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THE GENERALIZED t-COMTET NUMBERS AND SOME COMBINATORIAL APPLICATIONS

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Abstract. In the present article we use a combinatorial approach to generalize the Comtet numbers. In particular, we establish some combinatorial identities, recurrence relations and generating functions. Additionally, for some particular cases we study their relationship with t-successive associated Stirling numbers and their q-analogue.

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1. INTRODUCTION

It is well-known that the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of partitions of a set with n elements into k non-empty blocks. This sequence satisfies the recurrence relation

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\},$$

with the initial conditions $\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0$.

The Stirling numbers $\left\{ {n \atop k} \right\}$ can be generalized to the *associated Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}_{\geq m}$ (cf. [1, 4, 7, 8, 10, 11, 18, 20])) by means of a restriction on the size of the blocks. In particular, this sequence gives the number of partitions of n elements into k blocks, such that each block contains at least m elements. It is clear that $\left\{ {n \atop k} \right\}_{\geq 1} = \left\{ {n \atop k} \right\}$. This combinatorial sequence has been applied to the study of some special polynomials such as generalized Bernoulli and Cauchy polynomials, (see, e.g., [12–16]).

Recently, Belbachir and Tebtoub [2] considered a variation for the associated Stirling numbers. They introduced the *2-successive associated Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}^{[2]}$. This new sequence counts the number of partitions of n elements

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into k blocks, with the additional condition that each block contains at least two consecutive elements. Moreover, the last element n must either form a block with its predecessor or belong to another block satisfying the previous conditions. In [2], the authors derived the recurrence

$$\left\{ {n \atop k} \right\}^{[2]} = k \left\{ {n-1 \atop k} \right\}^{[2]} + \left\{ {n-2 \atop k-1} \right\}^{[2]}, \; n \geq 2k,$$

with the initial conditions $\left\{ {0 \atop 0} \right\}^{[2]} = 1$, $\left\{ {n \atop n-1} \right\}^{[2]} = 0$ and $\left\{ {n \atop 0} \right\} = 0$ for $n \geq 1$.

Inspired by these results, in this paper we aim to investigate the sequence $\{a^{[t]}(n,k)\}_{n,k\geq 0}$, defined by the recurrence relation

$$a^{[t]}(n,k) = u_k a^{[t]}(n-1,k) + a^{[t]}(n-t,k-1), \quad n \geq tk, \tag{1.1}$$

with the initial conditions $a^{[t]}(0,0) = 1, a^{[t]}(n,n-\ell) = 0$ for $\ell = 1,2,\ldots,t-1$ and $a^{[t]}(n,0) = 0$, for $n \geq 1$. Moreover, $\{u_n\}$ is a sequence of real numbers.

We will call the sequence $\{a^{[t]}(n,k)\}_{n,k\geq 0}$ the *generalized t-Comtet numbers*. The reason for this name is that for $t = 1$ we recover the Comtet numbers (see, e.g., [9, 21]). Note that if $u_k = k$, then $a^{[t]}(n,k) = \left\{ {n \atop k} \right\}^{[t]}$. This sequence is called by Belbachir and Tebtoub [3] as the *t-successive associated Stirling numbers*. If $t = 2$ and $u_k = k$, then $a^{[2]}(n,k) = \left\{ {n \atop k} \right\}^{[2]}$. If $t = 1$ and $u_k = k$, then $a^{[1]}(n,k) = \left\{ {n \atop k} \right\}$.

In this paper our goal is to give the recurrence relation, the generating function and some combinatorial identities. For some particular cases, we give combinatorial interpretations.

2. Basic properties

From the recurrence relation (1.1) we obtain the following generating function.

Theorem 1. *For $k \geq 1$,*

$$A_k^{[t]}(x) := \sum_{n\geq tk} a^{[t]}(n,k)x^n = \frac{x^{tk}}{(1-u_0x)(1-u_1x)(1-u_2x)\cdots(1-u_kx)}, \tag{2.1}$$

with $A_0^{[t]}(x) = \frac{1}{1-u_0x}$.

Proof. Multiplying both sides of (1.1) by x^n and summing over $n \geq tk$, we have

$$\begin{aligned} A_k^{[t]}(x) &= u_k \sum_{n\geq tk} a^{[t]}(n-1,k)x^n + \sum_{n\geq tk} a^{[t]}(n-t,k-1)x^n \\ &= u_k x \sum_{n\geq tk} a^{[t]}(n,k)x^n + \sum_{n\geq tk-t} a^{[t]}(n,k-1)x^{n+t} \\ &= u_k x A_k^{[t]}(x) + x^t A_{k-1}^{[t]}(x). \end{aligned}$$

Then

$$A_k^{[t]}(x) = \frac{x^t A_{k-1}^{[t]}(x)}{1 - u_k x}.$$

Iterating this last recurrence, we obtain (2.1). □

From the above relation, we have the following combinatorial expression.

