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MULTIPLE SOLUTIONS FOR ASYMPTOTICALLY LINEAR $2p$-ORDER HAMILTONIAN SYSTEMS WITH IMPULSIVE EFFECTS

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Abstract. In this paper, we are concerned with $2p$-order Hamiltonian systems with impulsive effects. We investigate the variational structure associated to this system. In addition, we obtain some results of multiple solutions for asymptotically linear $2p$-order Hamiltonian systems via variational methods and critical point theorems. Meanwhile, some examples are presented to illustrate our main results.

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1. INTRODUCTION AND MAIN RESULTS

In recent years, variational methods have been introduced to investigate various impulsive differential equations since papers [10, 13] appeared. As one kind of widely applicable differential equations, Hamiltonian systems with impulsive effects have been also concerned on widely and many new corresponding results have been obtained, see for instance [3, 7–9, 11, 12, 15]. However, in aforementioned papers Hamiltonian systems are second order. To the best of our knowledge, few authors have considered asymptotically linear $2p$-order Hamiltonian systems with impulsive effects. One difficulty is that the suitable impulsive effects associated to this system have been not found. Another is that the suitable critical point theorems have been not applied. In this paper, we present such impulsive conditions. More precisely, we investigate multiple solutions for

$$\begin{cases} (-1)^{p+1}u^{(2p)} + \nabla V(t,u) = 0, \quad a.e.\ t \in [0,T], \\ u^{(j)}(0) = u^{(j)}(T), \quad j = 0,1,\cdots,2p-1, \\ \Delta(u^{i^{(2p-j)}}(t_k)) = I_{ijk}(u^{i^{(j-1)}}(t_k)), \\ \qquad i = 1,2,\cdots,N, j = 1,2,\cdots,p, k = 1,2,\cdots,q, \end{cases} \tag{1.1}$$

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where p is a positive integer, $u(t) = (u^1(t), u^2(t), \cdots, u^N(t))$, $V : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ is measurable with respect to t, for every $u \in \mathbf{R}^N$, continuously differentiable in u, for almost every $t \in [0,T]$, $0 = t_0 < t_1 < \cdots < t_q < t_{q+1} = T$, t_k $(k = 1,2,\cdots,q)$ are the instants where the impulses occur, $\Delta(u^{i^{(2p-j)}}(t_k)) = u^{i^{(2p-j)}}(t_k^+) - u^{i^{(2p-j)}}(t_k^-)$ and $I_{ijk} : \mathbf{R} \to \mathbf{R}$ $(i = 1,2,\cdots,N,\ j = 1,2,\cdots,p,\ k = 1,2,\cdots,q)$ are continuous.

From now on, we write $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as $\{1,2,\cdots,N\}, \{1,2,\cdots,p\}$ and $\{1,2,\cdots,q\}$ respectively. In addition, $\mathcal{L}_s(\mathbf{R}^N)$ stands for the space of symmetric matrices of order N and I_N is the unit matrix in $\mathcal{L}_s(\mathbf{R}^N)$. For any $A_1, A_2 \in \mathcal{L}_s(\mathbf{R}^N)$, we denote by $A_1 \le A_2$ if $A_2 - A_1$ is positively semi-definite, and denote by $A_1 < A_2$ if $A_2 - A_1$ is positively definite. For any $A_1, A_2 \in L^\infty([0,T]; \mathcal{L}_s(\mathbf{R}^N))$, we denote by $A_1 \le A_2$ if $A_1(t) \le A_2(t)$ for $a.e.\ t \in]0,T[$, and denote by $A_1 < A_2$ if $A_1 \le A_2$ and $A_1(t) < A_2(t)$ on a subset of $]0,T[$ with nonzero measure.

