the square root of the matrix dimension, correspond to partial differential equation defined in a square domain where the five diagonals are not zero in band. If we apply to this matrix Gaussian elimination, we obtain matrix which has band with many non-zero elements. Therefore the resolvent set of the band operators is important for solving such problems.

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Spectral theory is the one of the most useful tools in science. There are many applications in mathematics and physics which contain matrix theory, control theory, function theory, differential and integral equations, complex analysis and quantum physics. For example, atomic energy levels are determined and therefore the frequency of a laser or the spectral signature of a star are obtained by it in quantum mechanics.

1.1. *The spectrum*

Let $L : X \to Y$ be a bounded linear operator where X and Y are Banach spaces. Denote the range of L, $R(L)$ and the set of all bounded linear operators on X into itself $B(X)$.

Assume that X be a Banach space and $L \in B(X)$. The adjoint operator $L^* \in B(X^*)$ of L is defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$ where X^* is the dual space X.

Let X is a complex normed linear space and $D(L) \subset X$ be domain of L where $L : D(L) \to X$ be a linear operator. For $L \in B(X)$ we determine a complex number λ by the operator $(\lambda I - L)$ denoted by L_λ which has the same domain $D(L)$, such that I is the identity operator. Recall that the resolvent operator of L_λ is $L_\lambda^{-1} := (\lambda I - L)^{-1}$.

Let $\lambda \in \mathbb{C}$. If L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X then λ is called a regular value of L.

The set $\rho(L, X)$ of all regular values of L is called the resolvent set of L. $\sigma(L, X) := \mathbb{C} \backslash \rho(L; X)$ is called the spectrum of L where $\mathbb{C}$ is the complex plane. Hence those values $\lambda \in \mathbb{C}$ for which L_λ is not invertible are contained in the spectrum $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: The point spectrum $\sigma_p(L, X)$ is the set such that L_λ^{-1} does not exist. Further $\lambda \in \sigma_p(L, X)$ is called the eigenvalue of L. We say that $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L, X)$ of L if the resolvent operator L_λ^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L, X)$ of L if the resolvent operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X; in this case L_λ^{-1} may be bounded or unbounded. Together with the point spectrum, these two subspectra form a disjoint subdivision

$$\sigma(L, X) = \sigma_p(L, X) \cup \sigma_c(L, X) \cup \sigma_r(L, X) \tag{1.1}$$

of the spectrum of L.

1.2. *Goldberg's classification of spectrum*

If $T \in B(X)$, then there are three possibilities for $R(T)$:

(I) $R(T) = X$, (II) $\overline{R(T)} = X$, but $R(T) \neq X$, (III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and continuous, (2) T^{-1} exists but discontinuous, (3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see [7]).

If λ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$, then $\lambda \in \rho(L,X)$. All scalar values of λ not in $\rho(L,X)$ comprise the spectrum of L. The further classification of $\sigma(L,X)$ gives rise to the fine spectrum of L. That is, $\sigma(L,X)$ can be divided into the subsets $I_2\sigma(L,X) = \varnothing$, $I_3\sigma(L,X)$, $II_2\sigma(L,X)$, $II_3\sigma(L,X)$, $III_1\sigma(L,X)$, $III_2\sigma(L,X)$, $III_3\sigma(L,X)$. For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L,X)$.

Throughout w denote the space of all real or complex valued sequences. The space of all bounded, convergent, null and bounded variation sequences are denoted by ℓ_∞, c, c_0 and bv, respectively. Also by ℓ_1, ℓ_p, bv_p we denote the spaces of all absolutely summable sequences, $p-$absolutely summable sequences and $p-$bounded variation sequences, respectively.

Many researchers have investigated the spectrum and the fine spectrum of linear operators defined by some matrices over certain sequence spaces. There are a lot of studies about spectrum and fine spectrum. For instance, the fine spectrum of the Cesàro operator on the sequence space ℓ_p for $(1 < p < \infty)$ has been examined by Gonzalez [8]. Also, Wenger [17] has studied the fine spectrum of the Hölder summability operator over c, and Rhoades [12] generalized this result to the weighted mean methods. Reade [11] has investigated the spectrum of the Cesàro operator on the sequence space c_0. The spectrum of the Rhaly operators on the sequence spaces c_0 and c has examined by Yildirim [19]. The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on ℓ_p, $(1 < p < \infty)$ has examined by Yildirim and Durna [20]. In [14], Tripathy and Das determined the spectrum and fine spectrum of the upper triangular matrix $U(r,s)$ on the sequence space

$$cs = \left\{ x = (x_n) \in w : \lim_{n\to\infty} \sum_{i=0}^{n} x_i \text{ exists} \right\},$$

