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IDENTITIES AND CONGRUENCES INVOLVING THE GEOMETRIC POLYNOMIALS

MILOUD MIHOUBI AND SAID TAHARBOUCHET

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Abstract. In this paper, we investigate the umbral representation of the geometric polynomials $\mathbf{w}_{\mathbf{x}}^{n} := w_n(x)$ to derive some properties involving these polynomials. Furthermore, for any prime number p and any polynomial f with integer coefficients, we show $(f(\mathbf{w}_{\mathbf{x}}))^p \equiv f(\mathbf{w}_{\mathbf{x}})$ (mod p) and we give other curious congruences.

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1. Introduction

The geometric numbers are quantities arising from enumerative combinatorics and have nice number-theoretic properties. In combinatorics, the n-th geometric number (named also the n-th ordered Bell number) counts the number of ways to partition the set $[n] := \{1, \ldots, n\}$ into ordered subsets [2, 3, 6]. The geometric polynomials are defined by $w_n(x) = \sum_{k=0}^{n} {n \brace k} k! x^k$ and satisfy the recurrence relation $(x+1)w_n(x) = x \sum_{j=0}^{n} \binom{n}{j} w_j(x)$, $n \geq 1$, [9], where ${n \brace k}$ is the (n,k)-th Stirling number of the second kind [2, 26]. These polynomials have attracted attention from many researchers, see for instance [9, 10, 15–17]. For $x = 1$ we obtain the geometric numbers $w_n := w_n(1) = \sum_{k=0}^{n} {n \brace k} k!$, for more information about these numbers, see [6–8, 11, 12, 14, 28, 29]. More generally, let $w_n(x;r,s)$ be the n-th (r,s)-geometric polynomial defined by

$$w_n(x;r,s) = \sum_{k=0}^{n} {n+r \brace k+r}_r (k+s)! x^k.$$

This polynomial generalizes the geometric polynomial $w_n(x) = w_n(x;0,0)$ and the polynomial $w_n(x;r,r)$ introduced by Mező [18]. Here, ${n \brace k}_r$ denotes the (n,k)-th r-Stirling number of the second kind [4]. One can see easily that

$$w_0(x;r,s) = s!,$$
$$w_1(x;r,s) = s!(r + (s+1)x),$$

$$w_2(x,r,s) = s!(r^2+(2r+1)(s+1)x+(s+1)(s+2)x^2).$$

We note that this generalization can be viewed as a particular case of that defined by Kargin et al. [16]. As it shown below, these polynomials are also linked to the absolute r-Stirling numbers of first kind denoted by $\left[{n \atop k}\right]_r$.
Recall that the r-Stirling numbers can be defined by [4, 26]

$$(x)_n = \sum_{k=0}^{n}(-1)^{n-k}\left[{n+r \atop k+r}\right]_r (x+r)^k \text{ and } (x+r)^n = \sum_{k=0}^{n}\left\{{n+r \atop k+r}\right\}_r (x)_k,$$

where $(\alpha)_n = \alpha\cdots(\alpha-n+1)$ if $n\geq 1$, $(\alpha)_0 = 1$.
This work is motivated by application of the umbral calculus method to determine identities and congruences involving Bell numbers and polynomials in the works of Gessel [13], Sun et al. [27], Mező et al. [19] and Benyattou et al. [1]. In this paper, we will talk about identities and congruences involving the (r,s)-geometric polynomials based on the geometric umbra defined by $\mathbf{w}_\mathbf{x}^n := w_n(x)$. For more information about umbral calculus, see [5, 13, 22–25].

2. Identities involving the (r,s)-geometric polynomials

The above recurrence relation is equivalent to $(x+1)\mathbf{w}_\mathbf{x}^n = x(\mathbf{w}_\mathbf{x}+1)^n, n\geq 1$. Furthermore, we have

Proposition 1. *Let f be a polynomial and r,s be non-negative integers. Then*

$$(x+1)f(\mathbf{w}_\mathbf{x}+r) = xf(\mathbf{w}_\mathbf{x}+r+1)+f(r),$$
$$(\mathbf{w}_\mathbf{x}+r)_{n+r} = (n+r)!x^n(x+1)^r,$$
$$(\mathbf{w}_\mathbf{x}+r-s)^n(\mathbf{w}_\mathbf{x})_s = x^s w_n(x;r,s),$$
$$(\mathbf{w}_\mathbf{x}+r)^n(\mathbf{w}_\mathbf{x}+s)_s = (x+1)^s w_n(x;r,s).$$