Let us have the space

$$H_1^p := \left\{ u \in H^p(]0,T[; \mathbf{R}^N) | u^{(j)}(0) = u^{(j)}(T), j = 0,1,\cdots,p-1 \right\}$$

with the inner product

$$\langle u, v \rangle = \int_0^T (u^{(p)}(t), v^{(p)}(t)) + (u(t), v(t))\, dt, \quad \forall u, v \in H_1^p,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbf{R}^N$. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T |u^{(p)}(t)|^2 + |u(t)|^2\, dt \right)^{\frac{1}{2}}, \quad \forall u \in H_1^p.$$

Suppose that I_{ijk} and V satisfy that the following some conditions:

(I_1) Every $I_{ijk}(i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C})$ is bounded and $I_{ijk}(0) = 0$.

(I_2) Every $I_{ijk}(i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C})$ is odd.

(V_1) $V(t,u)$ is twice continuously differentiable in u for a.e. $t \in [0,T]$.

(V_2) $\nabla V(t,0) \equiv 0$ and set $A_0(t) \equiv D_u^2 V(t,0)$.

(V_3) There exist $A_1, A_2 \in L^\infty([0,T]; \mathcal{L}_s(\mathbf{R}^N))$ and $r > 0$ such that

$$A_1(t) \le D_u^2 V(t,u) \le A_2(t)$$

for every $u \in \mathbf{R}^N$ with $|u| \ge r$, and a.e. $t \in [0,T]$.

(V_4) $V(t,-u) = V(t,u)$ for every $u \in \mathbf{R}^N$ and a.e. $t \in [0,T]$.

(V_5) $V(t,0) \equiv 0$.

Our main results are the following two theorems.

Theorem 1. *Suppose that* $(I_1), (V_1) - (V_3)$ *with* $i_p(A_1) = i_p(A_2) > 0$, $\nu_p(A_2) = 0$, *and* $i_p(A_1) \overline{\in} [i_p(A_0), i_p(A_0) + \nu_p(A_0)]$ *hold. Then,* (1.1) *has at least one nontrivial weak solution. Further, if* $\nu_p(A_0) = 0$ *and* $\left| i_p(A_1) - i_p(A_0) \right| \ge pN$, *then problem* (1.1) *has two nontrivial weak solutions.*

Theorem 2. *Suppose that* $(I_1),(I_2),(V_1)-(V_5)$ *with* $i_p(A_1)=i_p(A_2)>0$, $\nu_p(A_2)=0$, *and* $\nu_p(A_0)=0$ *hold. Then, problem* (1.1) *has at least* $|i_p(A_1)-i_p(A_0)|$ *distinct pairs of nontrivial weak solutions.*

Remark 1. Here $i_p(A)$ and $\nu_p(A)$ are called the index and nullity of A respectively. Indeed, for any $A\in L^\infty([0,T];\mathcal{L}_s(\mathbf{R}^N))$, we define

$$\phi_A(u,v)=\int_0^T (u^{(p)}(t),v^{(p)}(t))\,dt-\int_0^T (A(t)u(t),v(t))\,dt,\quad \forall u,v\in H_1^p.$$

For any $x,y\in H_1^p$ if $\phi_A(x,y)=0$, we say that x and y ϕ_A-orthogonal. If H_1,H_2 are the two subsets of H_1^p and for any $x\in H_1$ and $y\in H_2$, $\phi_A(x,y)=0$, we say that H_1 and H_2 ϕ_A-orthogonal. In addition, H_1^p has a ϕ_A-orthogonal decomposition $H_1^p=H_1^{p+}(A)\oplus H_1^{p0}(A)\oplus H_1^{p-}(A)$ such that ϕ_A is positive definite, zero and negative definite on $H_1^{p+}(A),H_1^{p0}(A)$ and $H_1^{p-}(A)$ respectively. Moreover, $H_1^{p0}(A)$ and $H_1^{p-}(A)$ are finitely dimensional. Hence, for any $A\in L^\infty([0,T];\mathcal{L}_s(\mathbf{R}^N))$, we define $i_p(A)=H_1^{p-}(A)$, $\nu_p(A)=\dim H_1^{p0}(A)$. These results are the immediate conclusions of Proposition 2.1.1, Definition 2.1.2 in [5].