which is a Banach space with respect to the norm $\|x\|_{cs} = \sup_n \left|\sum_{i=0}^{n} x_i\right|$. Also they determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $U(r,s)$ on the same space. In [16], Tripathy and Saikia determined the norm and spectrum of the Cesàro matrix considered as a bounded operator on $\overline{bv_0} \cap \ell_\infty$. In [15], Tripathy and Paul examined the spectra of the operator $D(r,0,0,s)$ on sequence spaces c_0 and c. In [9], Paul and Tripathy investigated the spectrum of the operator $D(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p. In [13], Tripathy and Das determined the spectra of the Rhaly operator on the class of bounded statistically null bounded variation sequence space. In [10], Paul and

Tripathy investigated the so-called fine spectrum of the operator $D(r,0,0,s)$ over a sequence space bv_0. In [3], Das and Tripathy determined the spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ on the sequence space cs.

2. FINE SPECTRUM

The upper triangular matrix $U(a;0;b)$ is an infinite matrix with the non-zero diagonals are the entries of an oscillatory sequence of the form

$$U(a;0;b)=\begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & a_2 & 0 & b_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_2 & 0 & b_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \tag{2.1}$$

where $b_0,b_1,b_2\neq 0$.

Lemma 1 (Wilansky [18], Example 8.4.5 A, Page 129)**.** *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(c_0)$ from c_0 to itself if and only if*
(i) the rows of A in ℓ_1 and their ℓ_1 norms are bounded,
(ii) the columns of A are in c_0.

The operator norm of T is the supremum of ℓ_1 norm values of the rows.

Corollary 1. *$U(a;0;b):c_0\to c_0$ is a bounded linear operator and*

$$\|U(a;0;b)\|_{(c_0:c_0)}=\max\{|a_0|+|b_0|,|a_1|+|b_1|,|a_2|+|b_2|\}.$$

Lemma 2 (Golberg [7, p.59])**.** *T has a dense range if and only if T^* is 1-1.*

Lemma 3 (Golberg [7, p.60])**.** *T has a bounded inverse if and only if T^* is onto.*

Theorem 1. $\sigma_p(U(a;0;b),c_0)=\{\alpha\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|<|b_0||b_1||b_2|\}$.

Proof. Let λ be an eigenvalue of the operator $U(a;0;b)$. Then there exists $x\neq\theta=(0,0,0,...)$ in c_0 such that $U(a;0;b)x=\lambda x$. Then

$$\begin{aligned} a_0x_0+b_0x_2&=\lambda x_0\\ a_1x_1+b_1x_3&=\lambda x_1\\ a_2x_2+b_2x_4&=\lambda x_2\\ a_0x_3+b_0x_5&=\lambda x_3\\ a_1x_4+b_1x_6&=\lambda x_4\\ &\vdots \end{aligned}$$

From above, we have

$$\begin{aligned}
x_{6n} &= q^n x_0,\\
x_{6n+1} &= q^n x_1,\\
x_{6n+2} &= \frac{\lambda - a_0}{b_0} q^n x_0,\\
x_{6n+3} &= \frac{\lambda - a_1}{b_1} q^n x_1,\\
x_{6n+4} &= \frac{(\lambda - a_0)(\lambda - a_2)}{b_0 b_2} q^n x_0,\\
x_{6n+5} &= \frac{(\lambda - a_0)(\lambda - a_1)}{b_0 b_1} q^n x_1
\end{aligned}$$

where $n \geq 0$ and $q = \frac{(\lambda - a_0)(\lambda - a_1)(\lambda - a_2)}{b_0 b_1 b_2}$. Clearly, the subsequences (x_{6n+r}), $r = \overline{0,5}$ of $x = (x_n)$ are in c_0 if and only if $|\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| < |b_0|\,|b_1|\,|b_2|$ and hence, $x = (x_n) \in c_0$ if and only if $|\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| < |b_0|\,|b_1|\,|b_2|$. Therefore, $\sigma_p(U(a;0;b), c_0) = \{\alpha \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| < |b_0|\,|b_1|\,|b_2|\}$. $\square$

We will use the following Lemma to find the adjoint of a linear transform on the sequence space c_0.

Lemma 4 (p.266 [17]). *Let $T : c_0 \longmapsto c_0$ be a linear map and define $T^* : \ell_1 \longmapsto \ell_1$, by $T^* g = g \circ T$, $g \in c_0^* \cong \ell_1$, then T must be given with the matrix A, moreover, T^* must be given with the matrix A^t.*

Theorem 2. $\sigma_p(U(a;0;b)^*, c_0^* \cong \ell_1) = \varnothing$.