Proof. It suffices to show the first identity for $f(x) = x^n$. For $r=0$ we have $(x+1)\mathbf{w}_\mathbf{x}^n - x(\mathbf{w}_\mathbf{x}+1)^n = \delta_{(n=0)}$. Assume it is true for $r-1$, then if we set

$$h_n(r) := (x+1)(\mathbf{w}_\mathbf{x}+r)^n - x(\mathbf{w}_\mathbf{x}+r+1)^n$$

we obtain $h_n(r) = \sum_{j=0}^{n}\binom{n}{j}h_j(r-1) = \sum_{j=0}^{n}\binom{n}{j}(r-1)^j = r^n$, which concludes the induction step. For the other identities, since $(x)_n = \sum_{k=0}^{n}(-1)^{n-k}\left[{n \atop k}\right]x^k$ and $(x)_n$ is a sequence of binomial type [20, 23], we obtain

$$(\mathbf{w}_\mathbf{x}+r)_{n+r} = \sum_{j=0}^{n+r}\binom{n+r}{j}(r)_j(\mathbf{w}_\mathbf{x})_{n+r-j} = (n+r)!x^n(x+1)^r.$$

So, the polynomials $x^s w_n(x;r,s)$ and $(x+1)^s w_n(x,r,s)$ must be, respectively,

$$\sum_{j=0}^{n}\left\{{n+r \atop j+r}\right\}_r(\mathbf{w}_\mathbf{x})_{j+s} = \sum_{j=0}^{n}\left\{{n+r \atop j+r}\right\}_r(\mathbf{w}_\mathbf{x}-s)_j(\mathbf{w}_\mathbf{x})_s = (\mathbf{w}_\mathbf{x}+r-s)^n(\mathbf{w}_\mathbf{x})_s,$$

$$\sum_{j=0}^{n} \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (\mathbf{w_x}+s)_{j+s} = \sum_{j=0}^{n} \left\{ \begin{matrix} n+r \\ j+r \end{matrix} \right\}_r (\mathbf{w_x})_j (\mathbf{w_x}+s)_s = (\mathbf{w_x}+r)^n (\mathbf{w_x}+s)_s.$$

□

The last two identities of Proposition 1 lead to:

Corollary 1. *Let r; s be non-negative integers and f be a polynomial. Then*

$$(x+1)^s f(\mathbf{w_x}+r-s)(\mathbf{w_x})_s = x^s f(\mathbf{w_x}+r)(\mathbf{w_x}+s)_s.$$

Proposition 2. *Let $\mathcal{P}_n$ and $\mathcal{T}_n$ be the polynomials*

$$\mathcal{P}_n(x;r) = \sum_{j=0}^{n} (-1)^j \binom{j+r}{r} x^{n-j} \quad \text{and} \quad \mathcal{T}_n(x;r) = \sum_{j=0}^{n} \binom{n+r}{j+r} x^j.$$

Then $(\mathbf{w_x}-r-1)_n = n!\mathcal{P}_n(x;r)$ *and* $(\mathbf{w_x}+n+r)_n = n!\mathcal{T}_n(x;r)$.

Proof. It suffices to observe that

$$(\mathbf{w_x}-r-1)_n = \sum_{j=0}^{n} \binom{n}{j} (-r-1)_j (\mathbf{w_x})_{n-j} = n! \sum_{j=0}^{n} (-1)^j \binom{j+r}{r} x^{n-j},$$

$$(\mathbf{w_x}+n+r)_n = \sum_{j=0}^{n} \binom{n}{j} (n+r)_{n-j} (\mathbf{w_x})_j = n! \sum_{j=0}^{n} \binom{n+r}{j+r} x^j.$$

□

The following theorem can be served to derive several identities and congruences for the (r,s)-geometric polynomials.