This paper is organized as follows. In Section 2, we first recall several critical point theorems. Then, we investigate the variational structure associated to problem (1.1) in H_1^p. Finally, we quote the two lemmas which are crucial in our argument. In Section 3, we verify our main results by applying variational methods and critical point theorems when V satisfies the generalized asymptotically linear conditions. Our results extends some conclusions directly in [4]. Analogously, by new definition of weak solution, one can be dealt with problem (1.1) when V satisfies some other conditions, such as the convex potential condition, the even type potential condition, the Ahmad-Lazer-Paul type coercive condition and its several generalizations, the sublinear potential condition, the superquadratic potential condition, the subquadratic potential condition and the asymptotically quadratic potential condition. In Section 4, we present some examples in order to illustrate our results.

2. Preliminaries

Let X be a Hilbert space. We first recall some critical point theorems in critical point theory. These theorems are due to K. C. Chang.

Theorem 3 ([1, Theorem 4.3.4]). *Let* $f\in C^1(X,\mathbf{R})$ *be even and* $f(0)=0$. *Assume* f *satisfies* $PS-$*condition and*
(i) there is an m-dimensional subspace X_1 *and a constant* $r>0$ *such that*

$$\sup_{x\in X_1\cap\partial U_r} f(x)<0,$$

(ii) there is a j-codimensional subspace X_2 such that

$$\inf_{x\in X_2} f(x) > -\infty.$$

Then f has at least $m-j$ distinct pairs of critical points provided $m-j>0$.

Theorem 4 ([1, Theorem 4.3.6]). *Let $f\in C^1(X,\mathbf{R})$ be even and $f(0)=0$. Assume f satisfies $PS-$condition and*
(i) there is a j-codimensional subspace X_1 and two constants $r,\alpha>0$ such that

$$f(x)\geq\alpha \quad for\ any\ x\in X_1\cap\partial U_r,$$

(ii) there is a m-dimensional subspace X_2 and a constant $R>0$ such that

$$f(x)\leq 0 \quad for\ any\ x\in X_2\backslash U_R.$$

Then f has at least $m-j$ distinct pairs of critical points provided $m-j>0$.

The last one is called three solutions theorem and can be verified by Theorem 5.1, Theorem 5.2 and Corollary 5.2 in [2]. One can find its proof in [6].

Theorem 5 ([6, Proposition 5.5.2]). *Assume $f\in C^2(X,\mathbf{R})$ and satisfies PS-condition, $f''(x)$ is Fredholm with finite Morse index for each critical point $x\in X$ and $f'(0)=0$. Suppose there is a positive integer γ such that*

$$\gamma\overline{\in}[m^-(f''(0)),m^0(f''(0))+m^-(f''(0))]$$

and $H_q(X,f_a;\mathbf{R})\cong\delta_{q\gamma}\mathbf{R}$ for some regular value $a<I(0)$, where $\delta_{q\gamma}=\begin{cases}1, & \delta=\gamma,\\ 0, & \delta\neq\gamma.\end{cases}$
Then, f have a critical point $x_0\neq 0$. Moreover, if 0 is a non-degenerate critical point, and $m^0(f''(x_0))\leq|\gamma-m^-(f''(0))|$, then f have another critical point $x_1\neq x_0,0$.

Remark 2. Here $f_a=\{x\in X|f(x)\leq a, a\in\mathbf{R}\}$. The Morse nullity and Morse index of f at $x\in X$ are defined as $\dim(\ker f''(x))$ and the supremum of the dimensions of the vector subsequence of X in which $f''(x)$ is negative definite respectively. Both are denoted by $m^0(f''(x))$ and $m^-(f''(x))$ respectively.

Next, we investigate the variational structure of (1.1). This idea comes from papers [10, 14].

If $u\in H_1^p$, then $u_i^{(2p-j)}(t_k^+)$ and $u_i^{(2p-j)}(t_k^-)(i\in\mathbf{A}, j\in\mathbf{B}, k\in\mathbf{C})$ may not exist. This leads to impulsive effects.

