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AN INEQUALITY FOR THE MODULUS OF THE RATIO OF TWO COMPLEX GAMMA FUNCTIONS

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Abstract. The Euler gamma function is closely connected with the theory of zeta-functions. We prove a new inequality for the modulus of the ratio of two complex gamma functions $\Gamma(s)/\Gamma(2-s)$, arising in problems of the size of Selberg zeta-functions at places symmetric with respect to the critical line. This inequality, used together with technics of estimation, allows us in a different way re-prove and extend the result of R. Garunkštis and A. Grigutis for the modified Selberg zeta-function.

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1. INTRODUCTION

The Euler gamma function is closely connected with zeta-functions, and its properties are of great importance to the theory of zeta-functions and applications. E.g., the Riemann zeta-function satisfies the well-known functional equation [3]

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s), \tag{1.1}$$

the Selberg zeta-function associated with the modular group $\mathrm{PSL}(2,\mathbb{Z})$ satisfies the functional equation [4]

$$Z_{\mathrm{PSL}(2,\mathbb{Z})}(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(1-s)\frac{\zeta(2s)}{\zeta(2(1-s))}\frac{\Gamma(2s)}{\Gamma(2(1-s))}(2\pi)^{1-2s}\times$$
$$\times\exp\left(\frac{\pi}{3}\int_0^{s-1/2} v\tan\pi v\,dv - \frac{\pi}{2}\int_0^{s-1/2}\frac{dv}{\cos\pi v} - \frac{4\pi}{3\sqrt{3}}\int_0^{s-1/2}\frac{\cos\pi v/3}{\cos\pi v}\mathrm{d}v\right).$$

It is known [1] that

$$|\zeta(1-s)| > |\zeta(s)|, \tag{1.2}$$

where $s=\sigma+it$, is true for $\sigma > 1/2$ and $t \geqslant 6.8$ except where $\zeta(s)=0$ (note that, if the inequality is valid without exceptions, then the Riemann hypothesis is true and vice versa).

In [2] R. Garunkštis and A. Grigutis have proved a similar theorem for the modified Selberg zeta-function

$$W(s) = Z_{\mathrm{PSL}(2,\mathbb{Z})}(s)/\zeta(2s). \tag{1.3}$$

Theorem 1 (R. Garunkštis and A. Grigutis)**.** *If* $1/2 < \sigma < 1$ *and* $t \geqslant 6.053$*, then*

$$|W(1-s)| > |W(s)|. \tag{1.4}$$

In the proof of the theorem R. Garunkštis and A. Grigutis established and used two lemmas for ratios of complex gamma functions [2].

Lemma 1 (R. Garunkštis and A. Grigutis)**.** *For* $t \in \mathbb{R}$ *the following inequality holds:*

$$\left|\frac{\Gamma(2+it)}{\Gamma(it)}\right| < \left|\frac{\sqrt{2}}{2} + it\right|^2. \tag{1.5}$$

Lemma 2 (R. Garunkštis and A. Grigutis)**.** *For* $1/2 < \sigma < 1$ *and* $t \in \mathbb{R}$ *the following inequality holds:*

$$\left|\frac{\Gamma(2s)}{\Gamma(2(1-s))}\right| \leqslant \left|2s - 2 - \frac{\sqrt{2}}{2}\right|^{4(\sigma-1/2)}. \tag{1.6}$$

However, in order to obtain more subtle results, the following lemma has to be proved.

Lemma 3. *Let* $s = \sigma + it$*. For* $1 < \sigma < 2$ *we have*

$$\left|\frac{\Gamma(s)}{\Gamma(2-s)}\right| \leqslant |s|^{2(\sigma-1)}. \tag{1.7}$$

We can prove this statement using the theorem of Phragmén and Lindelöf.

Theorem 2 (Phragmén-Lindelöf)**.** *Let* $f(z)$ *be analytic in the strip*

$$S(\alpha,\beta) = \{z | z = x + iy, \alpha < x < \beta\}. \tag{1.8}$$

Let us assume $|f(z)| \leqslant 1$ *on the boundaries* $x = \alpha$ *and* $x = \beta$*, and moreover*

$$|f(z)| < Ce^{e^{k|y|}}$$

for some $C > 0$ *and* $0 < k < \frac{\pi}{\beta-\alpha}$*. Then* $|f(z)| \leqslant 1$ *throughout the strip* $S(\alpha,\beta)$*.*

Proof. See Rademacher [5] for the proof. □

2. Proof of Lemma 3

Proof. Let $s = 2-w$, $\sigma = 2-\rho$. Then the statement of the theorem is equivalent to

$$\left|\frac{\Gamma(2-w)}{\Gamma(w)}\right| \leqslant |2-w|^{2(1-\rho)}, \tag{2.1}$$

here $0 < \rho < 1$. Let us denote

$$f(w) = \frac{\Gamma(2-w)}{\Gamma(w)(2-w)^{2(1-\rho)}} \,. \tag{2.2}$$

First consider the left boundary $\rho = 0$, $w = -it$. We obtain

$$f(w) = \frac{\Gamma(2+it)}{\Gamma(-it)(2+it)^2} \,. \tag{2.3}$$

Hence, since $\overline{\Gamma(z)} = \Gamma(\overline{z})$,

$$\begin{aligned} |f(w)| &= \frac{|\Gamma(2+it)|}{|\Gamma(it)||2+it|^2} = \frac{|(1+it)(it)|}{4+t^2} = \\ &= \frac{\sqrt{t^4+t^2}}{t^2+4} = \sqrt{\frac{t^4+t^2}{t^4+8t^2+16}}, \end{aligned} \tag{2.4}$$

yielding us $|f(w)| < 1$.