Theorem 1. *Let m,s be non-negative integers and f be a polynomial. Then*

$$(x+1)^m f(\mathbf{w_x}) - x^m f(\mathbf{w_x}+m) = \sum_{k=0}^{m-1} f(k)(x+1)^{m-1-k} x^k, \quad m \geq 1.$$

Proof. Set $f(x) = \sum_{k=0}^{n} a_k x^k$ and use Proposition 1 to obtain

$$(x+1) f(\mathbf{w_x}) - x f(\mathbf{w_x}+1) = f(0) + \sum_{k=0}^{n} a_k \Big((x+1)\mathbf{w_x}^k - x(\mathbf{w_x}+1)^k \Big) = f(0).$$

So, the identity is true for $m=1$. Assume it is true for m. Then

$$\begin{aligned}(x+1)^{m+1} f(\mathbf{w_x}) &= (x+1)\left(\sum_{k=0}^{m-1} (x+1)^{m-1-k} x^k f(k) + x^m f(\mathbf{w_x}+m) \right) \\ &= \sum_{k=0}^{m-1} (x+1)^{m-k} x^k f(k) + x^m (x+1) f(\mathbf{w_x}+m)\end{aligned}$$

and since $(x+1)f(\mathbf{w_x}+m)-xf(\mathbf{w_x}+m+1)=f(m)$, we can write

$$\begin{aligned}(x+1)^{m+1}f(\mathbf{w_x}) &= \sum_{k=0}^{m-1}(x+1)^{m-k}x^k f(k)+x^m\Big(xf(\mathbf{w_x}+m+1)+f(m)\Big)\\ &= \sum_{k=0}^{m-1}(x+1)^{m-k}x^k f(k)+x^m f(m)+x^{m+1}f(\mathbf{w_x}+m+1)\\ &= \sum_{k=0}^{m}(x+1)^{m-k}x^k f(k)+x^{m+1}f(\mathbf{w_x}+m+1)\end{aligned}$$

which concludes the induction step. □

We note that for $f(x)=x^n$ and $x=1$ in Theorem 1 we obtain Proposition 3.3 given in [8].

Corollary 2. *For any polynomial f there holds*

$$f(\mathbf{w_x})=\frac{1}{1+x}\sum_{k\geq 0}f(k)\left(\frac{x}{1+x}\right)^k,\quad x>-\frac{1}{2}.$$

Proof. For $m=1$ in Theorem 1, when we replace $f(x)$ by $f(x+r)$ we get the identity $f(r)=(x+1)f(\mathbf{w_x}+r)-xf(\mathbf{w_x}+r+1)$. Then

$$\begin{aligned}RHS &= \lim_{n\to\infty}\frac{1}{1+x}\sum_{k=0}^{n}\left(\frac{x}{1+x}\right)^k\Big((x+1)f(\mathbf{w_x}+k)-xf(\mathbf{w_x}+k+1)\Big)\\ &= \lim_{n\to\infty}\left(f(\mathbf{w_x})-\left(\frac{x}{1+x}\right)^{n+1}f(\mathbf{w_x}+n+1)\right)=f(\mathbf{w_x})\end{aligned}$$

which completes the proof. □

Corollary 3. *Let n,r,s be non-negative integers.*
For $f(x)=(x+r)^n(x+s)_s$ or $(x+r-s)^n(x)_s$ in Corollary 2 we obtain

$$w_n(x;r,s)=\frac{s!}{(1+x)^{s+1}}\sum_{k\geq 0}\binom{k+s}{s}(k+r)^n\left(\frac{x}{1+x}\right)^k,\quad x>-\frac{1}{2}.$$

Corollary 4. *For any integers $r\geq 0$, $s\geq 0$ and $n\geq 1$ the polynomial $w_n(x,r,s+r)$ has only real non-positive zeros.*

Proof. From Corollary 3 we may state

$$x^r(x+1)^s w_{n+1}(x;r,s+r)=x\frac{d}{dx}\Big(x^r(x+1)^{s+1}w_n(x;r,s+r)\Big)$$

and using the recurrence relation of r-Stirling numbers we conclude that this identity remains true for all real number x. So, by induction on n, it follows that $w_n(x;r,s+r)$, $n \geq 1$, has only real non-positive zeros. □

Lemma 1. *For any non-negative integers $n \geq 2$ there holds*

$$(1+x)w_{n-1}(x) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} (k-1)!x^k.$$

Proof. From the definition of geometric polynomials, we have

$$\begin{aligned}
(1+x)w_{n-1}(x) &= \sum_{k=1}^{n-1} \left\{ {n-1 \atop k} \right\} k!x^k + \sum_{k=1}^{n-1} \left\{ {n-1 \atop k} \right\} k!x^{k+1} \\
&= \sum_{k=1}^{n} \left(k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \right) (k-1)!x^k \\
&= \sum_{k=1}^{n} \left\{ {n \atop k} \right\} (k-1)!x^k.
\end{aligned}$$

□

For more explicit formulae for geometric polynomials, see for example [15].

