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CERTAIN DISCRETE BESSEL CONVOLUTIONS OF THE APPELL POLYNOMIALS

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Abstract. In this article, the Appell polynomials are combined with Bessel functions to introduce a hybrid family namely the Appell-Bessel functions. The infinite sums and Jacobi-Anger expansions for this family are established. The Bernoulli-Bessel and Euler-Bessel functions are introduced as particular cases of the Appell-Bessel functions and their properties are obtained.

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1. INTRODUCTION AND PRELIMINARIES

The importance of the generalized Bessel functions stems from their wide use in applications and from their implications in different fields of pure and applied mathematics. The scattering of free or weakly bounded electrons by intense laser fields is an example where generalized Bessel functions play an important role. The spectral properties of synchrotron radiation by relativistic electrons passing in magnetic undulators, the gain of free electron lasers operating on higher off-axis harmonics are the examples where generalized Bessel functions play a crucial role. The analytical and numerical study of generalized Bessel functions has revealed their interesting properties, which in some sense can be regarded as an extension of the properties of Bessel functions to a two-dimensional domain. In this connection, the relevance of generalized Bessel functions and their multi-variable extension in mathematical physics has been emphasized, since they provide analytical solutions to partial differential differential equations such as the multi-dimensional diffusion equation, the Schrӧdinger and Klein-Gordon equations. The algebraic structure underlying generalized Bessel functions can be recognized in full analogy with Bessel functions, thus providing a unifying view to the theory of both Bessel and generalized Bessel functions. Hence the interest for the generalized Bessel functions is justified, for details see [3] and references therein.

We recall that the 2-variable Bessel functions $J_n(x,y)$ are defined by the following generating function [3, p. 332 (2.7)]:

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right)=\sum_{n=-\infty}^{\infty} J_n(x,y)t^n. \tag{1.1}$$

The generalized Bessel functions of the above type are getting more and more importance in physics, mainly in connection with spontaneous or stimulated scattering processes for which the dipole approximation is inadequate.

Sequences of polynomials play a fundamental role in applied mathematics. One of the important classes of polynomial sequences is the class of Appell polynomial sequences [1]. The Appell polynomial sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. In the past few decades, there has been a renewed interest in Appell sequences.

Recall that, in 1880, Appell [1] introduced and studied sequences of n-degree polynomials $A_n(x),\ n=0,1,2,\cdots$, satisfying the recurrence relation

$$\frac{d}{dx}\,A_n(x)\ = n\ A_{n-1}(x), \quad n=1,2,\ldots \quad . \tag{1.2}$$

The exponential generating function of Appell polynomials is defined as:

$$A(t)e^{xt}=\sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.3}$$

where $A(t)$ is formal power series of the form

$$A(t)=\sum_{n=0}^{\infty} a_n\frac{t^n}{n!}, \quad a_0\neq 0. \tag{1.4}$$

The study of the properties of ordinary and generalized polynomials is, sometimes greatly simplified by the use of concepts and formalisms associated with the "monomiality principle" [2,7]. It has shown in [4] that the Appell polynomials $A_n(x)$ are *quasi-monomial* with respect to the following "multiplicative" and "derivative" operators:

$$\hat{M}_A = x+\frac{A'(\partial_x)}{A(\partial_x)}, \tag{1.5}$$

and

$$\hat{P}_A=\partial_x, \tag{1.6}$$

respectively, that is $\hat{M}_A$ and $\hat{P}_A$ satisfy the following relations:

$$\hat{M}_A\{A_n(x)\}=A_{n+1}(x), \tag{1.7}$$

$$\hat{P}_A\{A_n(x)\}=nA_{n-1}(x), \tag{1.8}$$

for all $n \in \mathbb{N}$. The operators $\hat{M}_A$ and $\hat{P}_A$ also satisfy the commutation relation

$$[\hat{P}_A, \hat{M}_A] = \hat{P}_A \hat{M}_A - \hat{M}_A \hat{P}_A = \hat{1} \tag{1.9}$$

and thus display the Weyl group structure. In view of the monomiality principle, other properties of the Appell polynomials can be derived from those of the $\hat{M}_A$ and $\hat{P}_A$ operators. In fact:

(i) Combining recurrences (1.7) and (1.8), it follows that

$$\hat{M}_A \, \hat{P}_A\{A_n(x)\} = n \; A_n(x), \tag{1.10}$$

which yields the following differential equation satisfied by $A_n(x)$:

$$\left(x\partial_x + \frac{A'(\partial_x)}{A(\partial_x)}\partial_x - n \right) A_n(x) = 0. \tag{1.11}$$

(ii) Since $A_0(x) = 1$, therefore $A_n(x)$ can be explicitly constructed as:

$$A_n(x) = \hat{M}_A^n\{1\}, \tag{1.12}$$

which yields the following series definition of $A_n(x)$:

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} A_k x^{n-k}. \tag{1.13}$$

(iii) Identity (1.12) implies that the exponential generating function of $A_n(x)$ can be given in the form:

$$\exp(t\hat{M}_A)\{1\} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \qquad |t| < \infty, \tag{1.14}$$

which yields generating function (1.3) of $A_n(x)$.

In this paper, the Appell-Bessel functions are introduced by means of generating function and their properties are studied. The corresponding results for the Bernoulli-Bessel and Euler-Bessel functions are also obtained.

2. Appell-Bessel functions ${}_AJ_n(x, y)$

First, we introduce the Appell-Bessel functions by means of generating function.

Replacing x by the multiplicative operator $\hat{M}_A$ of the Appell polynomials $A_n(x)$ in the l.h.s. of generating function (1.1) and denoting the resultant Appell-Bessel functions in the r.h.s. by ${}_AJ_n(x, y)$, we have

$$\exp\left(\frac{\hat{M}_A}{2}\left(t - \frac{1}{t} \right) \right) \exp\left(\frac{y}{2}\left(t^2 - \frac{1}{t^2} \right) \right) = \sum_{n=-\infty}^{\infty} {}_AJ_n(x, y) t^n, \tag{2.1}$$

which in view of equation (1.14) with t replaced by $\frac{1}{2}\left(t-\frac{1}{t}\right)$ becomes

$$\exp\left(\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right)\sum_{n=0}^{\infty}A_n(x)\frac{\left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)^n}{n!}=\sum_{n=-\infty}^{\infty}{}_AJ_n(x,y)t^n. \tag{2.2}$$

By virtue of equation (1.3), we find the following generating function for the Appell-Bessel functions ${}_AJ_n(x,y)$:

$$A\left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right)=\sum_{n=-\infty}^{\infty}{}_AJ_n(x,y)t^n. \tag{2.3}$$

Note: Since

$$\sum_{n=-\infty}^{\infty}J_n(x,y)=1, \tag{2.4}$$

therefore, in view of equation (2.3), we have

$$\sum_{n=-\infty}^{\infty}{}_AJ_n(x,y)=A(0). \tag{2.5}$$

Next, we establish certain properties of the Appell-Bessel functions ${}_AJ_n(x,y)$.

Differentiating generating function (2.3) partially w.r.t. x and y respectively, the following recurrence relations for the Appell-Bessel functions ${}_AJ_n(x,y)$ are obtained:

$$\frac{\partial}{\partial x}{}_AJ_n(x,y)=\frac{1}{2}\Big[{}_AJ_{n-1}(x,y)-{}_AJ_{n+1}(x,y)\Big]. \tag{2.6}$$

$$\frac{\partial}{\partial y}{}_AJ_n(x,y)=\frac{1}{2}\Big[{}_AJ_{n-2}(x,y)-{}_AJ_{n+2}(x,y)\Big]. \tag{2.7}$$