Proof. From Lemma 4, it is clear that the matrix of $U(a;0;b)^*$ is transpose of the matrix of $U(a;0;b)$. Let λ be an eigenvalue of the operator $U(a;0;b)^*$. Then there exists $x \neq \theta = (0,0,0,...)$ in ℓ_1 such that $U(a;0;b)^* x = \lambda x$.
Then, we have

$$\begin{aligned}
a_0 x_0 &= \lambda x_0 && (2.2)\\
a_1 x_1 &= \lambda x_1 && (2.3)\\
b_0 x_0 + a_2 x_2 &= \lambda x_2 && (2.4)\\
b_1 x_1 + a_0 x_3 &= \lambda x_3 && (2.5)\\
b_2 x_2 + a_1 x_4 &= \lambda x_4 && (2.6)\\
b_0 x_3 + a_2 x_5 &= \lambda x_5 && (2.7)\\
b_1 x_4 + a_0 x_6 &= \lambda x_6 && (2.8)\\
b_2 x_5 + a_1 x_7 &= \lambda x_7 && (2.9)\\
b_0 x_6 + a_2 x_8 &= \lambda x_8 && (2.10)\\
&\vdots
\end{aligned}$$

Then we have

$$n=3k,\ \ b_0x_n+a_2x_{n+2}=\lambda x_{n+2} \tag{2.11}$$
$$n=3k+1,\ \ b_1x_n+a_0x_{n+2}=\lambda x_{n+2} \tag{2.12}$$
$$n=3k+2,\ \ b_2x_n+a_1x_{n+2}=\lambda x_{n+2} \tag{2.13}$$

Let $x_0\neq 0$ then we get $\lambda=a_0$ from (2.2), $x_1=0$ from (2.5), $x_4=0$ from (2.8), $x_2=0$ from (2.6) and $x_0=0$ from (2.4). But this contradicts with our assumption.

Now let $x_0=0$ and $x_1\neq 0$ then we get $\lambda=a_1$ from (2.3), $x_2=0$ from (2.6), $x_5=0$ from (2.9), $x_3=0$ from (2.7), $x_1=0$ from (2.5). But this contradicts with our assumption.

Similarly let $x_0=0$, $x_1=0$ and $x_2\neq 0$ then we get $\lambda=a_1$ from (2.4), $x_6=0$ from (2.10), $x_4=0$ from (2.8), $x_2=0$ from (2.6). But this contradicts with our assumption.

Finally, let x_{3k+1} be the first non-zero of the sequence (x_n). If $n=3k$, then from (2.11) we have $\lambda=a_2$. Again from (2.11) for $n=3k+3$ we have $b_0x_{3k+3}+a_2x_{3k+5}=a_2x_{3k+5}$, then we get $x_{3k+3}=0$. But from (2.12) for $n=3k+1$ we have $b_1x_{3k+1}+a_0x_{3k+3}=a_2x_{3k+3}$, we have $x_{3k+1}=0$, a contradiction.

Similarly, if x_{3k} or $x_{3k}+2$ be the first non-zero of the sequence (x_n) we get a contradiction.

Hence, $\sigma_p(U(a;0;b)^*,c_0^*\cong\ell_1)=\varnothing$. □

Theorem 3. $\sigma_r(U(a;0;b),c_0)=\varnothing$.

Proof. Since, $\sigma_r(A)=\sigma_p(A^*,\ell_1)\backslash\sigma_p(A,c_0)$, Theorems 1 and 2 give us required result. □

Lemma 5.

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}a_kb_{nk}\right)=\sum_{k=0}^{\infty}a_k\left(\sum_{n=k+1}^{\infty}b_{nk}\right)$$

where (a_k) and (b_{nk}) are nonnegative real numbers.

Proof.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}a_kb_{nk}\right)&=\sum_{k=0}^{0}a_kb_{1k}+\sum_{k=0}^{1}a_kb_{2k}+\sum_{k=0}^{2}a_kb_{3k}+\sum_{k=0}^{3}a_kb_{4k}+\cdots\\&=a_0b_{10}+(a_0b_{20}+a_1b_{21})+(a_0b_{30}+a_1b_{31}+a_2b_{32})\\&\quad+(a_0b_{40}+a_1b_{41}+a_2b_{42}+a_3b_{43})+\cdots\\&=a_0\sum_{n=1}^{\infty}b_{n0}+a_1\sum_{n=2}^{\infty}b_{n1}+a_2\sum_{n=3}^{\infty}b_{n2}+\cdots\\&=\sum_{k=0}^{\infty}a_k\left(\sum_{n=k+1}^{\infty}b_{nk}\right).\end{aligned}$$