Proposition 3. *Let n, r, s be non-negative integers. Then*

$$\log\left(1 + \sum_{n\geq 1} \frac{w_n(x;r,s)}{s!} \frac{t^n}{n!}\right) = (r+(s+1)x)t + (s+1)(x+1) \sum_{n\geq 2} w_{n-1}(x) \frac{t^n}{n!}.$$

In particular, for $r = s = 0$ we get

$$\log\left(1 + \sum_{n\geq 1} w_n(x) \frac{t^n}{n!}\right) = xt + (x+1) \sum_{n\geq 2} w_{n-1}(x) \frac{t^n}{n!}.$$

Proof. One can verify easily that the exponential generating function of the polynomials $w_n(x;r,s)$ is to be $s!\exp(rt)(1-x(\exp(t)-1))^{-s-1}$. Then, upon using this generating function and the last Lemma, we can write

$$\begin{aligned}
LHS &= rt - (s+1)\ln(1-x(\exp(t)-1)) \\
&= rt + (s+1) \sum_{k\geq 1} \frac{x^k}{k} (\exp(t)-1)^k \\
&= rt + (s+1) \sum_{k\geq 1} (k-1)!x^k \sum_{n\geq k} \left\{ {n \atop k} \right\} \frac{t^n}{n!}
\end{aligned}$$

$$= rt + (s+1)xt + (s+1)\sum_{n\geq 2}\frac{t^n}{n!}\sum_{k=1}^{n}\left\{ n \atop k \right\}(k-1)!x^k$$
$$= (r+(s+1)x)t + (s+1)(x+1)\sum_{n\geq 2} w_{n-1}(x)\frac{t^n}{n!}.$$

□

3. CONGRUENCES INVOLVING THE (R,S)-GEOMETRIC POLYNOMIALS

In this section, we give some congruences involving the (r,s)-geometric polynomials. Let $\mathbb{Z}_p$ be the ring of p-adic integers and for two polynomials $f(x)$, $g(x) \in \mathbb{Z}_p[x]$, the congruence $f(x) \equiv g(x) \pmod{p\mathbb{Z}_p[x]}$ means that the corresponding coefficients of $f(x)$ and $g(x)$ are congruent modulo p. This congruence will be used later as $f(x) \equiv g(x)$ and we will use $a \equiv b$ instead $a \equiv b \pmod{p}$.

Proposition 4. *Let n, r, s be non-negative integers and p be a prime number. Then, for any polynomial f with integer coefficients there holds*

$$\sum_{k=0}^{p-1} f(k)(x+1)^{p-1-k}x^k \equiv f(\mathbf{w_x}).$$

In particular, for $f(x) = (x+r-s)^n(x)_s$ or $(x+r)^n(x+s)_s$ we get, respectively,

$$\sum_{k=0}^{p-1}(r-s+k)^n(k)_s(x+1)^{p-1-k}x^k \equiv x^s w_n(x;r,s),$$
$$\sum_{k=0}^{p-1}(r+k)^n(s+k)_s(x+1)^{p-1-k}x^k \equiv (x+1)^s w_n(x;r,s).$$

Proof. For $m = p$ be a prime number, Theorem 1 implies

$$LHS = (x+1)^p f(\mathbf{w_x}) - x^p f(\mathbf{w_x}+p) \equiv (x^p+1)f(\mathbf{w_x}) - x^p f(\mathbf{w_x}) = f(\mathbf{w_x}).$$

For the particular cases, use Proposition 1. □

Corollary 5. *Let n, r, s, m, q be non-negative integers and p be a prime number. Then, for any polynomials f and g with integer coefficients there holds*

$$(f(\mathbf{w_x}))^p g(\mathbf{w_x}) \equiv f(\mathbf{w_x})g(\mathbf{w_x}).$$

In particular, we have $w_{mp+q}(x;r,s) \equiv w_{m+q}(x;r,s)$.

Proof. By Fermat's little theorem and by twice application of Proposition 4 we may state

$$LHS \equiv \sum_{k=0}^{p-1}(f(k))^p g(k)(x+1)^{p-1-k}x^k \equiv \sum_{k=0}^{p-1} f(k)g(k)(x+1)^{p-1-k}x^k = RHS.$$

□

We note that, for $f(x) = x^m$, $g(x) = x^q$ and $x = 1$, Corollary 5 may be seen as a particular case of Theorem 3.1 given in [8].