Consequently, the following relations are obtained:

$$\frac{\partial^k}{\partial x^k}{}_AJ_n(x,y)=\frac{1}{2^k}\Big[{}_AJ_{n-k}(x,y)-\binom{k}{1}{}_AJ_{n-k+2}(x,y)+\binom{k}{2}{}_AJ_{n-k+4}(x,y)+\cdots+(-1)^k{}_AJ_{n+k}(x,y)\Big]. \tag{2.8}$$

$$\frac{\partial^k}{\partial y^k}{}_AJ_n(x,y)=\frac{1}{2^k}\Big[{}_AJ_{n-2k}(x,y)-\binom{k}{1}{}_AJ_{n-2k+4}(x,y)$$

$$+\binom{k}{2}{}_AJ_{n-2k+8}(x,y)+\cdots+(-1)^k{}_AJ_{n+2k}(x,y)\Big]. \tag{2.9}$$

Also, we have the following relations involving partial derivatives

$$\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^k\left[x^{\pm n}{}_AJ_n(x,y)\right]=x^{\pm n-k}(\pm 1)^k{}_AJ_{n-k}(x,y). \tag{2.10}$$

$$\left(\frac{1}{y}\frac{\partial}{\partial y}\right)^k\left[y^{\pm n}{}_AJ_n(x,y)\right]=y^{\pm n-k}(\pm 1)^k{}_AJ_{n-2k}(x,y). \tag{2.11}$$

As a consequence of the relations (2.8) and (2.9), we also have

$$\frac{\partial^{k+m}}{\partial x^k\partial y^m}{}_AJ_n(x,y)=\frac{1}{2^{k+m}}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\sum_{j=0}^{m}(-1)^j\binom{m}{j}{}_AJ_{n-k-2m+2i+4j}(x,y). \tag{2.12}$$

The Jacobi-Anger expansion is useful in physics (for example, in conversion of plane waves and the cylindrical waves) and in signal processing (to describe frequency modulation signals). The following Jacobi-Anger type expansion involving Appell-Bessel functions ${}_AJ_n(x,y)$ is obtained by setting $t=e^{i\phi}$ in generating function (2.3):

$$A(i\sin\phi)\,(\cos\alpha+i\sin\alpha)=\sum_{n=-\infty}^{\infty}{}_AJ_n(x,y)e^{in\phi}, \tag{2.13}$$

where $\alpha:=x\sin\phi+y\sin(2\phi)$.

In the next section, the Bernoulli-Bessel and Euler-Bessel functions are considered as members of the Appell-Bessel family.

3. BERNOULLI-BESSEL AND EULER-BESSEL FUNCTIONS

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are important members of the Appell family. The Bernoulli polynomials are employed in the integral representation of differentiable periodic functions and play an important role in the approximation of such functions by means of polynomials. They are also used in the remainder term of the composite Euler Maclaurin quadrature formula. The Euler polynomials are strictly connected with the Bernoulli ones and enter in the Taylor expansion in a neighborhood of the origin of the trigonometric and hyperbolic secant functions.

Since for $A(t)=\frac{t}{e^t-1}$, the Appell polynomials reduce to the Bernoulli polynomials, therefore taking $A\left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)=\frac{\frac{1}{2}\left(t-\frac{1}{t}\right)}{e^{\frac{1}{2}\left(t-\frac{1}{t}\right)}-1}$ in the l.h.s. of generating function (2.3) and denoting the resultant Bernoulli-Bessel functions in the r.h.s. by ${}_BJ_n(x,y)$, the following generating function of the Bernoulli-Bessel functions ${}_BJ_n(x,y)$ is obtained:

$$\frac{\frac{1}{2}\left(t-\frac{1}{t}\right)}{e^{\frac{1}{2}\left(t-\frac{1}{t}\right)}-1}\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right)=\sum_{n=-\infty}^{\infty}{}_BJ_n(x,y)t^n. \tag{3.1}$$