Corollary 1. *The generalized t-Comtet numbers are given by the explicit identity*

$$a^{[t]}(n,k) = \sum_{i_1+i_2+\cdots+i_k=n-tk} u_1^{i_1} u_2^{i_2} \cdots u_k^{i_k}, \tag{2.2}$$

for $n \geq tk$.

Theorem 2. *The generalized t-Comtet numbers satisfy the following recurrence relation*

$$a^{[t]}(n,k) = \sum_{i=0}^{n-tk} u_k^i a^{[t]}(n-i-t, k-1). \tag{2.3}$$

Proof. For $n \geq tk$,

$$\begin{array}{lll}
a(n,k) & = u_k a(n-1,k) & + a(n-t,k-1), \\
u_k a(n-1,k) & = u_k^2 a(n-2,k) & + u_k a(n-1-t,k-1), \\
u_k^2 a(n-2,k) & = u_k^3 a(n-3,k) & + u_k^2 a(n-2-t,k-1), \\
\vdots & \vdots \quad \vdots & \vdots \quad \vdots \\
u_k^{n-tk-1} a(tk+1,k) & = u_k^{n-tk} a(tk,k) & + u_k^{n-tk-1} a(tk+1-t,k-1), \\
u_k^{n-tk} a(tk,k) & = u_k^{n-tk+1} a(tk-1,k) & + u_k^{n-tk} a(t(k-1),k-1),
\end{array}$$

by summing, we get the result. □

Theorem 3. *We have the following rational explicit formula*

$$a^{[t]}(n+tk,k) = \sum_{j=0}^{k} \frac{u_j^{k+n}}{\prod_{i \neq j}(u_j - u_i)}, \tag{2.4}$$

which is independent from t.

Proof. We have

$$A_k^{[t]}(x) = \sum_{n \geq tk} a^{[t]}(n,k) x^n = x^{tk} \sum_{n \geq 0} a^{[t]}(n+tk,k) x^n,$$

then

$$\begin{aligned}\sum_{n\geq 0} a^{[t]}(n+tk,k)x^n &= \frac{1}{(1-u_0x)(1-u_1x)\cdots(1-u_kx)}\\ &= \sum_{j=0}^{k} \frac{\alpha_j}{1-u_jx}\\ &= \sum_{j=0}^{k} \frac{u_j^k}{\prod_{i\neq j}(u_j-u_i)} \sum_{n\geq 0} u_j^n x^n\\ &= \sum_{n\geq 0}\left(\sum_{j=0}^{k} \frac{u_j^{k+n}}{\prod_{i\neq j}(u_j-u_i)}\right)x^n,\end{aligned}$$

which gives the result. □

Corollary 2. *The dual expression depending on t*

$$a^{[t]}(n,k) = \sum_{j=0}^{k} \frac{u_j^{n+k(t-1)}}{\prod_{i\neq j}(u_j-u_i)}. \tag{2.5}$$

2.1. *Exponential generating function for the t-Comtet numbers*

Let $u_1,\dots,u_k$ be a sequence of complex numbers and let $(A_m)_{m=1,\dots,n}$ be the sequence of matrices such that A_m is $m\times m$-matrix

$$A_m = \begin{bmatrix} u_{k-m} & u_{k-m+1} & \cdots & u_{k-1}\\ u_{k-m+1} & \cdots & & u_k\\ \vdots & \ddots & \ddots & 0\\ u_{k-1} & u_k & 0 & 0 \end{bmatrix},$$

with the convention that $u_{<0}=0$.

Consider also

$$\sigma_j = (-1)^j \sum_{1\leq k_1<k_2<\cdots<k_j\leq k} u_{k_1}\cdots u_{k_j},$$

(the alternate sequence of elementary symmetric function associated to $u_1,u_2,\dots,u_k$).

We have $(p-u_1)(p-u_2)\cdots(p-u_k) = p^k+\sigma_1p^{k-1}+\sigma_2p^{k-2}+\cdots+\sigma_k$. Now we can state the following lemma which will be used to establish the main result of this subsection.

Lemma 1. *We have the following decomposition*

$$\frac{1}{p^n(p-u_1)\cdots(p-u_k)} = \sum_{i=0}^{n}\frac{\alpha_{n-i}}{p^i} + \sum_{j=1}^{k}\frac{\beta_j}{p-u_j}, \tag{2.6}$$

with $\alpha_i = \dfrac{(-1)^{\lfloor (i+1)/2 \rfloor}}{\sigma_k^{i+1}} \det(A_i)$, $\alpha_0 = 1/\sigma_k$, *and* $\beta_j = \dfrac{1}{u_j^n \prod_{\substack{i=1\\ i\neq j}}^{k} (u_j - u_i)}$.