$\square$

Theorem 4. *$\sigma_c(U(a;0;b),c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0||\lambda - a_1||\lambda - a_2| = |b_0||b_1||b_2|\}$ and $\sigma(U(a;0;b),c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0||\lambda - a_1||\lambda - a_2| \le |b_0||b_1||b_2|\}$.*

Proof. Let $y = (y_n) \in \ell_1$ be such that $(U(a;0;b) - \lambda I)^* x = y$ for some $x = (x_n)$. Then we get system of linear equations:

$$\begin{aligned}
(a_0 - \lambda)x_0 &= y_0 \\
(a_1 - \lambda)x_1 &= y_1 \\
b_0 x_0 + (a_2 - \lambda)x_2 &= y_2 \\
b_1 x_1 + (a_0 - \lambda)x_3 &= y_3 \\
b_2 x_2 + (a_1 - \lambda)x_4 &= y_4 \\
b_0 x_3 + (a_2 - \lambda)x_5 &= y_5 \\
&\vdots \\
b_0 x_{3n} + (a_2 - \lambda)x_{3n+2} &= y_{3n+2} \\
b_1 x_{3n+1} + (a_0 - \lambda)x_{3n+3} &= y_{3n+3} \\
b_2 x_{3n+2} + (a_1 - \lambda)x_{3n+4} &= y_{3n+4} \\
&\vdots
\end{aligned}$$

where $n \ge 0$. Solving these equations, we have

$$\begin{aligned}
x_0 &= \frac{1}{a_0 - \lambda} y_0 \\
x_1 &= \frac{1}{a_1 - \lambda} y_1 \\
x_2 &= \frac{1}{a_2 - \lambda} y_2 - \frac{b_0}{(a_0 - \lambda)(a_2 - \lambda)} y_0
\end{aligned}$$

$$\begin{aligned}
x_3 &= \frac{1}{a_0 - \lambda} y_3 - \frac{b_1}{(a_0 - \lambda)(a_1 - \lambda)} y_1 \\
x_4 &= \frac{1}{a_1 - \lambda} y_4 - \frac{b_2}{(a_1 - \lambda)(a_2 - \lambda)} y_2 + \frac{b_0 b_2}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_0 \\
x_5 &= \frac{1}{a_2 - \lambda} y_5 - \frac{b_0}{(a_0 - \lambda)(a_2 - \lambda)} y_3 + \frac{b_0 b_1}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_1 \\
x_6 &= \frac{1}{a_0 - \lambda} y_6 - \frac{b_1}{(a_0 - \lambda)(a_1 - \lambda)} y_4 + \frac{b_1 b_2}{(a_0 - \lambda)(a_1 - \lambda)(a_2 - \lambda)} y_2 \\
&\quad - \frac{b_0 b_1 b_2}{(a_0 - \lambda)^2 (a_1 - \lambda)(a_2 - \lambda)} y_0
\end{aligned}$$

$$x_7 = \frac{1}{a_1-\lambda}y_7 - \frac{b_2}{(a_1-\lambda)(a_2-\lambda)}y_5 + \frac{b_0 b_2}{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}y_3$$
$$- \frac{b_0 b_1 b_2}{(a_0-\lambda)(a_1-\lambda)^2(a_2-\lambda)}y_1$$
$$\vdots$$

Thus we get

$$x_{2n+t} = \frac{1}{a_{2n+t}-\lambda}\left[y_{2n+t} + \sum_{k=0}^{n-1}(-1)^{n+k} y_{2k+t} \prod_{\nu=1}^{n-k} \frac{b_{2n-2\nu+t}}{a_{2n-2\nu+t}-\lambda} \right],$$

$t = 0,1$; $n = 1,2,\ldots$. Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod 3)$. Therefore we get

$$\begin{aligned}
\sum_{v=0}^{\infty}|x_v| &= |x_0| + |x_1| + |x_2| + |x_3| + \cdots \\
&= |x_0| + |x_1| + \sum_{n=1}^{\infty}|x_{2n+t}| \\
&= \left|\frac{y_0}{a_0-\lambda}\right| + \left|\frac{y_1}{a_1-\lambda}\right| \\
&\quad + \sum_{n=1}^{\infty}\left|\frac{1}{a_{2n+t}-\lambda}\left[y_{2n+t} + \sum_{k=0}^{n-1}(-1)^{n+k} y_{2k+t} \prod_{\nu=1}^{n-k} \frac{b_{2n-2\nu+t}}{a_{2n-2\nu+t}-\lambda} \right]\right|
\end{aligned}$$