Let us take $y\in H_1^p$, multiply both sides of the equation in (1.1) by y, and integrate between 0 and T, then

$$\int_0^T((-1)^{p+1}u^{(2p)},v)\,dt+\int_0^T(\nabla V(t,u),v)\,dt=0. \tag{2.1}$$

Moreover, combing $u^{(2p-1)}(0)=u^{(2p-1)}(T)$, one has

$$
\begin{aligned}
&\int_0^T ((-1)^{p+1}u^{(2p)}, v)\,dt \\
&= (-1)^{p+1}\sum_{k=0}^{q}\int_{t_k}^{t_{k+1}} (u^{(2p)}, v)\,dt \\
&= (-1)^{p+1}\sum_{k=0}^{q}\Big((u^{(2p-1)}(t_{k+1}^-), v(t_{k+1}^-)) - (u^{(2p-1)}(t_k^+), v(t_k^+))\Big) \\
&\quad - (-1)^{p+1}\sum_{k=0}^{q}\int_{t_k}^{t_{k+1}} (u^{(2p-1)}, \dot{v})\,dt \\
&= (-1)^{p+1}\sum_{i=1}^{N}\sum_{k=0}^{q}\Big((u_i^{(2p-1)}(t_{k+1}^-), v_i(t_{k+1}^-)) - (u_i^{(2p-1)}(t_k^+), v_i(t_k^+))\Big) \\
&\quad + (-1)^{p+2}\sum_{k=0}^{q}\int_{t_k}^{t_{k+1}} (u^{(2p-1)}, \dot{v})\,dt \\
&= (-1)^{p+1}\Big((u^{(2p-1)}(T^-), v(T^-)) - (u^{(2p-1)}(0^+), v(0^+))\Big) \\
&\quad + (-1)^{p+2}\sum_{i=1}^{N}\sum_{k=1}^{q}\Delta(u_i^{(2p-1)}(t_k))v_i(t_k) + (-1)^{p+2}\int_0^T (u^{(2p-1)}, \dot{v})\,dt \\
&= \int_0^T (-1)^{p+2}(u^{(2p-1)}, \dot{v})\,dt + (-1)^{p+2}\sum_{i=1}^{N}\sum_{k=1}^{q} I_{i1k}(u_i(t_k))v_i(t_k).
\end{aligned}
$$

Similarly, combining $u^{(2p-j)}(0) = u^{(2p-j)}(T)$, $j \in \mathbf{B}\backslash\{1\}$, one has

$$
\begin{aligned}
&\int_0^T ((-1)^{p+2}u^{(2p-1)}, \dot{v})\,dt \\
&= \int_0^T ((-1)^{p+3}u^{(2p-2)}, \ddot{v})\,dt + (-1)^{p+3}\sum_{i=1}^{N}\sum_{k=1}^{q} I_{i2k}(\dot{u}_i(t_k))\dot{v}_i(t_k),
\end{aligned}
$$

$$\cdots,$$

$$
\begin{aligned}
&\int_0^T ((-1)^{2p}u^{(p+1)}, v^{(p-1)})\,dt \\
&= \int_0^T ((-1)^{2p+1}u^{(p)}, v^{(p)})\,dt + (-1)^{2p+1}\sum_{i=1}^{N}\sum_{k=1}^{q} I_{ipk}(u_i^{(p-1)}(t_k))v_i^{(p-1)}(t_k).
\end{aligned}
$$

Hence,

$$\int_0^T (u^{(p)}, v^{(p)})\,dt - \int_0^T (\nabla V(t,u), v)\,dt + \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q}(-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k))v_i^{(j-1)}(t_k) = 0. \tag{2.2}$$

Definition 1. Say that $u \in H_1^p$ is a weak solution for (1.1) if (2.2) holds for any $v \in H_1^p$.