Similarly, for $A(t)=\frac{2}{e^t+1}$, the Appell polynomials reduce to the Euler polynomials, therefore taking $A\left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)=\frac{2}{e^{\frac{1}{2}\left(t-\frac{1}{t}\right)}+1}$ in the l.h.s. of generating function (2.3) and denoting the resultant Euler-Bessel functions in the r.h.s. by ${}_EJ_n(x,y)$, the following generating function of the Euler-Bessel functions ${}_EJ_n(x,y)$ is obtained:

$$\frac{2}{e^{\frac{1}{2}\left(t-\frac{1}{t}\right)}+1}\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right)=\sum_{n=-\infty}^{\infty}{}_EJ_n(x,y)t^n. \tag{3.2}$$

For the Bernoulli-Bessel functions ${}_BJ_n(x,y)$ and Euler-Bessel functions ${}_EJ_n(x,y)$, we have

$$\lim_{t\to 1}\sum_{n=-\infty}^{\infty}{}_BJ_n(x,y)t^n=1 \tag{3.3}$$

and

$$\sum_{n=-\infty}^{\infty}{}_EJ_n(x,y)=1, \tag{3.4}$$

respectively.

Further, the following Jacobi-Anger type expansions of the Bernoulli-Bessel and Euler-Bessel functions are obtained:

$$\frac{\sin\phi\,(\sin(\sin\phi-\alpha)+\sin\alpha)}{4\sin^2\left(\frac{\sin\phi}{2}\right)}=\sum_{n=-\infty}^{\infty}{}_BJ_n(x,y)\cos(n\phi); \tag{3.5}$$

$$\frac{\sin\phi\,(\cos(\sin\phi-\alpha)-\cos\alpha)}{4\sin^2\left(\frac{\sin\phi}{2}\right)}=\sum_{n=-\infty}^{\infty}{}_BJ_n(x,y)\sin(n\phi) \tag{3.6}$$

and

$$\frac{\cos(\alpha-\sin\phi)+\cos\alpha}{2\cos^2\left(\frac{\sin\phi}{2}\right)}=\sum_{n=-\infty}^{\infty}{}_EJ_n(x,y)\cos(n\phi); \tag{3.7}$$

$$\frac{\sin(\alpha-\sin\phi)+\sin\alpha}{2\cos^2\left(\frac{\sin\phi}{2}\right)}=\sum_{n=-\infty}^{\infty}{}_EJ_n(x,y)\sin(n\phi), \tag{3.8}$$

where $\alpha:=x\sin\phi+y\sin(2\phi)$.

4. Concluding remarks

The Appell family generated by (1.3) is obviously rather restrictive; it does not allow the treatment of some other polynomial sets on the Laguerre or the Bessel polynomials within the context of the operational formalism. Recently, Dattoli *et. al.* [5] have shown that the extension of Appell family to Sheffer family [6] allows such a possibility. To give an illustration of this approach, we extend the Appell-Bessel functions to the Sheffer-Bessel functions.

A polynomial sequence $s_n(x)$ $(n = 0, 1, 2, ...)$, is called Sheffer polynomials [6], if $s_n(x)$ possesses the following exponential generating function:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{4.1}$$

where $A(t)$ has expansion (1.4) and $H(t)$ is a power series such that

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, \quad h_1 \neq 0. \tag{4.2}$$

Clearly the Appell polynomials belong to the Sheffer family. If $s_n(x)$ are of Sheffer type, then it is always possible to find the explicit representations of the multiplicative and derivative operators $\hat{M}$ and $\hat{P}$. Among the polynomials encountered in quantum mechanics, Hermite and Laguerre polynomials are of Sheffer type, whereas Legendre, Jacobi and Gegenbauer polynomials are not.