Corollary 6. *For any non-negative integers $m \geq 1, n, r, s$ and any prime number p, there hold*

$$(x+1)^{s+1}(w_{m(p-1)}(x;r,s) - s!) \equiv -(s-r')_s(x+1)^{r'}x^{p-r'}, \quad r' \neq 0,$$
$$(x+1)^{s+1}(w_{m(p-1)}(x;r,s) - s!) \equiv -s!(x^p+1), \quad r' = 0,$$

where $r' \equiv r$ and $r' \in \{0, 1, \ldots, p-1\}$.

Proof. Set $n = m(p-1)$ in Proposition 4. If $r' \neq 0$ we get

$$\begin{aligned}(x+1)^s w_{m(p-1)}(x;r,s) &\equiv \sum_{k=0}^{p-1}(r'+k)^{m(p-1)}(s+k)_s(x+1)^{p-1-k}x^k \\ &\equiv \sum_{k=0,\ r'+k\neq p}^{p-1} (s+k)_s(x+1)^{p-1-k}x^k \\ &= \sum_{k=0}^{p-1}(s+k)_s(x+1)^{p-1-k}x^k \\ &\quad -(s-r'+p)_s(x+1)^{r'-1}x^{p-r'} \\ &\equiv (x+1)^s w_0(x;0,s) - (s-r')_s(x+1)^{r'-1}x^{p-r'} \\ &\equiv s!(x+1)^s - (s-r')_s(x+1)^{r'-1}x^{p-r'}\end{aligned}$$

and if $r' = 0$ we get

$$\begin{aligned}(x+1)^{s+1}w_{m(p-1)}(x;r,s) &\equiv \sum_{k=1}^{p-1}(s+k)_s(x+1)^{p-k}x^k \\ &= \sum_{k=0}^{p-1}(s+k)_s(x+1)^{p-k}x^k - s!(x+1)^p \\ &= (x+1)^{s+1}w_0(x;0,s) - s!(x+1)^p \\ &= s!(x+1)^{s+1} - s!(x^p+1).\end{aligned}$$

which complete the proof. □

Remark 1. For $r = s = m-1 = 0$ in Corollary 6 or $n = p$ in Lemma 1 we obtain $(x+1)w_{p-1}(x) \equiv x - x^p$ which gives for $x = 1$ the known congruence $w_{p-1} \equiv 0$, see [8].

Now, we give some curious congruences on (r,s)-geometric polynomials and on $(r_1,\ldots,r_q)$-geometric polynomials defined below.

Theorem 2. *For any integers $n,m,r,s\geq 0$ and any prime number $p\nmid m$, there holds*

$$\sum_{k=1}^{p-1}\frac{w_{n+k}(x;r,s)}{(-m)^k}\equiv(-m)^n(w_{p-1}(x;r+m,s)-s!).$$

Proof. Upon using the identity $x^s w_n(x;r,s)=(\mathbf{w_x}+r-s)^n(\mathbf{w_x})_s$ and the known congruence $(-m)^{-k}\equiv\binom{p-1}{k}m^{p-1-k}$ we obtain

$$\begin{aligned}
x^s LHS &\equiv \sum_{k=0}^{p-1}\binom{p-1}{k}m^{p-1-k}(\mathbf{w_x}+r-s)^{n+k}(\mathbf{w_x})_s\\
&=(\mathbf{w_x}+r-s)^n(\mathbf{w_x}+r+m-s)^{p-1}(\mathbf{w_x})_s\\
&=\sum_{j=0}^{n}\binom{n}{j}(-m)^{n-j}(\mathbf{w_x}+r+m-s)^{j+p-1}(\mathbf{w_x})_s\\
&=(-m)^n(\mathbf{w_x}+r+m-s)^{p-1}(\mathbf{w_x})_s\\
&+\delta_{(n\geq 1)}\sum_{j=1}^{n}\binom{n}{j}(-m)^{n-j}(\mathbf{w_x}+r+m-s)^{j+p-1}(\mathbf{w_x})_s\\
&=x^s(-m)^n w_{p-1}(x;r+m,s)\\
&+\delta_{(n\geq 1)}x^s\sum_{j=1}^{n}\binom{n}{j}(-m)^{n-j}w_{p+j-1}(x;r+m,s)\\
&\equiv x^s(-m)^n w_{p-1}(x;r+m,s)\\
&+\delta_{(n\geq 1)}x^s\sum_{j=1}^{n}\binom{n}{j}(-m)^{n-j}w_{j}(x;r+m,s)\\
&=x^s(-m)^n w_{p-1}(x;r+m,s)+\delta_{(n\geq 1)}x^s(w_n(x;r,s)-(-m)^n s!)\\
&=x^s[(-m)^n w_{p-1}(x;r+m,s)+w_n(x;r,s)-(-m)^n s!],
\end{aligned}$$

where δ is the Kronecker's symbol, i.e. $\delta_{(n\geq 1)}=1$ if $n\geq 1$ and 0 otherwise. □