Consider the functional $\varphi : H_1^p \to \mathbf{R}$ defined by putting

$$\varphi(u) := \frac{1}{2}\int_0^T |u^{(p)}(t)|^2\,dt - \int_0^T V(t,u(t))\,dt + \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q}\int_0^{u_i^{(j-1)}(t_k)} I_{ijk}(s)\,ds. \tag{2.3}$$

Then, φ is Gâteaux differential at any $u \in H_1^p$ and

$$\langle \varphi'(u), v\rangle = \int_0^T (u^{(p)}, v^{(p)})\,dt - \int_0^T (\nabla V(t,u), v)\,dt + \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q}(-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k))v_i^{(j-1)}(t_k) \tag{2.4}$$

for any $\nabla V(t,u) \in H_1^p$. Hence, the following lemma holds by (2.4) and Definition 1.

Lemma 1. *If $u \in H_1^p$ is a critical point of φ, then u is a weak solution for (1.1).*

Finally, we quote the two important lemmas. The first lemma is identical with Proposition in [6] when $T = 1$ and it's proof is absolutely similar to the proof of Proposition 5.3.1.

Lemma 2. *If (V_1)-(V_3) hold, then for any $\varepsilon > 0$, there exists $A : [0,T]\times\mathbf{R}^N \to \mathcal{L}_s(\mathbf{R}^N)$ and $g : [0,T]\times\mathbf{R}^N \to \mathbf{R}^N$ such that*

$$\nabla v(t,u) = A(t,u)u + g(t,u), \tag{2.5}$$

where

$$A_1 - \varepsilon I_N \le A(t,u) \le A_2 + \varepsilon I_N, \text{ for any } u \in \mathbf{R}^N \text{ a.e. } t \in [0,T], \tag{2.6}$$

$$A(\cdot,u(\cdot)) \in L^\infty([0,T],\mathcal{L}_s(\mathbf{R}^N)) \text{ for all } u \in L^2([0,T],\mathbf{R}^N), \tag{2.7}$$

and

$$g(\cdot,u(\cdot)) \in L^\infty([0,T],\mathbf{R}^N) \text{ is bounded for all } u \in L^2([0,T],\mathbf{R}^N). \tag{2.8}$$

Remark 3. Say then V satisfies asymptotically linear conditions if (2.5)-(2.8) hold. Lemma 2 shows that if (V_1)-(V_3) hold, then F satisfies these conditions.

The second lemma is a immediate corollary of Proposition 2.1.3 in [5].

Lemma 3. *For any $A \in L^\infty([0,T];\mathcal{L}_s(\mathbf{R}^N))$, we have*
(i) $\nu_p(A)$ is the dimension of the solution subspace of

$$\begin{cases} (-1)^{p+1}u^{(2p)} + A(t)u = 0, \quad t \in [0,T], \\ u^{(j)}(0) = u^{(j)}(T), \ j = 0,1,\cdots,2p-1, \end{cases} \tag{2.9}$$

and $\nu_p(A) \in \{0,1,\cdots,pN\}$.
(ii) $i_p(A) = \sum_{\lambda<0} \nu_p(A+\lambda I_N)$.
(iii) If $i_p(A) = 0$, then

$$\int_0^T \left|u^{(p)}(t)\right|^2 dt \geq \int_0^T (A(t)u(t),u(t))\,dt, \qquad \forall u \in H_1^p.$$

And the equality holds if and only if $u \in H_1^{p\,0}(A)$.
(iv) $\nu_p(A) = m^0(\phi_A)$, $i_p(A) = m^-(\phi_A)$.
For any $A_1, A_2 \in L^\infty([0,T];\mathcal{L}_s(\mathbf{R}^N))$, we have
(v) If $A_1 \leq A_2$, then $i_p(A_1) \leq i_p(A_2)$ and $i_p(A_1)+\nu_p(A_1) \leq i_p(A_2)+\nu_p(A_2)$; if $A_1 < A_2$, then $i_p(A_1)+\nu_p(A_1) \leq i_p(A_2)$.
(vi) If $i_p(A_1) = i_p(A_2) > 0$, $\nu_p(A_2) = 0$, then $H_1^p = H_1^{p-}(A_1) \oplus H_1^{p+}(A_2)$.