$$\begin{aligned}
&\le \left|\frac{y_0}{a_0-\lambda}\right| + \left|\frac{y_1}{a_1-\lambda}\right| \\
&\quad + \frac{1}{|a_{2n+t}-\lambda|}\sum_{n=1}^{\infty}\left[|y_{2n+t}| + \sum_{k=0}^{n-1}|y_{2k+t}| \prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu+t}}{a_{2n-2\nu+t}-\lambda}\right| \right] \\
&= \left|\frac{y_0}{a_0-\lambda}\right| + \left|\frac{y_1}{a_1-\lambda}\right| + \frac{1}{|a_{2n+t}-\lambda|}\sum_{n=1}^{\infty}|y_{2n+t}| \\
&\quad + \frac{1}{|a_{2n+t}-\lambda|}\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-1}|y_{2k+t}| \prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu+t}}{a_{2n-2\nu+t}-\lambda}\right| \right].
\end{aligned}$$

Suppose $t=0$ and consider the series $\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-1}|y_{2k}|\prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu}}{a_{2n-2\nu}-\lambda}\right|\right]$. In Lemma 5 if we take $a_k=|y_{2k}|$ and $b_{nk}=\prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu}}{a_{2n-2\nu}-\lambda}\right|$ then we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-1}|y_{2k}|\prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu}}{a_{2n-2\nu}-\lambda}\right|\right]&=\sum_{k=0}^{\infty}\left[\sum_{n=k+1}^{\infty}|y_{2k}|\prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu}}{a_{2n-2\nu}-\lambda}\right|\right]\\&=\sum_{k=0}^{\infty}\left[|y_{2k}|\sum_{n=k+1}^{\infty}\prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu}}{a_{2n-2\nu}-\lambda}\right|\right]\end{aligned}$$

Also since $\prod_{\nu=1}^{n-k}\left|\frac{b_{2n-2\nu}}{a_{2n-2\nu}-\lambda}\right|\sim M\left[\frac{b_2b_1b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}\right]^{(n-k-1)/3}$ (M constant) as $n\to\infty$, the last equation turns into the series

$$\sum_{k=0}^{\infty}\left[|y_{2k}|\sum_{n=0}^{\infty}\left[\frac{b_2b_1b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}\right]^{n/3}\right]. \tag{2.14}$$

Since $y=(y_n)\in\ell_1$, the series $\sum_{k=0}^{\infty}|y_{2k}|$ is convergent. Hence the series (2.14) is convergent if and only if $\left|\frac{b_2b_1b_0}{(a_2-\lambda)(a_1-\lambda)(a_0-\lambda)}\right|<1$. Consequently, if $\lambda\in\mathbb{C}$, $|a_2-\lambda||a_1-\lambda||a_0-\lambda|>|b_2||b_1||b_0|$, then $(x_n)\in\ell_1$. Therefore, the operator $(U(a;0;b)-\lambda I)^*$ is onto if $|\lambda-a_0||\lambda-a_1||\lambda-a_2|>|b_0||b_1||b_2|$. Then by Lemma 3 $U(a;0;b)-\lambda I$ has a bounded inverse if $|\lambda-a_0||\lambda-a_1||\lambda-a_2|>|b_0||b_1||b_2|$. So, $\sigma_c(U(a;0;b),c_0)\subseteq\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|\le|b_0||b_1||b_2|\}$.
Since $\sigma(L,c_0)$ is the disjoint union of $\sigma_p(L,c_0)$, $\sigma_r(L,c_0)$ and $\sigma_c(L,c_0)$, therefore

$$\sigma(U(a;0;b),c_0)\subseteq\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|\le|b_0||b_1||b_2|\}.$$

By Theorem 1, we get

$$\begin{aligned}&\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|<|b_0||b_1||b_2|\}\\&=\sigma_p(U(a;0;b),c_0)\subset\sigma(U(a;0;b),c_0)\end{aligned}$$

Since, $\sigma(L,c_0)$ is a compact set, so it is closed and thus,

$$\begin{aligned}\overline{\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|<|b_0||b_1||b_2|\}}&\subset\overline{\sigma(U(a;0;b),c_0)}\\&=\sigma(U(a;0;b),c_0)\end{aligned}$$

and $\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|\le|b_0||b_1||b_2|\}\subset\sigma(U(a;0;b),c_0)$.
Hence, $\sigma(U(a;0;b),c_0)=\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|\le|b_0||b_1||b_2|\}$ and so $\sigma_c(U(a;0;b),c_0)=\{\lambda\in\mathbb{C}:|\lambda-a_0||\lambda-a_1||\lambda-a_2|=|b_0||b_1||b_2|\}$. □

Theorem 5. *If* $|\lambda-a_0||\lambda-a_1||\lambda-a_2|<|b_0||b_1||b_2|$, *then* $\lambda\in I_3\sigma(U(a;0;b),c_0)$.