Let $\mathbf{r}_q=(r_1,\ldots,r_q)$ be a vector of non-negative integers and let

$$w_n(x;\mathbf{r}_q)=\sum_{j=0}^{n+|\mathbf{r}_{q-1}|}\left\{ {n+|\mathbf{r}_q| \atop j+r_q} \right\}_{\mathbf{r}_q}(j+r_q)!x^j,\ \ 0\leq r_1\leq\cdots\leq r_q,$$

where ${n+|\mathbf{r}_q| \brace j+r_q}_{\mathbf{r}_q}$ are the $(r_1,\dots,r_q)$-Stirling numbers defined by Mihoubi et al. [21]. This polynomial is a generalization of the r-geometric polynomials $w_n(x;r) := w_n(x;r,r)$.

Proposition 5. *For any non-negative integers n,m and any prime $p \nmid m$, there holds*

$$x^{r_q}\sum_{k=1}^{p-1}\frac{w_{n+k}(x;\mathbf{r}_q)}{(-m)^k} \equiv (-m)^n(-m)_{r_1}\cdots(-m)_{r_q}(w_{p-1}(x;m,0)-1).$$

In particular, for $q=1$ and $r_q=r$ we obtain

$$x^{r}\sum_{k=1}^{p-1}\frac{w_{n+k}(x;r,r)}{(-m)^k} \equiv (-m)^n(-m)_{r}(w_{p-1}(x;m,0)-1).$$

Proof. By the identity $(\mathbf{w}_\mathbf{x})_n = n!x^n$ and by [21, Th. 10] we have

$$\begin{aligned}
x^{r_q}w_n(x;\mathbf{r}_q) &= \sum_{j=0}^{n+|\mathbf{r}_{q-1}|}{n+|\mathbf{r}_q| \brace j+r_q}_{\mathbf{r}_q}(\mathbf{w}_\mathbf{x})_{j+r_q}\\
&= \sum_{j=0}^{n+|\mathbf{r}_{q-1}|}{n+|\mathbf{r}_q| \brace j+r_q}_{\mathbf{r}_q}(\mathbf{w}_\mathbf{x}-r_q)_j(\mathbf{w}_\mathbf{x})_{r_q}\\
&= \mathbf{w}_\mathbf{x}^n(\mathbf{w}_\mathbf{x})_{r_1}\cdots(\mathbf{w}_\mathbf{x})_{r_q}\\
&= \sum_{k=0}^{|\mathbf{r}_q|}a_k(\mathbf{r}_q)\mathbf{w}_\mathbf{x}^{n+k}\\
&= \sum_{j=0}^{|\mathbf{r}_q|}a_j(\mathbf{r}_q)w_{n+j}(x),
\end{aligned}$$

where $\sum_{k=0}^{|\mathbf{r}_q|}a_k(\mathbf{r}_q)u^k = (u)_{r_1}\cdots(u)_{r_q}$. So, by application of Theorem 2 we get

$$\begin{aligned}
x^{r_q}\sum_{k=1}^{p-1}\frac{w_{n+k}(x;\mathbf{r}_q)}{(-m)^k} &= \sum_{j=0}^{|\mathbf{r}_q|}a_j(\mathbf{r}_q)\sum_{k=1}^{p-1}\frac{w_{n+j+k}(x;0,0)}{(-m)^k}\\
&\equiv \sum_{j=0}^{|\mathbf{r}_q|}a_j(\mathbf{r}_q)(-m)^{n+j}(w_{p-1}(x;m,0)-1)\\
&= (-m)^n(-m)_{r_1}\cdots(-m)_{r_q}(w_{p-1}(x;m,0)-1).
\end{aligned}$$