Now consider the right boundary $\rho = 1$, $w = 1-it$. We obtain

$$f(w) = \frac{\Gamma(1+it)}{\Gamma(1-it)} \,. \tag{2.5}$$

Hence,

$$|f(w)| = \frac{|\Gamma(1+it)|}{|\Gamma(1-it)|} = 1 \,. \tag{2.6}$$

Next, let us consider the modulus of the function $f(w)$. Since $|\Gamma(z)| \leqslant \Gamma(\Re z)$ and $\Gamma(2-\rho) = \Gamma(\sigma) \leqslant 1$, we obtain

$$|f(w)| = \frac{|\Gamma(2-w)|}{|\Gamma(w)||2-w|^{2(1-\rho)}} \leqslant \frac{\Gamma(2-\rho)}{((2-\rho)^2+t^2)^{(1-\rho)}}\left|\frac{1}{\Gamma(w)}\right| \leqslant \left|\frac{1}{\Gamma(w)}\right|. \tag{2.7}$$

It is known [5], that if t is sufficiently large (i.e. $|t| \geqslant 1$), then the reciprocal gamma function

$$\frac{1}{\Gamma(w)} = O(e^{\frac{\pi}{2}|t|}|t|^{\frac{1}{2}-\rho}). \tag{2.8}$$

Since the reciprocal gamma function is an entire function, it is bounded in every compact subset of the complex plane (in particular, for $0 \leqslant \rho \leqslant 1$ and $0 \leqslant t \leqslant 1$ the modulus of the reciprocal gamma function $|1/\Gamma(w)| \leqslant 2$).

Thus, $|f(w)| = O(e^{e^{|t|}})$ for $t \in \mathbb{R}$, which yields us the statement of Lemma 3. □

3. Theorem for the modified Selberg zeta-function

The inequality (1.7) of Lemma 3, used together with technics of estimation, allows us (see Theorem 3) in a different way re-prove and extend the result of R. Garunkštis and A. Grigutis for the modified Selberg zeta-function (cf. Theorem 1).

Theorem 3. *For* $1/2 < \sigma < 1$ *and* $t \in [0, t_1) \cup (t_2, \infty)$ *we have*

$$|W(1-s)| > |W(s)|. \tag{3.1}$$

Here $t_1 = 1.740440...$ *and* $t_2 = 6.088036...$ *are the roots of the function*

$$L_1(t) = \log\left(\frac{1+t^2}{\pi \cosh\log\left(2\sinh\frac{\pi t}{6}\right)}\right). \tag{3.2}$$

By the definition of the modified Selberg zeta-function (1.3) and Lemma 3 we have

$$\left|\frac{W(s)}{W(1-s)}\right| = \left|\frac{\Gamma(2s)}{\Gamma(2(1-s))}(2\pi)^{1-2s}e^{Q(s)}\right| \leqslant \left(\frac{|2s|^2}{2\pi}\right)^{2\sigma-1} e^{\Re(Q(s))}, \tag{3.3}$$

here

$$Q(s) = \int_0^{s-1/2} \frac{\pi}{3}v\tan\pi v - \frac{\pi}{2\cos\pi v} - \frac{4\pi}{3\sqrt{3}}\frac{\cos\pi v/3}{\cos\pi v}\mathrm{d}v. \tag{3.4}$$

Integral (3.4) can be evaluated using triangular contour with vertices at $A(0,0)$, $B(\sigma - 1/2, t)$ and $C(\sigma - 1/2, 0)$: $\int_{AC} + \int_{CB} + \int_{BA} = 0$. Hence,

$$\underbrace{\Re(Q(s))}_{\Re\int_{AB}} = \underbrace{I_1(\sigma)}_{\int_{AC}} + \underbrace{R(\sigma,t)}_{\Re\int_{CB}}. \tag{3.5}$$

Here

$$I_1(\sigma) = \int_0^{\sigma-1/2} \frac{\pi}{3}\theta\tan\pi\theta - \frac{\pi}{2\cos\pi\theta} - \frac{4\pi}{3\sqrt{3}}\frac{\cos\pi\theta/3}{\cos\pi\theta}\mathrm{d}\theta \tag{3.6}$$

and

$$R(\sigma,t) = \Re\left\{\int_0^t \frac{\frac{\pi i}{3}\left(\sigma - \frac{1}{2} + i\theta\right)}{\cot\left(\pi\left(\sigma - \frac{1}{2} + i\theta\right)\right)} - \frac{\frac{\pi i}{2} + \frac{4\pi i}{3\sqrt{3}}\cos\left(\frac{\pi}{3}\left(\sigma - \frac{1}{2} + i\theta\right)\right)}{\cos\left(\pi\left(\sigma - \frac{1}{2} + i\theta\right)\right)}\mathrm{d}\theta\right\}. \tag{3.7}$$

Let us denote

$$L(\sigma,t) = (2\sigma - 1)\log\frac{\sigma^2 + t^2}{\pi/2} + I_1(\sigma) + R(\sigma,t). \tag{3.8}$$

Hence (cf. (3.3) and (3.5)),

$$\log\left|\frac{W(s)}{W(1-s)}\right| \leqslant L(\sigma,t). \tag{3.9}$$

Calculating function $I_1(\sigma)$ (3.6), we obtain

$$\begin{aligned} I_1(\sigma) &= \int_0^{\sigma-1/2} \frac{\pi}{3}\theta\tan\pi\theta - \frac{\pi}{2\cos\pi\theta} - \frac{4\pi}{3\sqrt{3}}\frac{\cos\pi\theta/3}{\cos\pi\theta}\mathrm{d}\theta = \\ &= -\frac{1}{6\pi}\mathrm{Cl}_2(2\pi\sigma) - \frac{2\sigma-1}{6}\log 2 - \frac{2\sigma-1}{6}\log\sin\pi\sigma - \\ &-\frac{1}{2}\log\tan\frac{\pi\sigma}{2} + \frac{2}{3}\log\left(\frac{\sqrt{3}}{2}\cot\frac{\pi\sigma}{3} - \frac{1}{2}\right). \end{aligned} \tag{3.10}$$