Proof. We leave the proof to the reader. □

Let $C_k^{[t]}(x) := \sum_{n\geq tk} a^{[t]}(n,k)\frac{x^n}{n!}$, with $C_0^{[t]}(x) = 1$. We have

$$\frac{\partial^t}{\partial x^t} C_k^{[t]}(x) = \sum_{n\geq t(k-1)} a^{[t]}(n+t,k)\frac{x^n}{n!},$$

which gives using relation (1.1),

$$\frac{\partial^t}{\partial x^t} C_k^{[t]}(x) = u_k \frac{\partial^{t-1}}{\partial x^{t-1}} C_k^{[t]}(x) + C_{k-1}^{[t]}(x). \tag{2.7}$$

To solve the linear recurrence differential equation we use Laplace transform. Using the fact that

$$C_k^{[t]}(0) = \frac{\partial}{\partial x} C_k^{[t]}(y)\bigg|_{y=0} = \cdots = \frac{\partial^{t-1}}{\partial x^{t-1}} C_k^{[t]}(y)\bigg|_{y=0} = 0,$$

we get

$$\prod_{i=1}^{k} (p^t - u_i p^{t-1}) \mathcal{L}(C_k^{[t]}(y)) = \mathcal{L}(C_{k-1}^{[t]}),$$

where $\mathcal{L}(C_k^{[t]}(y)) = \int_0^\infty C_k^{[t]} e^{py} \partial y$.

Thus by recursion, we get

$$p^{(t-1)k} \prod_{i=1}^{k} (p - u_i) \mathcal{L}(C_k^{[t]}(y)) = \mathcal{L}(C_0^{[t]}(y)) = \mathcal{L}(u(y)),$$

where $u(t)$ is the Heaviside function. Using Lemma 1, we have

$$\mathcal{L}(C_k^{[t]}(y))) = \mathcal{L}(u(y)) \left[\sum_{i=0}^{(t-1)k} \frac{\alpha_i}{p^i} + \sum_{j=1}^{k} \frac{\beta_j}{p - u_j} \right].$$

The inverse Laplace transform gives,

$$C_k^{[t]}(y) = \sum_{i=1}^{(t-1)k} \alpha_i \frac{y^{i-1}}{(i-1)!} + \sum_{j=1}^{k} \beta_j e^{u_j y}. \tag{2.8}$$

Theorem 4. *The exponential generating function of t-Comtet numbers is given by*

$$\sum_{n\geq tk} a^{[t]}(n,k)\frac{x^n}{n!} = \sum_{i=1}^{(t-1)k} \alpha_i \frac{x^{i-1}}{(i-1)!} + \sum_{j=1}^{k} \beta_j e^{u_j x}. \tag{2.9}$$

3. The 2-successive associated r-Whitney numbers

In this section, we study the particular case $u_k = km + r$. Let $n, r \geq 0$ be integers. Let $\Pi_r(n,k)$ denote the set of partitions of the set $[n+r] := \{1,\ldots,n,n+1,\ldots,n+r\}$ into $k+r$ blocks, such that, the first r elements are in distinct blocks. The elements $\{1,2,\ldots,r\}$ will be called *special elements*. A block of a partition of the above set is called *special* if it contains special element. The cardinality of $\Pi_r(n,k)$ is the r-Stirling numbers of the second kind [5].

The 2-*successive associated r-Whitney numbers of the second kind*, denoted $W_{m,r}^{[2]}(n,k)$, count the number of partitions in $\Pi_r(n,k)$, such that:

- the k non-special blocks contain at least two consecutive numbers,
- all the elements but the last one and its predecessor in non-special blocks are coloured with one of m colours independently,
- the elements in the special blocks are not coloured,
- the last element $n+r$ must either form a block with its predecessor or belong to another block (special or not-special) satisfying the previous conditions.

We denote by $\Pi_{r,m}^{[2]}(n,k)$ the set of partitions in $\Pi_r(n,k)$ that satisfying the previous conditions. It is clear that if $r=0$ and $m=1$, then $W_{1,0}^{[2]}(n,k) = \left\{ {n \atop k} \right\}^{[2]}$, (see [2]).