3. Proofs of main results

Proof of Theorem 1. By Theorem 5 and Lemma 3(iv), we complete the whole proof by three steps.
Step 1: $\varphi \in C^2(H_1^p,\mathbf{R})$ and 0 is a non-degenerate critical point of φ.

If (I_1) and (V_1) hold, then by the continuity of $I_{ijk}(i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C})$, for every $u \in H_1^p$, $\varphi''(u)$ is determined by

$$\begin{aligned}\langle \varphi''(u)v,w\rangle = \int_0^T (w^{(p)}(t),v^{(p)}(t))\,dt - \int_0^T (D_u^2V(t,u(t))w(t),v(t))\,dt \\ + \sum_{i=1}^N\sum_{j=1}^p\sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k))w_i^{(j-1)}(t_k)\end{aligned} \tag{3.1}$$

for all $v,w \in H_1^p$. Moreover, we have $\varphi \in C^2(H_1^p,\mathbf{R})$. Meanwhile, (2.4) together with (I_1) and (V_2) implies $\varphi'(0) = 0$. Namely, 0 is a critical point of φ. In addition, by (3.1), we have

$$\langle \varphi''(0)v,w\rangle = \int_0^T (w^{(p)}(t),v^{(p)}(t))\,dt - \int_0^T (A_0(t)w(t),v(t))\,dt.$$

If $\varphi''(0)v = 0$, then for all $w \in H_1^p$,

$$\int_0^T (w^{(p)}(t),v^{(p)}(t))\,dt - \int_0^T (A_0(t)w(t),v(t))\,dt = 0.$$

It implies that v is a solution of (2.9) where $A = A_0$. Because of $\nu_p(A_0) = 0$, we get $v = 0$. The only thing left to do is to prove that for all $w \in H_T^p$, $\varphi''(0)v = w$ has one solution in H_1^p. Indeed, $\varphi''(0) = Id - K$ where $K : H_1^p \to H_1^p$ defined by

$$\langle Kv, w\rangle = \int_0^T (A_0(t)w(t), v(t))\,dt \quad \forall v, w \in H_1^p.$$

Considering that K is compact, we obtain that $R(\varphi''(0)) = H_1^p$ from $\ker \varphi''(0) = \{0\}$. Hence, $\varphi''(0)$ has a bounded inverse. Namely, 0 is a non-degenerate critical point of φ.

Step 2: φ satisfies PS-condition.

If (I_1) and $(V_1)-(V_3)$ with $i(A_1) = i(A_2) > 0$, $\nu(A_2) = 0$ hold, then φ satisfies PS-condition.

Let $\{\varphi(u_n)\}$ be a bounded sequence such that $\varphi'(u_n) \to 0$. We first prove that $\{u_n\}$ is bounded in H_T^p. Indeed, by (2.4), we have

$$\begin{aligned}\int_0^T (u_n^{(p)}, v^{(p)})\,dt = \langle \varphi'(u_n), v\rangle + \int_0^T (\nabla V(t, u_n), v)\,dt \\ - \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_{n_i}^{(j-1)}(t_k))v_i^{(j-1)}(t_k). \end{aligned}\tag{3.2}$$

Let $F = C([0,T]; \mathbf{R}^N)$ with the norm $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$. We only need to verify that $\{u_n\}$ will be bounded in F. If not, we can assume $\|u_n\|_\infty \to +\infty$ and set $v_n = u_n/\|u_n\|_\infty$.