Here $\mathrm{Cl}_2(x)$ is the Clausen function of order 2,

$$\mathrm{Cl}_2(x) = -\int_0^x \log\left|2\sin\frac{t}{2}\right|\mathrm{d}t. \tag{3.11}$$

Noticing that (cf. (3.7))

$$\Re\left\{\frac{i(\sigma-1/2+i\theta)}{\cot(\pi(\sigma-1/2+i\theta))}\right\} = \frac{\theta\sin 2\pi\sigma + (1/2-\sigma)\sinh 2\pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}, \tag{3.12}$$

$$\Re\left\{\frac{i/2}{\cos\left(\pi\left(\sigma-1/2+i\theta\right)\right)}\right\} = \frac{\cos\pi\sigma\sinh\pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma} \tag{3.13}$$

and

$$\Re\left\{\frac{i\cos\left(\frac{\pi}{3}\left(\sigma-\frac{1}{2}+i\theta\right)\right)}{\cos\left(\pi\left(\sigma-\frac{1}{2}+i\theta\right)\right)}\right\} = \frac{\cos\left(\frac{2\pi\sigma}{3}+\frac{\pi}{6}\right)\sinh\frac{4\pi\theta}{3} + \cos\left(\frac{4\pi\sigma}{3}-\frac{\pi}{6}\right)\sinh\frac{2\pi\theta}{3}}{\cosh 2\pi\theta - \cos 2\pi\sigma}. \tag{3.14}$$

we calculate function $R(\sigma,t)$ (3.7), obtaining

$$\begin{aligned} R(\sigma,t) = &\underbrace{\frac{\pi}{3}\sin 2\pi\sigma\int_0^t \frac{\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}\mathrm{d}\theta}_{=I_2(\sigma,t)} + \\ &+\underbrace{\frac{\pi}{3}\left(\frac{1}{2}-\sigma\right)\int_0^t \frac{\sinh 2\pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}\mathrm{d}\theta}_{=I_3(\sigma,t)} + \\ &+\underbrace{\pi\cos\pi\sigma\int_0^t \frac{-\sinh\pi\theta}{\cosh 2\pi\theta - \cos 2\pi\sigma}\mathrm{d}\theta}_{=I_4(\sigma,t)} + \\ &+\underbrace{\frac{-4\pi}{3\sqrt{3}}\int_0^t \frac{\cos\left(\frac{2\pi\sigma}{3}+\frac{\pi}{6}\right)\sinh\frac{4\pi\theta}{3} + \cos\left(\frac{4\pi\sigma}{3}-\frac{\pi}{6}\right)\sinh\frac{2\pi\theta}{3}}{\cosh 2\pi\theta - \cos 2\pi\sigma}\mathrm{d}\theta}_{=I_5(\sigma,t)}. \end{aligned} \tag{3.15}$$

Note that $R(\sigma,t)$ is even function by t, thus, it suffices to consider non-negative t values. Calculating summands of the function $R(\sigma,t)$ (3.15), we obtain

$$I_2(\sigma,t)=\frac{\pi}{3}\sin 2\pi\sigma\int_0^t\frac{\theta}{\cosh 2\pi\theta-\cos 2\pi\sigma}\mathrm{d}\theta, \tag{3.16}$$

$$\begin{aligned}I_3(\sigma,t)=-&\frac{2\sigma-1}{12}\log(\cosh 2\pi t-\cos 2\pi\sigma)+\\&+\frac{2\sigma-1}{6}\log\sin\pi\sigma+\frac{2\sigma-1}{12}\log 2,\end{aligned} \tag{3.17}$$

$$I_4(\sigma,t)=\frac{1}{2}\log\tan\frac{\pi\sigma}{2}-\frac{1}{4}\log\frac{\cosh\pi t-\cos\pi\sigma}{\cosh\pi t+\cos\pi\sigma}, \tag{3.18}$$

$$\begin{aligned}I_5(\sigma,t)=-&\frac{2}{3}\log\left(\frac{\sqrt{3}}{2}\cot\frac{\pi\sigma}{3}-\frac{1}{2}\right)-\\&-\frac{1}{3}\log\frac{\cosh\frac{2\pi t}{3}-\cos\frac{2\pi\sigma}{3}}{\cosh\frac{2\pi t}{3}-\cos\frac{2\pi(\sigma-1)}{3}}.\end{aligned} \tag{3.19}$$

Expressions (3.16)-(3.19) allow us to calculate function $L(\sigma,t)$ (3.8),

$$\begin{aligned}L(\sigma,t)=&(2\sigma-1)\log\left(\frac{2}{\pi}(\sigma^2+t^2)\right)-(2\sigma-1)\frac{\log 2}{12}-\\&-\frac{1}{6\pi}\mathrm{Cl}_2(2\pi\sigma)+\frac{\pi}{3}\sin 2\pi\sigma\int_0^t\frac{\theta}{\cosh 2\pi\theta-\cos 2\pi\sigma}\mathrm{d}\theta-\\&-\frac{2\sigma-1}{12}\log(\cosh 2\pi t-\cos 2\pi\sigma)+\frac{1}{4}\log\frac{\cosh\pi t+\cos\pi\sigma}{\cosh\pi t-\cos\pi\sigma}+\\&+\frac{1}{3}\log\frac{\cosh\frac{2\pi t}{3}-\cos\frac{2\pi(\sigma-1)}{3}}{\cosh\frac{2\pi t}{3}-\cos\frac{2\pi\sigma}{3}}.\end{aligned} \tag{3.20}$$

Next we will establish several auxiliary lemmas concerning the behaviour of the function $L(\sigma,t)$.