For example, $W_{2,3}^{[2]}(5,2) = 15$ with the partitions being (the $m=2$ different colours of the elements will be fixed as **red** and *blue*, and the $r=3$ special elements are $\overline{1}, \overline{2}$ and $\overline{3}$):

$$\{\{\overline{1}\},\{\overline{2}\},\{\overline{3}\},\{\mathbf{4},5,6\},\{7,8\}\}, \quad \{\{\overline{1}\},\{\overline{2}\},\{\overline{3}\},\{\mathit{4},5,6\},\{7,8\}\},$$
$$\{\{\overline{1},6\},\{\overline{2}\},\{\overline{3}\},\{4,5\},\{7,8\}\}, \quad \{\{\overline{1}\},\{\overline{2},6\},\{\overline{3}\},\{4,5\},\{7,8\}\},$$
$$\{\{\overline{1}\},\{\overline{2}\},\{\overline{3},6\},\{4,5\},\{7,8\}\}, \quad \{\{\overline{1},4\},\{\overline{2}\},\{\overline{3}\},\{5,6\},\{7,8\}\},$$
$$\{\{\overline{1}\},\{\overline{2},4\},\{\overline{3}\},\{5,6\},\{7,8\}\}, \quad \{\{\overline{1}\},\{\overline{2}\},\{\overline{3},4\},\{5,6\},\{7,8\}\},$$
$$\{\{\overline{1},8\},\{\overline{2}\},\{\overline{3}\},\{4,5\},\{6,7\}\}, \quad \{\{\overline{1}\},\{\overline{2},8\},\{\overline{3}\},\{4,5\},\{6,7\}\},$$
$$\{\{\overline{1}\},\{\overline{2}\},\{\overline{3},8\},\{4,5\},\{6,7\}\}, \quad \{\{\overline{1}\},\{\overline{2}\},\{\overline{3}\},\{\mathbf{4},5,8\},\{6,7\}\},$$
$$\{\{\overline{1}\},\{\overline{2}\},\{\overline{3}\},\{\mathit{4},5,8\},\{6,7\}\}, \quad \{\{\overline{1}\},\{\overline{2}\},\{\overline{3}\},\{4,5\},\{\mathbf{6},7,8\}\},$$
$$\{\{\overline{1}\},\{\overline{2}\},\{\overline{3}\},\{4,5\},\{\mathit{6},7,8\}\}.$$

Theorem 5. *For $n \geq 2k$, we have*

$$W_{m,r}^{[2]}(n,k) = (km+r)W_{m,r}^{[2]}(n-1,k) + W_{m,r}^{[2]}(n-2,k-1). \tag{3.1}$$

Proof. For any set partition of $\Pi_{r,m}^{[2]}(n,k)$, there are three options: either $n+r$ form a block with its predecessor $(n+r-1)$, or $n+r$ is in a special block or $n+r$ is in a non-special block. In the first case, there are $W_{m,r}^{[2]}(n-2,k-1)$ possibilities. In the second case, the element $n+r$ can be place into one of the r special blocks and the remaining elements can be chosen in $W_{m,r}^{[2]}(n-1,k)$. Altogether, we have $rW_{m,r}^{[2]}(n-1,k)$ possibilities. For the third case, we can follow a similar argument, then we obtain $kmW_{m,r}^{[2]}(n-1,k)$ possibilities. □

A comparison of (3.1) and (1.1) shows that $a^{[2]}(n,k) = W_{m,r}^{[2]}(n,k)$ for $u_k = km + r$. Therefore, from Theorem 1 and Corollary 1 we get the following corollaries.

Corollary 3. *For $k \geq 1$,*

$$W_k^{[2]}(x) := \sum_{n\geq 2k} W_{m,r}^{[2]}(n,k)x^n$$
$$= \frac{x^{2k}}{(1-rx)(1-(m+r)x)(1-(2m+r)x)\cdots(1-(km+r)x)}, \tag{3.2}$$

with $W_0^{[2]}(x) = \frac{1}{1-rx}$. Moreover, the 2-successive associated r-Whitney numbers of the second kind are given by the explicit identity

$$W_{m,r}^{[2]}(n,k) = \sum_{i_0+i_1+i_2+\cdots+i_k=n-2k} r^{i_0}(m+r)^{i_1}\cdots(km+r)^{i_k}, \tag{3.3}$$

for $n \geq 2k$.

In particular, for $m = 1$ and $r = 0$ we obtain the generating function of the 2-successive associated Stirling numbers of the second kind.

Corollary 4. *(see [2, Theorem 2.3 and Corollary 2.4] and [3, Theorem 18]) For $k \geq 1$,*

$$A_k(x) := \sum_{n\geq 2k} \left\{ {n \atop k} \right\}^{[2]} x^n = \frac{x^{2k}}{(1-x)(1-2x)\cdots(1-kx)}, \tag{3.4}$$

with $A_0(x) = 1$. Moreover,

$$\left\{ {n \atop k} \right\}^{[2]} = \sum_{i_1+i_2+\cdots+i_k=n-2k} 1^{i_1}2^{i_2}\cdots k^{i_k}.$$

Our next identity expresses $W_{m,r}^{[2]}(n,k)$ in terms of $\left\{ {i \atop k} \right\}^{[2]}$ for $i \leq n$.