By (3.2) and (2.5), we get

$$\begin{aligned}&\int_0^T (u_n^{(p)}, v^{(p)})\,dt \\ &= \|u_n\|_\infty^{-1}\langle \varphi'(u_n), v\rangle + \int_0^T (A(t, u_n)v_n(t), v)\,dt + \|u_n\|_\infty^{-1}\int_0^T (g(t, u_n), v)\,dt \\ &\quad - \|u_n\|_\infty^{-1} \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q (-1)^{j+p} I_{ijk}(u_i^{(j-1)}(t_k))v_i^{(j-1)}(t_k). \end{aligned}\tag{3.3}$$

It implies that $\{v_n\}$ is bounded in H_1^p. Without loss of generality, we assume $v_n \rightharpoonup v_0$ in H_1^p, then $v_n \to v_0$ in F. By mean of $\|v_n\|_\infty = 1$, we have $\|v_0\|_\infty = 1$. In addition, (3.3) can become

$$\begin{aligned}&\int_0^T \Big((u_n^{(p)}, v^{(p)}) - (A(t, u_n)v_n, v)\Big)\,dt \\ &= \|u_n\|_\infty^{-1}\langle \varphi'(u_n), v\rangle + \|u_n\|_\infty^{-1}\int_0^T (g(t, u_n), v)\,dt\end{aligned}$$

$$-\|u_n\|_\infty^{-1}\sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q}(-1)^{j+p}I_{ijk}(u_i^{(j-1)}(t_k))v_i^{(j-1)}(t_k). \tag{3.4}$$

However, by (2.6) and (2.7), there exists $\tilde{A}\in L^\infty([0,T];\mathcal{L}_s(\mathbf{R}^N))$ such that

$$\int_0^T (A(t,u_n)v(t),w(t))\,dt \to \int_0^T (\tilde{A}(t)v(t),w(t))\,dt \quad \forall v,w\in L^2([0,T],\mathbf{R}^N), \tag{3.5}$$

by going to subsequences if necessary, and

$$A_1-\varepsilon I_N \le \tilde{A} \le A_2+\varepsilon I_N. \tag{3.6}$$

Since $\varphi'(u_n)\to 0$ and $\|u_n\|_\infty \to +\infty$, the right side of (3.4) tends to zero. (3.4) and (3.5) imply that

$$\int_0^T \Big((v_0^{(p)}(t),v^{(p)}(t))-(\tilde{A}(t)v_0(t),v(t))\Big)\,dt=0, \qquad \forall v\in H_1^p. \tag{3.7}$$

Hence, v_0 is the solution of the following problem

$$\begin{cases}(-1)^{p+1}v^{(2p)}(t)+\tilde{A}(t)v(t)=0,\\ v^{(j)}(0)=v^{(j)}(T), \quad j=0,1,\cdots,2p-1.\end{cases} \tag{3.8}$$

By (3.6) and Lemma 3(v), we have $i_p(\tilde{A})\le i_p(A_2+\varepsilon I_N)$ and $i_p(\tilde{A})+\nu_p(\tilde{A})\le i_p(A_2+\varepsilon I_N)+\nu_p(A_2+\varepsilon I_N)$. But $\nu_p(A_2+\varepsilon I_N)=\nu_p(A_2)=0$. Hence, $\nu_p(\tilde{A})=0$ and (3.8) has only trivial solution. This contradicts $\|v_0\|_\infty=1$.

Next, we prove that $\{u_n\}$ contains a convergent subsequence. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_m}\}$ such that $u_{n_m}\rightharpoonup u_0$ in H_1^p. Then $u_{n_m}\to u_0$ uniformly in $[0,1]$ and

$$u_{n_m}\to u_0 \tag{3.9}$$

in$L^2([0,T]),\mathbf{R}^N)$. By (3.2), we have

$$\int_0^T (u_{n_m}^{(p)},v^{(p)})\,dt=\langle\varphi'(u_{n_m}),v\rangle+\int_0^T(\nabla V(t,u_{n_m}),v)\,dt$$
$$-\sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q}(-1)^{j+p}I_{ijk}({u_{n_m}}_i^{(j-1)}(t_k))v_i^{(j-1)}(t_k).$$