□

Remark 2. Since $x^{r_q} w_n(x;\mathbf{r}_q) = \mathbf{w}_\mathbf{x}^n (\mathbf{w}_\mathbf{x})_{r_1} \cdots (\mathbf{w}_\mathbf{x})_{r_q}$, then, for $g(x) = x^q (x)_{r_1} \cdots (x)_{r_q}$ and $f(x) = x^m$ in Corollary 5 we obtain

$$w_{mp+q}(x;\mathbf{r}_q) \equiv w_{m+q}(x;\mathbf{r}_q),$$

$$w_{m(p-1)}(x;\mathbf{r}_q) \equiv w_0(x;\mathbf{r}_q), \quad r_1 \cdots r_q \neq 0, \ m \geq 0.$$

Corollary 7. *Let $a_0(x), \ldots, a_t(x)$ be polynomials with integer coefficients,*

$$\mathcal{R}_{n,t}(x;r,s) = \sum_{i=0}^{t} a_i(x) w_{n+i}(x;r,s) \ \text{ and } \ \mathcal{L}_t(x,y) = \sum_{i=0}^{t} a_i(x) y^i.$$

Then, for any non-negative integers n, m, r, s and any prime $p \nmid m$, there hold

$$\sum_{k=1}^{p-1} \frac{\mathcal{R}_{n+k,t}(x;r,s)}{(-m)^k} \equiv (-m)^n \mathcal{L}_t(x,-m)(w_{p-1}(x;r+m,s) - s!).$$

Proof. Theorem 2 implies

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{\mathcal{R}_{n+k,t}(x;r,s)}{(-m)^k} &= \sum_{j=0}^{t} a_j(x) \sum_{k=1}^{p-1} \frac{w_{n+k+j}(x;r,s)}{(-m)^k} \\
&\equiv \sum_{j=0}^{t} a_j(x)(-m)^{n+j} (w_{p-1}(x;r+m,s) - s!) \\
&= (-m)^n \mathcal{L}_t(x,-m)(w_{p-1}(x;r+m,s) - s!).
\end{aligned}$$

□

4. Congruences involving $w_n(x;r,s)$, $\mathcal{P}_n(x,r)$ and $\mathcal{T}_n(x,r)$

The following theorem gives connection in congruences between the polynomials w_n and $\mathcal{P}_n$.

Theorem 3. *Let n, r be non-negative integers and p be a prime number. Then, for $m \in \{0, \ldots, p-1\}$ there holds*

$$\sum_{k=m}^{p-1} (-x)^k \frac{w_n(x;r+k,k)}{(k-m)!} \equiv (-1)^m m!(r+m)^n \mathcal{P}_{p-1}(x,m).$$

In particular, for $m = 0$, we get

$$\sum_{k=0}^{p-1} (-x)^k \frac{w_n(x;r+k,k)}{k!} \equiv r^n (1 + x + \cdots + x^{p-1}).$$

Proof. For $k < m$ we get $\langle m+1\rangle_{p-1-k} = 0$ and for $m \le k \le p-1$ we have

$$\langle m+1\rangle_{p-1-k} = \frac{(m+p-k-1)!}{m!} = \frac{(p-1-(k-m))!}{m!} \equiv -\frac{1}{m!}\frac{(-1)^{k-m}}{(k-m)!}.$$

where $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$ if $n \ge 1$ and $\langle x\rangle_0 = 1$. Then

$$\begin{aligned}
LHS &\equiv -(-1)^m m! \sum_{k=0}^{p-1} \langle m+1\rangle_{p-1-k} x^k w_n(x;r+k,k) \\
&\equiv -(-1)^m m! \sum_{k=0}^{p-1} \langle m-p+1\rangle_{p-1-k} (\mathbf{w_x}+r)^n (\mathbf{w_x})_k \\
&\equiv -(-1)^m m! \sum_{k=0}^{p-1} \binom{p-1}{k} \langle m-p+1\rangle_{p-1-k} (\mathbf{w_x}+r)^n \langle -\mathbf{w_x}\rangle_k \\
&= -(-1)^m m! \langle m-p+1-\mathbf{w_x}\rangle_{p-1} (\mathbf{w_x}+r)^n \\
&= -(-1)^m m! (\mathbf{w_x}-m+p-1)_{p-1} (\mathbf{w_x}+r)^n \\
&= -(-1)^m m! (\mathbf{w_x}-m+r+m)^n (\mathbf{w_x}-m+p-1)_{p-1} \\
&= -(-1)^m m! \sum_{j=0}^{n} \left\{ {n+r+m \atop j+r+m} \right\}_{r+m} (\mathbf{w_x}-m)_j (\mathbf{w_x}-m+p-1)_{p-1}.
\end{aligned}$$