Theorem 6. *Let* $n,k \geq 0$,

$$W_{m,r}^{[2]}(n,k) = \sum_{i=2k}^{n} r^{n-i} \binom{n-k}{n-i} m^{i-2k} \left\{ {i \atop k} \right\}^{[2]}. \tag{3.5}$$

Proof. From (3.2) we have

$$\begin{aligned}
\sum_{n\geq 2k} W_{m,r}^{[2]}(n,k)x^n &= \frac{x^{2k}}{(1-rx)(1-(m+r)x)(1-(2m+r)x)\cdots(1-(km+r)x)} \\
&= \frac{x^{2k}}{(1-rx)^{k+1}\left(1-\frac{mx}{1-rx}\right)\left(1-\frac{2mx}{1-rx}\right)\cdots\left(1-\frac{kmx}{1-rx}\right)} \\
&= \frac{(1-rx)^{k-1}}{m^{2k}} \frac{\left(\frac{mx}{1-rx}\right)^{2k}}{\left(1-\frac{mx}{1-rx}\right)\left(1-\frac{2mx}{1-rx}\right)\cdots\left(1-\frac{kmx}{1-rx}\right)} \\
&= \frac{(1-rx)^{k-1}}{m^{2k}} \frac{y^{2k}}{(1-y)(1-2y)\cdots(1-kmy)},
\end{aligned}$$

where $y = \frac{mx}{1-rx}$.

Therefore from (3.4), we have

$$\begin{aligned}
\sum_{n\geq 2k} W_{m,r}^{[2]}(n,k)x^n &= \frac{(1-rx)^{k-1}}{m^{2k}} \sum_{i\geq 2k} \left\{ {i \atop k} \right\}^{[2]} y^i \\
&= \sum_{i\geq 2k} \left\{ {i \atop k} \right\}^{[2]} \frac{m^{i-2k}x^i}{(1-rx)^{i-k+1}} \\
&= \sum_{i\geq 2k} \sum_{j\geq 0} m^{i-2k} \binom{i-k+j}{i-k} r^j \left\{ {i \atop k} \right\}^{[2]} x^{i+j}.
\end{aligned}$$

Comparing the coefficients of x^n, we obtain (3.5). □

Combinatorial proof: We can construct any set partition of $\Pi_{r,m}^{[2]}(n,k)$ as follows: we put $n-i$ elements in the special blocks. Then there are $\binom{n-k}{i-k} r^{n-i}$ possibilities. Note that we have to subtract k elements of n because in the non-special blocks there are at least two consecutive numbers. The remaining i elements ($i \geq 2k$) can be chosen in $m^{i-2k}\left\{ {i \atop k} \right\}$ ways. The factor m^{i-2k} accounts for the $i-2k$ non-minimal elements within these blocks that are each to be colored in one of m ways.

From Theorem 5 and by induction on n we obtain the following identity.

Theorem 7. *For $n \geq 2k$ we have*

$$W_{m,r}^{[2]}(n,k) = \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj+r)^{n-k}.$$

Proof. Let,

$$\begin{aligned}
W_{m,r}^{[2]}(n,k) &= \frac{(mk+r)}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj+r)^{n-1-k} \\
&\quad + \frac{1}{m^{k-1} k!} \sum_{j=0}^{k} (-1)^{k-1-j} \binom{k-1}{j} (mj+r)^{n-1-k} \\
&= \frac{(mk+r)}{m^{k-1} k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} (mj+r)^{n-1-k} \\
&\quad + \frac{r}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj+r)^{n-1-k} \\
&= \frac{1}{m^k k!} \sum_{i=0}^{n-1-k} r^{n-1-k-i} m^i \left[\sum_{j=0}^{k} (-1)^{k-j} j^{i+1} \binom{k}{j} m \right. \\
&\quad \left. + \sum_{j=0}^{k} (-1)^{k-j} j^i r \binom{k}{j} \right] \\
&= \frac{1}{m^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (mj+r)^{n-k}.
\end{aligned}$$

□

3.1. *Relations with the r-Whitney numbers*

The r-Whitney numbers of the second kind $W_{m,r}(n,k)$ were defined by Mező [17] as the connecting coefficients between some particular polynomials.

For non-negative integers n,k and r with $n \geq k \geq 0$ and for any integer $m > 0$

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}}, \tag{3.6}$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ for $n \geq 1$, and $x^{\underline{0}} = 1$.

The r-Whitney numbers of the second kind satisfy the recurrence [17]

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r) W_{m,r}(n-1,k). \tag{3.7}$$

Comparing (3.7) and (3.1) we have the following relation.