Proof. Suppose $|\lambda-a_0||\lambda-a_1||\lambda-a_2|<|b_0||b_1||b_2|$ and so from Theorem 1, $\lambda\in\sigma_p(U(a_0,a_1,a_2;),c_0)$. Hence, λ satisfies Golberg's condition 3. We shall show that $U(a;0;b)-\lambda I$ is onto when $|\lambda-a_0||\lambda-a_1||\lambda-a_2|<|b_0||b_1||b_2|$.
Let $y=(y_n)\in c_0$ be such that $(U(a;0;b)-\lambda I)x=y$ for $x=(x_n)$. Then,

$$\begin{aligned}
(a_0-\lambda)x_0+b_0x_2&=y_0\\
(a_1-\lambda)x_1+b_1x_3&=y_1\\
(a_2-\lambda)x_2+b_2x_4&=y_2\\
(a_0-\lambda)x_3+b_0x_5&=y_3\\
(a_1-\lambda)x_4+b_1x_6&=y_4\\
(a_2-\lambda)x_5+b_2x_7&=y_5\\
(a_0-\lambda)x_6+b_0x_8&=y_6\\
&\vdots
\end{aligned}$$

Calculating x_k, we get

$$\begin{aligned}
x_2&=\frac{1}{b_0}y_0+\frac{\lambda-a_0}{b_0}x_0\\
x_3&=\frac{1}{b_1}y_1+\frac{\lambda-a_1}{b_1}x_1\\
x_4&=\frac{1}{b_2}y_2+\frac{\lambda-a_2}{b_0b_2}y_0+\frac{(\lambda-a_0)(\lambda-a_2)}{b_0b_2}x_0
\end{aligned}$$

$$\begin{aligned}
x_5&=\frac{1}{b_0}y_3+\frac{\lambda-a_0}{b_0b_1}y_1+\frac{(\lambda-a_0)(\lambda-a_1)}{b_0b_1}x_1\\
x_6&=\frac{1}{b_1}y_4+\frac{\lambda-a_1}{b_1b_2}y_2+\frac{(\lambda-a_1)(\lambda-a_2)}{b_0b_1b_2}y_0+\frac{(\lambda-a_0)(\lambda-a_1)(\lambda-a_2)}{b_0b_1b_2}x_0\\
x_7&=\frac{1}{b_2}y_5+\frac{\lambda-a_2}{b_0b_2}y_3+\frac{(\lambda-a_0)(\lambda-a_2)}{b_0b_1b_2}y_1+\frac{(\lambda-a_0)(\lambda-a_1)(\lambda-a_2)}{b_0b_1b_2}x_1\\
x_8&=\frac{1}{b_0}y_6+\frac{\lambda-a_0}{b_0b_1}y_4+\frac{(\lambda-a_0)(\lambda-a_1)}{b_0b_1b_2}y_2+\frac{(\lambda-a_0)(\lambda-a_1)(\lambda-a_2)}{b_0^2b_1b_2}y_0\\
&\quad+\frac{(\lambda-a_0)^2(\lambda-a_1)(\lambda-a_2)}{b_0^2b_1b_2}x_0\\
&\vdots
\end{aligned}$$

From above, we have

$$x_{2n+t}=\frac{1}{b_{2n+1+t}}\left[y_{2n-2+t}+\sum_{k=0}^{n-2}y_{2k+t}\prod_{v=1}^{n-k-1}\frac{\lambda-a_{2n-2v+t}}{b_{2n-2v+t+1}}\right]$$

$$+x_t \prod_{\nu=1}^{n} \frac{\lambda - a_{2n-2\nu+t}}{b_{2n-2\nu+t}},$$

where $t = 0,1$; $n = 2,3,\ldots$. Herein $a_x = a_y$, $b_x = b_y$ for $x \equiv y(mod 3)$.
Since

$$\prod_{\nu=1}^{n} \frac{\lambda - a_{2n-2\nu+t}}{b_{2n-2\nu+t}} \sim M_1 \left[\frac{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}{b_0b_1b_2}\right]^{(n-1)/3} \quad \text{as } n \to \infty,$$

where M_1 is a constant, we have

$$x_{2n+t} \sim \frac{1}{b_{2n+1+t}}\left[y_{2n-2+t} + \sum_{k=0}^{n-2} y_{2k+t} \prod_{\nu=1}^{n-k-1} \frac{\lambda - a_{2n-2\nu+t}}{b_{2n-2\nu+t+1}}\right.$$
$$\left. + x_t M_2 \left[\frac{(a_0-\lambda)(a_1-\lambda)(a_2-\lambda)}{b_0b_1b_2}\right]^{(n-1)/3}\right., \tag{2.15}$$

as $n \to \infty$. Since $y = (y_n) \in c_0$, from (2.15)

$$y = (y_n) \in c_0 \text{ iff } \left|\frac{(\lambda - a_0)(a_1-\lambda)(a_2-\lambda)}{b_0b_1b_2}\right| < 1.$$