But for $j \ge 1$ we have

$$\begin{aligned}
&(\mathbf{w_x}-m)_j (\mathbf{w_x}-m+p-1)_{p-1} = (\mathbf{w_x}-m+p-1)_{j+p-1} \\
&\equiv (\mathbf{w_x}-m-1)_{j+p-1} = (j+p-1)! \mathscr{P}_{j+p-1}(x,m+1) \\
&\equiv -\delta_{(j=0)} \mathscr{P}_{p-1}(x,m+1),
\end{aligned}$$

hence, it follows $LHS \equiv (-1)^m m! (r+m)^n \mathscr{P}_{p-1}(x,m)$. □

A connection in congruences between the polynomials w_n and $\mathscr{T}_n$ is to be:

Theorem 4. *For any integers $n,m,r \ge 0$ and any prime p, there holds*

$$\sum_{k=0}^{p-1} (-m)_{p-1-k} (x+1)^k w_n(x;r+m,k) \equiv -r^n \mathscr{T}_{p-1}(x;m).$$

Proof. Upon using the identity $(x+1)^s w_n(x;r,s) = (\mathbf{w_x}+r)^n (\mathbf{w_x}+s)_s$ and the known congruence $(m)_{p-1-k} \equiv \binom{p-1}{k} \langle -m\rangle_{p-1-k}$ we obtain

$$LHS \equiv \sum_{k=0}^{p-1} \binom{p-1}{k} \langle m\rangle_{p-1-k} (\mathbf{w_x}+r+m)^n (\mathbf{w_x}+k)_k$$

$$
\begin{aligned}
&\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} \langle m \rangle_{p-1-k} (\mathbf{w_x} + r + m)^n \langle \mathbf{w_x} + 1 \rangle_k \\
&= (\mathbf{w_x} + r + m)^n \langle \mathbf{w_x} + m + 1 \rangle_{p-1} \\
&\equiv (\mathbf{w_x} + m + r)^n (\mathbf{w_x} + m + p - 1)_{p-1} \\
&= \sum_{j=0}^{n} \left\{ {n+r \atop j+r} \right\}_r (\mathbf{w_x} + m)_j (\mathbf{w_x} + m + p - 1)_{p-1} \\
&= \sum_{j=0}^{n} \left\{ {n+r \atop j+r} \right\}_r (\mathbf{w_x} + m + p - 1)_{j+p-1} \\
&= \sum_{j=0}^{n} \left\{ {n+r \atop j+r} \right\}_r (j + p - 1)! \mathcal{T}_{j+p-1}(x; m - j) \\
&= (p-1)! \mathcal{T}_{p-1}(x; m) + \sum_{j=1}^{n} \left\{ {n+r \atop j+r} \right\}_r (j + p - 1)! \mathcal{T}_{j+p-1}(x; m - j) \\
&\equiv -r^n \mathcal{T}_{p-1}(x; m).
\end{aligned}
$$

□

Corollary 8. *Let $\mathcal{R}_{n,t}(x;r,s)$ be as in Corollary 7. Then, for any non-negative integers n,m,r,s and any prime $p \nmid m$, there holds*

$$
\sum_{k=m}^{p-1} (-x)^k \binom{k}{m} \frac{\mathcal{R}_{n,t}(x; r+k, k)}{k!} \equiv (-1)^m (r+m)^n \mathcal{L}_t(x, r+m) \mathcal{P}_{p-1}(x, m).
$$

Proof. Theorem 3 implies

$$
\begin{aligned}
LHS &= \sum_{j=0}^{t} a_j(x) \sum_{k=m}^{p-1} (-x)^k \binom{k}{m} \frac{w_{n+j}(x; r+k, k)}{k!} \\
&\equiv \sum_{j=0}^{t} a_j(x) (-1)^m (r+m)^{n+j} \mathcal{P}_{p-1}(x, m) \\
&\equiv (-1)^m (r+m)^n \mathcal{L}_t(x, r+m) \mathcal{P}_{p-1}(x, m).
\end{aligned}
$$

□

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Authors' addresses

Miloud Mihoubi
USTHB, Faculty of Mathematics, RECITS Laboratory, P. O. Box 32 El Alia 16111 Algiers, Algeria
E-mail address: mmihoubi@usthb.dz, miloudmihoubi@gmail.com

Said Taharbouchet
USTHB, Faculty of Mathematics, RECITS Laboratory, P. O. Box 32 El Alia 16111 Algiers, Algeria
E-mail address: staharbouchet@usthb.dz, said.taharbouchet@gmail.com