4. The derivatives of the function $L(\sigma,t)$

Lemma 4. *For $1/2<\sigma<1$ and fixed $1/2\leqslant t<\infty$ the function $L(\sigma,t)$ (3.20) is convex by σ.*

Proof. Let us calculate partial derivatives of the function $L(\sigma,t)$ with respect to the variable σ,

$$\begin{aligned} L'_\sigma(\sigma,t) = {} & \underbrace{2\log\frac{\sigma^2+t^2}{\pi/2}}_{=F_1(\sigma,t)} + \underbrace{\frac{2\sigma(2\sigma-1)}{\sigma^2+t^2}}_{=F_2(\sigma,t)} + \underbrace{\frac{-\pi t}{3}\frac{\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=F_3(\sigma,t)} + \\ & + \underbrace{\frac{\pi}{6}(2\sigma-1)\frac{-\sin 2\pi\sigma}{\cosh 2\pi t-\cos 2\pi\sigma}}_{=F_4(\sigma,t)} + \underbrace{\frac{-\pi\sin\pi\sigma\cosh\pi t}{\cosh 2\pi t-\cos 2\pi\sigma}}_{=F_5(\sigma,t)} + \\ & + \underbrace{\frac{\pi}{3\sqrt{3}}\frac{1-2\cosh\frac{2\pi t}{3}\cos\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)}{\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi\sigma}{3}\right)}}_{=F_6(\sigma,t)}. \end{aligned} \tag{4.1}$$

$$\begin{aligned} L''_{\sigma\sigma}(\sigma,t) = {} & \frac{12\sigma-2}{\sigma^2+t^2} - \frac{4\sigma^2(2\sigma-1)}{(\sigma^2+t^2)^2} + \\ & + \underbrace{\frac{-\pi}{3}\frac{\sin 2\pi\sigma + \pi(2\sigma-1)\cos 2\pi\sigma + 3\pi\cos\pi\sigma\cosh\pi t}{\cosh 2\pi t-\cos 2\pi\sigma}}_{=G_1(\sigma,t)} + \\ & + \underbrace{\frac{\pi^2}{3}\sin 2\pi\sigma\frac{(2\sigma-1)\sin 2\pi\sigma + 6\sin\pi\sigma\cosh\pi t + 2t\sinh 2\pi t}{(\cosh 2\pi t-\cos 2\pi\sigma)^2}}_{=G_2(\sigma,t)} + \\ & + \underbrace{\frac{2\pi^2}{9\sqrt{3}}\frac{\sin\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)}{\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi(\sigma-1)}{3}\right)^2\left(\cosh\frac{2\pi t}{3}-\cos\frac{2\pi\sigma}{3}\right)^2}}_{=G_3(\sigma,t)} \times \\ & \times \underbrace{\left(2\cos\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right) - \cosh\frac{2\pi t}{3}\left(\frac{7}{2}+\cos\left(\frac{4\pi\sigma}{3}-\frac{2\pi}{3}\right)\right) + 2\cosh^3\frac{2\pi t}{3}\right)}_{=G_4(\sigma,t)}. \end{aligned} \tag{4.2}$$

First consider the interval $1/2 \leqslant t \leqslant 2$. Let us give a lower bounds of the functions $G_k(\sigma,t)$, which are defined in (4.2). Denote

$$v_1(\sigma,t) = \cosh 2\pi t - \cos 2\pi\sigma,$$
$$u_1(\sigma,t) = \sin 2\pi\sigma + \pi(2\sigma-1)\cos 2\pi\sigma + 3\pi\cos\pi\sigma\cosh\pi t.$$

Note that $v_1(\sigma,t)$ is positive for $t\in\mathbb{R}$ and $1/2<\sigma<1$. Next, $u_1(\sigma,t)$ is negative for $t\in\mathbb{R}$ and $1/2<\sigma\leqslant 3/4$. For $3/4<\sigma<1$, we have $0<\cos 2\pi\sigma<1$ and $\cos\pi\sigma<0$.

Hence,

$$u_1(\sigma,t) \leqslant \underbrace{\sin 2\pi\sigma + \pi(2\sigma-1) + 3\pi\cos\pi\sigma}_{=w_1(\sigma)}.$$

The function $w_1(\sigma)$ is convex, because

$$w_1''(\sigma) = -4\pi^2 \underbrace{\sin 2\pi\sigma}_{\leqslant 0} - 3\pi^3 \underbrace{\cos\pi\sigma}_{\leqslant 0} > 0,$$

for $3/4 < \sigma < 1$. With $w_1(3/4) < 0$ and $w_1(1) < 0$ it yields us $u_1(\sigma,t) < 0$ for $t \in \mathbb{R}$ and $1/2 < \sigma < 1$. Thus,

$$G_1(\sigma,t) = -\frac{\pi}{3}\frac{u_1(\sigma,t)}{v_1(\sigma,t)} > 0 \tag{4.3}$$

for $t \in \mathbb{R}$ and $1/2 < \sigma < 1$.

The function

$$G_2(\sigma,t) > \underbrace{\frac{2\pi^2}{-3}\frac{(3+2t\sinh\pi t)\cosh\pi t}{(\cosh 2\pi t - 1)^2}}_{=u_2(t)-increasing} \geqslant u_2(1/2) > \varepsilon = -0.8, \tag{4.4}$$

for $t \geqslant 1/2$ and $1/2 < \sigma < 1$.