Thus from (2.15), $(x_n) \in c_0$ iff $|\lambda - a_0||\lambda - a_1||\lambda - a_2| < |b_0||b_1||b_2|$. Therefore, $U(a;0;b) - \lambda I$ is onto. So, $\lambda \in I$. Hence we get the required result. $\square$

3. Subdivision of the spectrum

The spectrum $\sigma(L,X)$ is partitioned into three sets which are not necessarily disjoint as follows:

If there exists a sequence (x_n) in X shuch that $\|x_n\| = 1$ and $\|Lx_n\| \to 0$ as $n \to \infty$ then (x_n) is called Weyl sequence for L.

We call the set

$$\sigma_{ap}(L,X) := \{\lambda \in \mathbb{C} : \text{there exists a Weyl sequence for } \lambda I - L\} \tag{3.1}$$

the approximate point spectrum of L. Moreover, the set

$$\sigma_\delta(L,X) := \{\lambda \in \sigma(L,X) : \lambda I - L \text{ is not surjective}\} \tag{3.2}$$

is called defect spectrum of L. Finally, the set

$$\sigma_{co}(L,X) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\} \tag{3.3}$$

is called compression spectrum in the literature.

The following Proposition is very useful for calculating the separation of the spectrum of linear operator in Banach spaces.

TABLE 1. Subdivisions of spectrum of a linear operator.

		1	2	3
		L_λ^{-1} exists and is bounded	L_λ^{-1} exists and is unbounded	L_λ^{-1} does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L,X)$	–	$\lambda \in \sigma_p(L,X)$ $\lambda \in \sigma_{ap}(L,X)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L,X)$	$\lambda \in \sigma_c(L,X)$ $\lambda \in \sigma_{ap}(L,X)$ $\lambda \in \sigma_\delta(L,X)$	$\lambda \in \sigma_p(L,X)$ $\lambda \in \sigma_{ap}(L,X)$ $\lambda \in \sigma_\delta(L,X)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L,X)$ $\lambda \in \sigma_\delta(L,X)$ $\lambda \in \sigma_{co}(L,X)$	$\lambda \in \sigma_r(L,X)$ $\lambda \in \sigma_{ap}(L,X)$ $\lambda \in \sigma_\delta(L,X)$ $\lambda \in \sigma_{co}(L,X)$	$\lambda \in \sigma_p(L,X)$ $\lambda \in \sigma_{ap}(L,X)$ $\lambda \in \sigma_\delta(L,X)$ $\lambda \in \sigma_{co}(L,X)$

Proposition 1 ([1], Proposition 1.3)**.** *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(L^*,X^*) = \sigma(L,X)$, *(b)* $\sigma_c(L^*,X^*) \subseteq \sigma_{ap}(L,X)$,
(c) $\sigma_{ap}(L^*,X^*) = \sigma_\delta(L,X)$, *(d)* $\sigma_\delta(L^*,X^*) = \sigma_{ap}(L,X)$,
(e) $\sigma_p(L^*,X^*) = \sigma_{co}(L,X)$, *(f)* $\sigma_{co}(L^*,X^*) \supseteq \sigma_p(L,X)$,
(g) $\sigma(L,X) = \sigma_{ap}(L,X) \cup \sigma_p(L^*,X^*) = \sigma_p(L,X) \cup \sigma_{ap}(L^*,X^*)$.

By the definitions given above, we can write following table

Many authors have examined spectral divisions of generalized difference matrices. For example, Paul and Tripathy, [9] have studied the spectrum of the operator $D\,(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p.

The above-mentioned articles, concerned with the decomposition of spectrum defined by Goldberg. However, in [6] Durna and Yildirim have investigated subdivision of the spectra for factorable matrices on c_0 and in [2] Basar, Durna and Yildirim have investigated subdivisions of the spectra for generalized difference operator on the sequence spaces c_0 and c, in [4] Durna, have studied subdivision of the spectra for the generalized upper triangular double-band matrices Δ^{uv} over the sequence spaces c_0 and c and in [5] Durna, have studied subdivision of the spectra for the generalized difference operator $\Delta_{a,b}$ on the sequence space ℓ_p, $(1 < p < \infty)$

Corollary 2. $III_1\sigma(U(a;0;b),c_0) = III_2\sigma(U(a;0;b),c_0) = \varnothing$.