Corollary 5. *[2, Theorem 4.1] For $n \geq 2k$,*

$$W_{m,r}^{[2]}(n,k) = W_{m,r}(n-k,k). \tag{3.8}$$

Mező and Ramírez [19] studied the r-Whitney matrices of the second and the first kind and they derived several identities for these matrices. In particular, the r-Whitney matrix of the second kind is defined by

$$[W_{m,r}(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 \\ r^2 & m+2r & 1 & 0 & 0 \\ r^3 & m^2+3rm+3r^2 & 3m+3r & 1 & 0 \\ r^4 & m^3+4rm^2+6r^2m+4r^3 & 7m^2+12rm+6r^2 & 6m+4r & 1 \\ \vdots & & \vdots & & \vdots \end{bmatrix}.$$

Notice that the sequence $(W_{m,r}^{[2]}(n,k))_k$ corresponds with the sequence of elements on rays in direction $(1,1)$ over the r-Whitney matrix of the second kind.

4. The t-successive associated r-Whitney numbers

In this section, we consider the rays in direction $(s,1)$, i.e., we are going to study the sequence $\{W_{m,r}(n-sk,k)\}$. We denote by $W_{m,r}^{[t]}(n,k)$ the number $W_{m,r}(n-sk,k)$, where $t = s+1$. We call this new sequence *the t-successive associated r-Whitney numbers of the second kind*. It is possible to show that the t-successive associated r-Whitney numbers count the number of partitions in $\Pi_r(n,k)$, such that:

- the k non-special blocks contain at least t consecutive numbers,
- all the elements but the last one and its $t-1$ predecessors in non-special blocks are coloured with one of m colours independently,
- the elements in the special blocks are not coloured,
- the last element $n+r$ must either form a block with its $t-1$-predecessors or belong to another block (special or not-special) satisfying the previous conditions.

Reasoning in a similar manner as in Theorem 5 we obtain the following results.

Theorem 8. *For $n \geq tk$, we have*

$$W_{m,r}^{[t]}(n,k) = (km+r)W_{m,r}^{[t]}(n-1,k) + W_{m,r}^{[t]}(n-t,k-1). \tag{4.1}$$

For $k \geq 1$,

$$W_k^{[t]}(x) := \sum_{n\geq 0} W_{m,r}^{[t]}(n,k)x^n$$
$$= \frac{x^{tk}}{(1-rx)(1-(m+r)x)(1-(2m+r)x)\cdots(1-(km+r)x)}, \tag{4.2}$$

with $W_0^{[t]}(x) = \frac{1}{1-rx}$*. Moreover, for* $n \geq tk$ *we have*

$$W_{m,r}^{[t]}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(mj+r)^{n-(t-1)k}. \tag{4.3}$$

As corollary for $t=2$ we get [3, Theorem 4, Theorem 6 and Theorem 7].

It is not difficult to generalize the relation given in Theorem 6.

Theorem 9. *If* $n,k \geq 0$*, then*

$$W_{m,r}^{[t]}(n,k) = \sum_{i=tk}^{n} r^{n-i}\binom{n-k}{n-i} m^{i-tk} \left\{ {i \atop k} \right\}^{[t]}. \tag{4.4}$$

Consequence. From Equation (4.2) we deduce that $W_{m,r}^{[t]}(n+(t-1)k,k)$ are the classical r-Whitney numbers $W_{m,r}(n,k)$.

From the explicit formula given in (4.3) we get the exponential generating function of the t-successive associated r-Whitney numbers.

Theorem 10. *The exponential generating function of the t-successive associated r-Whitney numbers is*

$$W_k^{[t]}(x) := \sum_{n\geq tk} W_{m,r}^{[t]}\frac{x^n}{n!} = \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}}{k!m^k}\frac{e^{(jm+r)x}}{(jm+r)^{(t-1)k}}. \tag{4.5}$$

Corollary 6. *For the* 2*-successive associated r-Whitney numbers,*

$$W_k^{[2]}(x) = \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}}{k!m^k}\frac{e^{(jm+r)x}}{(jm+r)^{k}}. \tag{4.6}$$

These two result are more specified expressions as relation (2.9) of Theorem 4.