Proceeding on the same lines as in the case of Appell-Bessel functions, following generating function for the Sheffer-Bessel functions ${}_sJ_n(x,y)$ is obtained:

$$A\left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)\exp\left(xH\left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right) = \sum_{n=-\infty}^{\infty} {}_sJ_n(x,y)t^n. \tag{4.3}$$

We note that for $A(t) = \exp(-t^2)$ and $H(t) = 2t$, the Sheffer polynomials reduce to the Hermite polynomials $H_n(x)$ [6]. Hence from equation (4.3), the following generating function for the Hermite-Bessel functions ${}_HJ_n(x,y)$ is obtained:

$$\exp\left(x\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)-\frac{1}{4}\left(t-\frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_HJ_n(x,y)t^n. \tag{4.4}$$

The following recurrence relations for the Hermite-Bessel functions ${}_HJ_n(x,y)$ are obtained:

$$\frac{\partial}{\partial x}{}_HJ_n(x,y) = ({}_HJ_{n-1}(x,y) - {}_HJ_{n+1}(x,y)); \tag{4.5}$$

$$\frac{\partial}{\partial y} {}_H J_n(x,y) = \frac{1}{2}\left({}_H J_{n-2}(x,y) - {}_H J_{n+2}(x,y)\right). \tag{4.6}$$

On taking $t = e^{i\phi}$ in equation (4.4), the following Jacobi-Anger type expansions of the Hermite-Bessel functions ${}_H J_n(x,y)$ are obtained:

$$\cos\beta \exp\left(\sin^2\phi\right) = \sum_{n=-\infty}^{\infty} {}_H J_n(x,y)\cos(n\phi) \tag{4.7}$$

and

$$\sin\beta \exp\left(\sin^2\phi\right) = \sum_{n=-\infty}^{\infty} {}_H J_n(x,y)\sin(n\phi), \tag{4.8}$$

where $\beta := 2x\sin\phi + y\sin(2\phi)$.

From equation (4.4), the following infinite sums for the Hermite-Bessel functions of even and odd indices are obtained:

$$\cosh\left(x\left(t-\frac{1}{t}\right)\right)\exp\left(\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)-\frac{1}{4}\left(t-\frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n}(x,y)t^{2n} \tag{4.9}$$

and

$$\sinh\left(x\left(t-\frac{1}{t}\right)\right)\exp\left(\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)-\frac{1}{4}\left(t-\frac{1}{t}\right)^2\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n+1}(x,y)t^{2n+1}. \tag{4.10}$$

Setting $t = e^{i\phi}$ in equations (4.9) and (4.10), the following relations are deduced:

$$\cos(2x\sin\phi)\cos(y\sin(2\phi))\exp(\sin^2\phi) = \sum_{n=-\infty}^{\infty} {}_H J_{2n}(x,y)\cos(2n\phi); \tag{4.11}$$

$$\cos(2x\sin\phi)\sin(y\sin(2\phi))\exp\left(\sin^2\phi\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n}(x,y)\sin(2n\phi); \tag{4.12}$$

$$-\sin(2x\sin\phi)\sin(y\sin(2\phi))\exp\left(\sin^2\phi\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n+1}(x,y)\cos((2n+1)\phi); \tag{4.13}$$

$$\sin(2x\sin\phi)\cos(y\sin(2\phi))\exp\left(\sin^2\phi\right) = \sum_{n=-\infty}^{\infty} {}_H J_{2n+1}(x,y)\sin((2n+1)\phi). \tag{4.14}$$

Also from equation (4.4), we find the following integral representation for the Hermite-Bessel functions ${}_H J_n(x,y)$ as

$${}_H J_n(x,y) = \frac{1}{\pi}\int_0^{\pi} e^{\sin^2\phi}\cos\left(2x\sin\phi + y\sin(2\phi) - n\phi\right)d\phi. \tag{4.15}$$

The Sheffer polynomials, which include Appell polynomials as a special case along with the underlying operational formalism, offer a powerful tool for investigation of the properties of a wide class of special functions. Here, we have considered Hermite-Bessel functions as one of the members of the Sheffer-Bessel family. A general approach to generate Sheffer-Bessel functions will be discussed in a forthcoming investigation.

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