The function $G_3(\sigma,t)$ is positive for $1/2 < \sigma < 1$.

Next let us show that $G_4(\sigma,t)$ is increasing by σ and by t. Indeed, consider the derivatives

$$(G_4)'_\sigma = -\frac{4\pi}{3}\sin\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right) + \frac{4\pi}{3}\cosh\frac{2\pi t}{3}\sin\left(\frac{4\pi\sigma}{3}-\frac{2\pi}{3}\right) =$$

$$= \frac{4\pi}{3}\underbrace{\sin\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)}_{>0}\left(\underbrace{2\cos\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)}_{\in(1,2)}\underbrace{\cosh\frac{2\pi t}{3}}_{\geqslant 1} - 1\right) > 0.$$

$$(G_4)'_t = \frac{2\pi}{3}\sinh\frac{2\pi t}{3}\left(6\cosh^2\frac{2\pi t}{3} - \left(\frac{7}{2} + \cos\left(\frac{4\pi\sigma}{3}-\frac{2\pi}{3}\right)\right)\right) > 0$$

for $t > 0$. Thus $G_4(\sigma,t) \geqslant G_4(1/2,1/2) > 0$. Hence,

$$L''_{\sigma\sigma}(\sigma,t) \geqslant \underbrace{\frac{12\sigma-2}{\sigma^2+t^2} - \frac{4\sigma^2(2\sigma-1)}{(\sigma^2+t^2)^2} - 0.8}_{=B(\sigma,t)}.$$

Note that the function $B(\sigma,t)$ is decreasing by t. Consider

$$B'_t(\sigma,t) = (12\sigma-2)\frac{-2t}{(\sigma^2+t^2)^2} + (8\sigma^3-4\sigma^2)\frac{4t}{(\sigma^2+t^2)^3} =$$
$$= \frac{4t}{(\sigma^2+t^2)^3}\left(\underbrace{(2\sigma-3)}_{<0}\sigma^2-(6\sigma-1)t^2\right)<0$$

for $t>0$. Thus, $B(\sigma,t)\geqslant B(\sigma,2)=B_2(\sigma)$ and

$$B_2(\sigma)=\frac{12\sigma-2}{\sigma^2+4}-\frac{4\sigma^2(2\sigma-1)}{(\sigma^2+4)^2}-0.8=4\frac{\sigma^3(1-0.2\sigma)+(12\sigma-1.1\sigma^2-5.2)}{(\sigma^2+4)^2}>0$$

for $1/2<\sigma<1$, yielding us the statement of the lemma for $1/2\leqslant t\leqslant 2$.

Next consider the interval $t\geqslant C$. Let $C=\sqrt{\frac{1}{2}+\sqrt{\frac{11}{12}}}=1.20724\ldots$. Let us consider the second partial derivative $L''_{\sigma\sigma}$ (4.2). We have shown that the function $G_1(\sigma,t)$ (4.3) and the product $G_3(\sigma,t)G_4(\sigma,t)$ are positive for $t>0$. Hence, using (4.4), we obtain

$$L''_{\sigma\sigma}(\sigma,t)\geqslant\underbrace{\frac{12\sigma-2}{\sigma^2+t^2}-\frac{4\sigma^2(2\sigma-1)}{(\sigma^2+t^2)^2}-\frac{\pi^2}{6}\frac{(3+2t\sinh\pi t)\cosh\pi t}{\sinh^4\pi t}}_{=D(\sigma,t)}.$$

Let us show that $D(\sigma,t)$ is increasing by σ. Indeed, consider the derivative

$$D'_\sigma(\sigma,t)=4\frac{P(\sigma,t)}{(\sigma^2+t^2)^3},$$

here

$$P(\sigma,t)=-\sigma^4-\sigma^3-6t^2\sigma^2+3t^2\sigma+3t^4.$$

For $1/2<\sigma<1$ the polynomial $P(\sigma,t)$ is concave by σ, because the second derivative

$$P''_{\sigma\sigma}=-12\sigma^2-6\sigma-12t^2<0.$$

The function is nonnegative at the endpoints of the interval,

$$P(1/2,t)=3t^4-3/16\geqslant 0$$

for $t\geqslant 1/2$, and

$$P(1,t)=3t^4-3t^2-2\geqslant 0$$

for $t \geqslant C$. Hence, $P(\sigma,t) \geqslant 0$ and $D'_\sigma(\sigma,t) \geqslant 0$ for $t \geqslant C$, yielding us $D(\sigma,t) > D(1/2,t)$. Consider

$$\begin{aligned} D(1/2,t) =& \frac{4}{t^2+1/4} - \frac{\pi^2}{6}\frac{(3+2t\sinh\pi t)\cosh\pi t}{\sinh^4\pi t} = \\ =& \frac{4}{t^2}\frac{1}{1+1/(4t^2)} - \frac{4\pi^2}{3}te^{-2\pi t}\frac{(1+3e^{-\pi t}/t-e^{-2\pi t})(1+e^{-2\pi t})}{(1-e^{-2\pi t})^4} \geqslant \\ \geqslant& \frac{4H}{t^2} - 4Ate^{-2\pi t} = \frac{4}{t^2}\underbrace{(H - At^3e^{-2\pi t})}_{=E(t)}. \end{aligned}$$

Here

$$H = \frac{1}{1+1/(4C^2)} = 0.854\ldots, \quad A = \frac{\pi^2}{3}\frac{(1+3e^{-\pi C}/C)(1+e^{-2\pi C})}{(1-e^{-2\pi C})^4} = 3.483\ldots.$$

The function $E(t)$ is increasing for $t \geqslant C$ and $E(C)$ is positive, yielding us the statement of the lemma. □