Proof. (Theorem 10) We use the derivation $(t-1)k$ times according to x and using the consequence property, we get

$$\begin{aligned}
\frac{\partial^{(t-1)k}}{\partial^{(t-1)k}x}W_k(x) &= \sum_{n\geq tk-(t-1)k} W_{m,r}^{[t]}(n+k(t-1),k)\frac{x^n}{n!} \\
&= \frac{1}{k!m^k}e^{rx}(e^{mx}-1)^k \\
&= \frac{1}{k!m^k}e^{rx}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{jmx} \\
&= \sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{k-j}}{k!m^k}e^{(jm+r)x}.
\end{aligned}$$

□

Theorem 11. *For $n \geq k$, we have*

$$W_{m,r}^{[t]}(n+(t-1)k,k) = \sum_{\substack{i_1+\cdots+i_n=n-k \\ i_1,\cdots,i_n\in\{0,1\}}} \prod_{j=0}^{n-1}(r+m(j-\sum_{\ell=1}^{j} i_\ell))^{i_{j+1}}. \tag{4.7}$$

Proof. By induction over n, we suppose that the identity is true until $n-1$

$$\sum_{\substack{i_1+\cdots+i_n=n-k \\ i_1,\cdots,i_n\in\{0,1\}}} \prod_{j=0}^{n-1}(r+m(j-\sum_{l=1}^{j} i_l))^{i_{j+1}}$$

$$= \sum_{\substack{i_1+\cdots+i_{n-1}=(n-1)-(k-1) \\ i_1,\cdots,i_{n-1}\in\{0,1\}}} \prod_{j=0}^{n-2}(r+m(j-\sum_{l=1}^{j} i_l))^{i_{j+1}}$$

$$+\left(\sum_{\substack{i_1+\cdots+i_{n-1}=(n-1)-k \\ i_1,\cdots,i_{n-1}\in\{0,1\}}} \prod_{j=0}^{n-2}(r+m(j-\sum_{l=1}^{j} i_l))^{i_{j+1}}\right)(r+mk).$$

We have

$$W_{m,r}^{[t]}(n+(t-1)k,k)$$
$$= (mk+r)W_{m,r}^{[t]}(n-1+(t-1)k,k) + W_{m,r}^{[t]}(n-1+(t-1)(k-1),k),$$

which gives the desired result. □

Corollary 7. *For $n \geq tk$, we have*

$$W_{m,r}^{[t]}(n,k) = \sum_{\substack{i_1+\cdots+i_{n-(t-1)k}=n-tk \\ i_1,\cdots,i_{n-(t-1)k}\in\{0,1\}}} \prod_{j=0}^{n-(t-1)k-1} (r+m(j-\sum_{\ell=1}^{j} i_\ell))^{i_{j+1}}, \tag{4.8}$$

with empty sum equal zero.

Example 1. For $k = t = 2$ we have the following formula

$$W_{m,r}^{[2]}(n,2) = \sum_{i_1+\cdots+i_{n-2}=n-4} r^{i_1}(r+m(1-i_1))^{i_2}(r+m(2-i_1-i_2))^{i_3}\times$$
$$\cdots\times(r+m(n-3-i_1-i_2-\cdots-i_{n-3}))^{i_{n-2}},$$

$$W_{m,r}^{[2]}(4,2) = \sum_{i+j=0} r^i(r+m(1-i))^j = 1,$$

$$W^{[2]}_{m,r}(5,2) = \sum_{i+j+k=1} r^i(r+m(1-i))^j(r+m(2-i-j))^k = 3(r+m),$$

$$\begin{aligned}W^{[2]}_{m,r}(6,2) &= \sum_{i+j+k+\ell=2} r^i(r+m(1-i))^j(r+m(2-i-j))^k(r+m(3-i-j-k))^\ell \\ &= 6r^2+12mr+7m^2.\end{aligned}$$

Theorem 12. *We have the following explicit formula*

$$W^{[t]}_{m,r}(n+tk,k) = \frac{1}{m^k k!}\sum_{j=0}^{k-j}(-1)^{k-j}\binom{k}{j}(mj+r)^{n+k}, \tag{4.9}$$

and thus

$$W^{[t]}_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(mj+r)^{n-(t-1)k}. \tag{4.10}$$

Proof. It suffices to set $u_k = mk+r$ then $\left[u_j - u_i = m(j-i)\right]$ in Theorem 3. □

For $t=2$ we get Theorem 7.

Theorem 13. *Expression of t-successive r-Whitney numbers in terms of binomials and Stirling numbers.*

$$W^{[t]}_{m,r}(n+tk,k) = \left(\frac{r}{m}\right)^k \sum_{i=0}^{n+k}\binom{n+k}{i} r^{n-i} m^i \left\{ {i \atop k} \right\}. \tag{4.11}$$

Proof.

$$\begin{aligned}W^{[t]}_{m,r}(n+tk,k) &= \frac{1}{m^k k!}\sum_{j=0}^{k}\sum_{i=0}^{n+k}(-1)^{k-j}\binom{k}{j}\binom{n+k}{i}(mj)^i r^{n+k-i} \\ &= \frac{1}{m^k k!}\sum_{i=0}^{n+k} m^i r^{n+k-i}\binom{n+k}{i}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^i \\ &= \frac{1}{m^k}\sum_{i=0}^{n+k} m^i r^{n+k-i}\binom{n+k}{i}\left\{ {i \atop k} \right\}.\end{aligned}$$

Notice that $\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^i = 0$ for $i<k$. □

5. A q-ANALOGUE OF THE t-SUCCESSIVE ASSOCIATED STIRLING NUMBERS

Finally, we considerer a q-analogue of the t-successive associated Stirling numbers of the second kind. For this purpose, we use a similar statistic studied by Carlitz [6], see also [21].