Proof. Since $\sigma_r(L,c_0) = III_1\sigma(L,c_0) \cup III_2\sigma(L,c_0)$ from Table 1, the required result is obtained from Theorem 3. $\square$

Corollary 3. $II_3\sigma(U(a;0;b),c_0) = III_3\sigma(U(a;0;b),c_0) = \varnothing$.

Proof. Since $\sigma_p(L,c_0) = I_3\sigma(L,c_0) \cup II_3\sigma(L,c_0) \cup III_3\sigma(L,c_0)$ from Table 1, the required result is obtained from Theorem 1 and Theorem 5. □

Theorem 6. *The following statements are hold*

$$(a)\sigma_{ap}(U(a;0;b),c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \leq |b_0|\,|b_1|\,|b_2|\},$$
$$(b)\sigma_{\delta}(U(a;0;b),c_0) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| = |b_0|\,|b_1|\,|b_2|\},$$
$$(c)\sigma_{co}(U(a;0;b),c_0) = \varnothing.$$

Proof. (a) From Table 1, we get

$$\sigma_{ap}\left(U(a;0;b),c_0\right) = \sigma\left(U(a;0;b),c_0\right) \backslash III_1\sigma\left(U(a;0;b),c_0\right).$$

Hence $\sigma_{ap}\left(U(a;0;b),c_0\right) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \leq |b_0|\,|b_1|\,|b_2|\}$ from Corollary 2.

(b) From Table 1, we have

$$\sigma_{\delta}\left(U(a;0;b),c_0\right) = \sigma\left(U(a;0;b),c_0\right) \backslash I_3\sigma\left(U(a;0;b),c_0\right).$$

By using Theorem 4 and 5, we get the required result.

(c) By Proposition 1 (e), we have

$$\sigma_p(U^*(a;0;b),c_0^*) = \sigma_{co}\left(U(a;0;b),c_0\right).$$

From Theorem 2, we get the required result. □

Corollary 4. *The following statements are hold*

$$(a)\sigma_{ap}(U(a;0;b)^*,c_0^* \cong \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| = |b_0|\,|b_1|\,|b_2|\}$$
$$(b)\sigma_{\delta}(U(a;0;b)^*,c_0^* \cong \ell_1) = \{\lambda \in \mathbb{C} : |\lambda - a_0|\,|\lambda - a_1|\,|\lambda - a_2| \leq |b_0|\,|b_1|\,|b_2|\}.$$

Proof. Using Proposition 1 (c) and (d), we have

$$\sigma_{ap}(U(a;0;b)^*,c_0^* \cong \ell_1) = \sigma_{\delta}(U(a;0;b),c_0)$$

and

$$\sigma_{\delta}(U(a;0;b)^*,c_0^* \cong \ell_1) = \sigma_{ap}(U(a;0;b),c_0).$$

From Theorem 6 (a) and (b), we get the required results. □

4. Results

We can generalize our operator as follows.

$$U(a_0,a_1,\ldots,a_{n-1};0;b_0,b_1,\ldots,b_{n-1})$$

$$= \begin{bmatrix} a_0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & 0 & \ddots & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_{n-1} & 0 & b_{n-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & a_0 & 0 & b_0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 & b_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \tag{4.1}$$

where $b_0, b_1, \ldots, b_{n-1} \neq 0$.

One can get parallel all our results obtained in before section as follows.

Theorem 7. *The following statements are provided where* $S = \left\{ \lambda \in \mathbb{C} : \prod_{k=0}^{n-1} \left| \frac{\lambda - a_k}{b_k} \right| \leq 1 \right\}$, *$\mathring{S}$ be the interior of the set S and ∂S be the boundary of the set S*

(1) $\sigma_p(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \mathring{S}$,
(2) $\sigma_p(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1})^*, c_0^* \cong \ell_1) = \varnothing$,
(3) $\sigma_r(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \varnothing$,
(4) $\sigma_c(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \partial S$,
(5) $\sigma(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = S$,
(6) $I_3\sigma(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \mathring{S}$,
(7) $III_1\sigma(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \varnothing$,
(8) $III_2\sigma(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \varnothing$,
(9) $III_3\sigma(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \varnothing$,
(10) $II_3\sigma(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \varnothing$,
(11) $\sigma_{ap}(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = S$,
(12) $\sigma_\delta(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \partial S$,
(13) $\sigma_{co}(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1}), c_0) = \varnothing$,
(14) $\sigma_{ap}(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1})^*, c_0^* \cong \ell_1) = \partial S$,
(15) $\sigma_\delta(U(a_0, a_1, \ldots, a_{n-1}; 0; b_0, b_1, \ldots, b_{n-1})^*, c_0^* \cong \ell_1) = S$.

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