Lemma 5. *For $1/2 < \sigma < 1$, the derivative $L'_t(\sigma,t)$*

(1) *is positive for $t \in (0, 3.53]$,*
(2) *is negative for $t \in [3.77, \infty)$.*

Proof. Let us calculate the first partial derivative

$$\begin{aligned} L'_t(\sigma,t) = & \underbrace{(2\sigma-1)\frac{2t}{\sigma^2+t^2}}_{=N_1(\sigma,t)} + \underbrace{\frac{\pi}{3}\frac{t\sin 2\pi\sigma}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_2(\sigma,t)} + \\ & + \underbrace{\frac{\pi}{6}(2\sigma-1)\frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_3(\sigma,t)} + \underbrace{\frac{-\pi\cos\pi\sigma\sinh\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=N_4(\sigma,t)} + \\ & + \underbrace{\frac{2\pi}{9}\frac{\sinh\frac{2\pi t}{3}\left(\cos\frac{2\pi(\sigma-1)}{3} - \cos\frac{2\pi\sigma}{3}\right)}{\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}\right)}}_{=N_5(\sigma,t)}. \end{aligned} \tag{4.5}$$

Estimating functions in (4.5) we obtain $N_5(\sigma,t) > 0$, $N_1(\sigma,t) > 0$. Now let us show that $N_2(\sigma,t) + N_3(\sigma,t) + N_4(\sigma,t) > 0$. It is sufficient to prove that

$$\underbrace{\frac{1}{3}t\sin 2\pi\sigma - \frac{2\sigma-1}{6}\sinh 2\pi t - \cos\pi\sigma\sinh\pi t}_{=N(\sigma,t)} > 0.$$

Consider the derivative

$$N_t'(\sigma,t) = \frac{1}{3}\sin 2\pi\sigma - \frac{2\sigma-1}{3}\pi\cosh 2\pi t - \pi\cos\pi\sigma\cosh\pi t =$$
$$= -\frac{1}{3}(2\pi(2\sigma-1)\cosh^2\pi t + 3\pi\cos\pi\sigma\cosh\pi t - (\sin 2\pi\sigma + (2\sigma-1)\pi)).$$

The positive root of the quadratic equation

$$r(\sigma) = \frac{-3\pi\cos\pi\sigma + \sqrt{9\pi^2\cos^2\pi\sigma + 8\pi(2\sigma-1)\sin 2\pi\sigma + 8\pi^2(2\sigma-1)^2}}{4\pi(2\sigma-1)}.$$

For $1/2 < \sigma < 1$,

$$\frac{3+\sqrt{17}}{4} < r(\sigma) < \frac{3\pi}{4}.$$

For $0 < t \leqslant 0.37$

$$1 < \cosh 2\pi t \leqslant 1.755 < \frac{3+\sqrt{17}}{4}.$$

Hence, $N_t'(\sigma,t) > 0$ and $N(\sigma,t) > N(\sigma,0) = 0$, yielding us the statement of the lemma for $t \in (0, 0.37]$.

Consider the first partial derivative in the interval $t \in [0.37, 3.53]$,

$$\begin{aligned} L_t'(\sigma,t) = (2\sigma-1)\frac{2t}{\sigma^2+t^2} &+ \underbrace{\frac{\pi}{3}\frac{\cos\pi\sigma(2t\sin\pi\sigma - 3\sinh\pi t)}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=H_2(\sigma,t)} + \\ &+ \underbrace{\frac{\pi}{6}(2\sigma-1)\frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma}}_{=H_3(\sigma,t)} + \\ &+ \underbrace{\frac{2\pi}{9}\frac{\sinh\frac{2\pi t}{3}\left(\cos\frac{2\pi(\sigma-1)}{3} - \cos\frac{2\pi\sigma}{3}\right)}{\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi(\sigma-1)}{3}\right)\left(\cosh\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}\right)}}_{=H_4(\sigma,t)}. \end{aligned} \tag{4.6}$$

Estimating $H_2(\sigma,t)$ and $H_4(\sigma,t)$ we obtain $H_2(\sigma,t) > 0$ and $H_4(\sigma,t) > 0$ for $1/2 < \sigma < 1$. Thus,

$$\begin{aligned} L_t'(\sigma,t) >& (2\sigma-1)\frac{2t}{\sigma^2+t^2} + \frac{\pi}{6}(2\sigma-1)\frac{-\sinh 2\pi t}{\cosh 2\pi t - \cos 2\pi\sigma} > \\ >& (2\sigma-1)\left(\frac{2t}{\sigma^2+t^2} - \frac{\pi}{6}\frac{\sinh 2\pi t}{\cosh 2\pi t - 1}\right) = \\ =& (2\sigma-1)\underbrace{\left(\frac{2t}{\sigma^2+t^2} - \frac{\pi}{6}\coth\pi t\right)}_{=M(\sigma,t)}. \end{aligned}$$

The function $M(\sigma,t)$ is decreasing by σ, hence

$$M(\sigma,t) \geqslant \frac{2t}{1+t^2} - \frac{\pi}{6}\coth \pi t \geqslant \underbrace{\frac{2t}{1+t^2} - \frac{\pi}{6}A_k}_{=M_k(t)}.$$

Here

$$A_k = \begin{cases} \coth 0.37\pi & \text{for } 0.37 \leqslant t < 2.77, \\ \coth 2.77\pi & \text{for } 2.77 \leqslant t \leqslant 3.53. \end{cases}$$

Now

$$M_k'(t) = 2\frac{1-t^2}{(1+t^2)^2}.$$

For $0.37 \leqslant t < 2.77$ the function $M_1(t)$ increases in the interval $(0.37, 1)$ and decreases in the interval $(1, 2.77)$. At endpoints the function is positive, $M_1(0.37) > 0$ and $M_1(2.77) > 0$.