Let $\pi = B_1/B_2/\cdots/B_k$ be any block representation of a set partition in $\Pi_{0,1}^{[t]}(n,k) := \Pi^{[t]}(n,k)$, with $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. We define the following statistic on the set $\Pi^{[t]}(n,k)$.

$$w^{[t]}(\pi) := \sum_{i=1}^{k}(i-1)(|B_i| - t + 1).$$

We now define the *q-analogue of the t-successive associated Stirling numbers of the second kind.*

Definition 1. Define $\left\{ n \atop k \right\}_q^{[t]}$ as the distribution polynomial for the $w^{[t]}$ statistic on the set $\Pi^{[t]}(n,k)$, that is,

$$\left\{ n \atop k \right\}_q^{[t]} = \sum_{\pi \in \Pi^{[t]}(n,k)} q^{w^{[t]}(\pi)}, \qquad n,k \geq 0,$$

where q is an indeterminate.

It is clear that $\left\{ n \atop k \right\}_1^{[t]} = \left\{ n \atop k \right\}^{[t]}$.

For example, in the set $\Pi^{[2]}(7,3)$ we have the following partitions:

$$\{\{1,2\},\{3,4,5\},\{6,7\}\}, \quad \{\{1,2,3\},\{4,5\},\{6,7\}\}, \quad \{\{1,2,5\},\{3,4\},\{6,7\}\},$$
$$\{\{1,2\},\{3,4\},\{5,6,7\}\}, \quad \{\{1,2\},\{3,4,7\},\{5,6\}\}, \quad \{\{1,2,7\},\{3,4\},\{5,6\}\}.$$

Therefore,

$$\left\{ 7 \atop 3 \right\}_q^{[2]} = q^4 + q^3 + q^3 + q^5 + q^4 + q^3 = 3q^3 + 2q^4 + q^5.$$

Let us introduce the following notations.

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q[2]_q \cdots [n]_q \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The last coefficient is called q-binomial coefficient. If $q = 1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$.

Theorem 14. *For $n \geq tk$, we have*

$$\left\{ n \atop k \right\}_q^{[t]} = [k]_q \left\{ n-1 \atop k \right\}_q^{[t]} + q^{k-1} \left\{ n-t \atop k-1 \right\}_q^{[t]}. \tag{5.1}$$

Proof. For any set partition of $\Pi^{[t]}(n,k)$, there are two options: either n form a block with its $t-1$-predecessors, or n is in a block that satisfies the conditions. In the first case, there are $q^{k-1}\left\{ n-t \atop k-1 \right\}_q^{[t]}$ possibilities. In this case, the size of the last block B_k is t, then this block contributes a factor q^{k-1}. In the second case, the element n can be place into one of the k blocks and thus contributes a factor $1+q+q^2+\cdots+q^{k-1}=[k]_q$. Moreover, the remaining elements can be chosen in $\left\{ {n-1 \atop k} \right\}_q^{[t]}$ ways. Altogether, we have $[k]_q \left\{ {n-1 \atop k} \right\}_q^{[t]}$ possibilities. □

From above theorem, we obtain the following corollaries.

Corollary 8. *For $k \geq 1$,*

$$\sum_{n \geq tk} \left\{ {n \atop k} \right\}_q^{[t]} x^n = \frac{x^{tk} q^{\binom{k}{2}}}{(1-x)(1-[2]_q x)(1-[3]_q x)\cdots(1-[k]_q x)}. \tag{5.2}$$

Moreover, the q-analogue of the t-successive associated r-Stirling numbers are given by the explicit identity

$$\left\{ {n \atop k} \right\}_q^{[t]} = q^{\binom{k}{2}} \sum_{i_1+i_2+\cdots+i_k=n-tk} [1]_q^{i_1} [2]_q^{i_2} \cdots [k]_q^{i_k}, \tag{5.3}$$

for $n \geq tk$.

Corollary 9. *For $n \geq tk$ we have*

$$\left\{ {n \atop k} \right\}_q^{[t]} = \frac{1}{[k]_q!} \sum_{j=1}^{k} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} ([j]_q)^{n-(t-1)k}.$$

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