For $2.77 \leqslant t \leqslant 3.53$ the function $M_2(t)$ decreases. At the endpoint the function is positive, $M_2(3.53) > 0$, yielding us the statement of the lemma for the interval $0.37 \leqslant t \leqslant 3.53$.

Estimating functions $N_k(\sigma,t)$ in (4.5) for $t \geqslant 3.77$ we obtain (note that $N_2(\sigma,t) < 0$)

$$L_t'(\sigma,t) < \frac{2t(2\sigma-1)}{\sigma^2+t^2} - \frac{\pi(2\sigma-1)}{6} - \frac{\pi\cos\pi\sigma}{2\sinh\pi t} + \frac{2\pi}{3\sqrt{3}}\frac{\coth\frac{\pi t}{3}\sin\left(\frac{2\pi\sigma}{3}-\frac{\pi}{3}\right)}{\cosh\frac{2\pi t}{3} - \cos\frac{2\pi\sigma}{3}} <$$

$$< \underbrace{(2\sigma-1)\left(\frac{2t}{\sigma^2+t^2} + \underbrace{\left(-\frac{\pi}{6} + \frac{\pi^2}{4\sinh\pi t} + \frac{2\pi^2}{9\sqrt{3}}\frac{\coth\frac{\pi t}{3}}{\cosh\frac{2\pi t}{3} - \frac{1}{2}}\right)}_{=\mu(t)}\right)}_{=K(\sigma,t)}.$$

Consider the derivative of the function $\mu(t)$

$$\mu'(t) = \underbrace{\frac{\pi^3}{-4}\frac{\cosh\pi t}{\sinh^2\pi t}}_{<0} + \underbrace{\frac{-2\pi^3}{27\sqrt{3}}\frac{\sinh^{-2}\frac{\pi t}{3}\left(\cosh\frac{2\pi t}{3} - \frac{1}{2}\right) + 2\coth\frac{\pi t}{3}\sinh\frac{2\pi t}{3}}{\left(\cosh\frac{2\pi t}{3} - \frac{1}{2}\right)^2}}_{<0}.$$

Hence,

$$K_t'(\sigma,t) = (2\sigma-1)\left(\underbrace{\frac{2(\sigma^2-t^2)}{(\sigma^2+t^2)^2}}_{<0} + \underbrace{\mu'(t)}_{<0}\right) < 0$$

the function $K(\sigma,t)$ is decreasing by t and $K(\sigma,t) \leqslant K(\sigma,3.77) < 0$, yielding us the statement of the lemma. □

5. Auxiliary lemmas

Let us denote

$$L_1(t) = L(1,t). \tag{5.1}$$

Lemma 6. *The function* $L_1(t)$

(1) is negative *for* $t \in (0,t_1) \cup (t_2,\infty)$,
(2) is positive *for* (t_1,t_2),
(3) has unique maximum point *at* $t^* \in (t_1,t_2)$.

Here $t_1 = 1.740440\ldots$ *and* $t_2 = 6.088036\ldots$ *are the roots of the function.*

Proof. By (5.1) and (3.20) we obtain

$$\begin{aligned} L_1(t) =& \log\left(\frac{2}{\pi}(1+t^2)\right) - \frac{\log 2}{12} - \frac{1}{12}\log(\cosh 2\pi t - 1) + \\ &+\frac{1}{4}\log\frac{\cosh \pi t - 1}{\cosh \pi t + 1} + \frac{1}{3}\log\frac{\cosh\frac{2\pi t}{3} - 1}{\cosh\frac{2\pi t}{3} - \frac{1}{2}} = \\ &= \log\left(\frac{1+t^2}{\pi}\frac{\sinh\frac{\pi t}{6}}{\sinh^2\frac{\pi t}{6} + \left(\frac{1}{2}\right)^2}\right) = \log\left(\frac{1+t^2}{\pi\cosh\log\left(2\sinh\frac{\pi t}{6}\right)}\right). \end{aligned} \tag{5.2}$$

Note that

$$\lim_{t\to 0+} L_1(t) = -\infty, \qquad \lim_{t\to +\infty} L_1(t) = -\infty. \tag{5.3}$$

Next, $L_1(t) = 0$ iff

$$\underbrace{\cosh\log\left(2\sinh\frac{\pi t}{6}\right)}_{\geqslant 1} = \frac{1+t^2}{\pi}. \tag{5.4}$$

Hence, there are no zeros in the interval $(0,t_0)$. The function $L_1(t)$ is negative in the interval (cf. (5.3)). Here $t_0 = \sqrt{\pi - 1} = 1.463418\ldots$. By (5.2),

$$\begin{aligned} L_1(t) =& \log\left(\frac{2}{\pi}(1+t^2)e^{-\frac{\pi t}{6}}\frac{1}{1+\frac{e^{-\frac{2\pi t}{3}}}{1-e^{-\frac{\pi t}{3}}}}\right) = \\ &= \underbrace{\log\frac{2}{\pi} + \log(1+t^2) - \frac{\pi t}{6}}_{=\Psi_1(t)} + \underbrace{\log\left(1 - \frac{1}{e^{\frac{2\pi t}{3}} - e^{\frac{\pi t}{3}} + 1}\right)}_{=\Psi_2(t)}. \end{aligned} \tag{5.5}$$

The function $\Psi_1(t)$ is concave for $t > 1$. Indeed, consider the second derivative,

$$\Psi_1''(t) = 2\frac{1-t^2}{(1+t^2)^2} < 0. \tag{5.6}$$

Next, consider the second derivative of $\Psi_2(t)$,

$$\Psi_2''(t) = \frac{-\pi^2 e^{\frac{2\pi t}{3}}}{9\left(e^{\frac{2\pi t}{3}} - e^{\frac{\pi t}{3}} + 1\right)^2 \left(e^{\frac{\pi t}{3}} - 1\right)^2} \underbrace{\left(4e^{\frac{2\pi t}{3}} - 7e^{\frac{\pi t}{3}} + 4\right)}_{>0} < 0. \tag{5.7}$$

Combining (5.6) and (5.7) we obtain, that the function $L_1(t)$ is concave for $t > 1$. Thus, the concave function on an open set takes negative, then positive (e.g. $L_1((6/\pi)\log 3) > 0$), then again negative values (cf. (5.3)). Hence, it has unique positive maximum in the interval (t_1, t_2). Here t_1 and t_2 are the roots of the function $L_1(t)$. The values of the roots we obtain numerically with any sufficient accuracy. □

Next, let us denote

$$L_0(\sigma) = L(\sigma, 0). \tag{5.8}$$

Lemma 7. *For $1/2 < \sigma < 1$, the function $L_0(\sigma)$ is negative.*

Proof. By (3.8) and (3.15) we have

$$L_0(\sigma) = (2\sigma - 1)\log\frac{2}{\pi}\sigma^2 + I_1(\sigma).$$

Calculating the derivative of the function (cf. (3.10)) we obtain

$$\begin{aligned} L_0'(\sigma) = {} & \underbrace{2\log\frac{2}{\pi}\sigma^2 + \frac{2\sigma - 2}{\sigma}}_{=Q_1(\sigma)} + \underbrace{\frac{-\pi}{6}(2\sigma - 1)\cot\pi\sigma + \frac{-\pi}{2\sin\pi\sigma}}_{=Q_2(\sigma)} + \\ & + \underbrace{\frac{4\pi}{3\sqrt{3}}\frac{1}{1 - 2\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)}}_{=Q_3(\sigma)} + 2. \end{aligned} \tag{5.9}$$

Let us estimate $Q_k(\sigma)$ functions from above. For $1/2 < \sigma < 1$ the function $Q_1(\sigma)$ is negative since the derivative

$$Q_1'(\sigma) = \frac{4}{\sigma} + \frac{2}{\sigma^2} > 0$$

and $Q_1(1) < 0$.

For $1/2 < \sigma < 1$ the function

$$Q_2(\sigma) = \underbrace{\frac{-\pi}{6\sin\pi\sigma}}_{<0} \underbrace{((2\sigma - 1)\cos\pi\sigma + 3)}_{=q_2(\sigma)}.$$

The derivative

$$q_2'(\sigma) = 2\cos\pi\sigma - (2\sigma - 1)\pi\sin\pi\sigma < 0,$$

while $q_2(1) = 2 > 0$, hence $q_2(\sigma) > 0$, and $Q_2(\sigma)$ is negative.

For $1/2 < \sigma < 1$ the function

$$Q_3(\sigma) = \frac{4\pi}{3\sqrt{3}} \frac{1}{1 - 2\cos\left(\frac{2\pi\sigma}{3} - \frac{\pi}{3}\right)} + 2 < 0.$$

Hence, $L_0'(\sigma) < 0$ with $L_0(1/2) = 0$, yielding us the statement of the lemma. □

6. Proof of the theorem for the modified Selberg zeta-function

Now we can prove the Theorem 3.

Proof. Consider *max* value of the function $L(\sigma,t)$ in the rectangle $(\sigma,t) \in (1/2,1) \times (0,t_1)$. By Lemma 5, the function $L(\sigma,t)$ has no stationary points in the interior of the rectangle, so it suffices to investigate the behaviour of the function on vertices of the rectangle. By Lemma 4, the function $L(\sigma,t)$ is convex by σ and the derivative by t is positive, hence we must consider the first zero of the function $L_1(t)$ (cf. Lemma 6). Note that

$$\lim_{\sigma \to 1/2+} L(\sigma,t) = 0. \tag{6.1}$$

Next let us consider *max* value of the function $L(\sigma,t)$ in the strip $(\sigma,t) \in (1/2,1) \times (t_2,\infty)$. By Lemma 5, function $L(\sigma,t)$ has no stationary points in the interior of the strip, so it suffices to investigate the behaviour of the function on vertices. By Lemma 4, the function $L(\sigma,t)$ is convex by σ and the derivative by t is negative, hence we must consider the second zero of the function $L_1(t)$ (cf. Lemma 6). By Lemma 7 and (6.1), it gives us $L(\sigma,t) < 0$. Consequently (cf. (3.9))

$$\log\left|\frac{W(s)}{W(1-s)}\right| < 0,$$

yielding us the statement of the theorem. □

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References

[1] R. D. Dixon and L. Schoenfeld, "The size of the Riemann zeta-function at places symmetric with respect to the point 1/2." *Duke Math. J.*, vol. 33, no. 2, pp. 291—292, 1966, doi: 10.1215/S0012-7094-66-03333-3.

[2] R. Garunkštis and A. Grigutis, "The size of the Selberg zeta-function at places symmetric with respect to the line Re(s)= 1/2." *Results. Math.*, vol. 70, no. 1, pp. 271—281, 2016, doi: 10.1007/s00025-015-0486-7.

[3] H. Iwaniec, *Lectures on the Riemann Zeta Function.* American Mathematical Society, 2014.
[4] M. Minamide, “On zeros of the derivative of the modified Selberg zeta function for the modular group.” *J. Ind. Math. Soc.*, vol. 80, no. 3-4, pp. 275—312, 2013.
[5] H. Rademacher, *Topics in analytic number theory.* Springer-Verlag Berlin Heidelberg, 1973. doi: 10.1007/978-3-642-80615-5.

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