Moreover, $\varphi'(u_{n_m})\to 0$ implies

$$\int_0^T (u_0^{(p)},v^{(p)})\,dt=\int_0^T(\nabla V(t,u_0),v)\,dt$$
$$-\sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q}(-1)^{j+p}I_{ijk}(u_{0_i}^{(j-1)}(t_k))v_i^{(j-1)}(t_k), \quad \forall v\in H_1^p.$$

Hence,

$$\int_0^T \left(\left(u_{n_m}^{(p)}(t) - u_0^{(p)}(t)\right), v^{(p)}(t)\right) dt$$
$$= \langle \varphi'(u_{n_m}), v\rangle + \int_0^T ((\nabla V(t, u_{n_m}(t)) - \nabla V(t, u_0(t))), v(t))\, dt$$
$$- \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q} (-1)^{j+p} \left(I_{ijk}(u_{n_m}{}_i^{(j-1)}(t_k)) - I_{ijk}(u_{0_i}^{(j-1)}(t_k)\right) v_i^{(j-1)}(t_k).$$

It follows that

$$\|u_{n_m} - u_0\|$$
$$= \|u_{n_m}^{(p)} - u_0^{(p)}\|_{L^2} + \|u_{n_m} - u_0\|_{L^2}$$
$$= \sup_{\|v\|\le 1} \int_0^1 \left(\left(u_{n_m}^{(p)}(t) - u_0^{(p)}(t)\right), v^{(p)}\right) dt + \|u_{n_m} - u_0\|_{L^2}$$
$$= \sup_{\|v\|\le 1} \Bigg(\varphi'(u_{n_m}), v + \int_0^1 \left(V'(t, u_{n_m}) - V'(t, u_0)\right) v\, dt$$
$$- \sum_{i=1}^{N}\sum_{j=1}^{p}\sum_{k=1}^{q} (-1)^{j+p} \left(I_{ijk}(u_{n_m}{}_i^{(j-1)}(t_k)) - I_{ijk}(u_{0_i}^{(j-1)}(t_k)\right) v_i^{(j-1)}(t_k)\Bigg)$$

By $\varphi'(u_{n_m}) \to 0$, (V_1), the continuity of $I_{ijk}(i \in \mathbf{A}, j \in \mathbf{B}, k \in \mathbf{C})$ and (3.9), we have $\|u_{n_m} - u_0\| \to 0$. Namely, $u_{n_m} \to u_0$ in H_1^p.
Step 3: For some regular value $a < \varphi(0)$,

$$H_q(H_1^p, \varphi_a; \mathbf{R}) \cong \delta_{q\gamma}\mathbf{R}, \quad where \quad \gamma = i_p(A_1). \tag{3.10}$$

By Lemma 2, we know that

$$\nabla V(t,u) = A(t,u)u + g(t,u), \tag{3.11}$$

where

$$A_1 - \varepsilon I_N \le A(t,u) \le A_2 + \varepsilon I_N$$

and $g(t,u)$ is bounded since $(V_1)-(V_3)$ with $i_p(A_1) = i_p(A_2) > 0$ hold. In addition, if let $A_3(t) = A_1(t) - \varepsilon I_N$, $A_4(t) = A_2(t) + \varepsilon I_N$, then

$$A_3(t) \le A(t,u) \le A_4(t). \tag{3.12}$$

Moreover, let $H_1 = H_1^{p-}(A_3)$, $H_2 = H_1^{p+}(A_4)$, then $H_1^p = H_1 \oplus H_2$ by Lemma 3(vi). Denote

$$\|u\|_1 := (-\phi_{A_3}(u,u))^{\frac{1}{2}}, \forall u \in H_1, \quad \|u\|_2 := \phi_{A_4}(u,u))^{\frac{1}{2}}, \forall u \in H_2.$$

From Lemma 3(iii), we can verify that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent to $\|\cdot\|$ on H_1 and H_2 respectively. Hence, by (2.4), for any $u = u_1 + u_2$ with $u_1 \in H_1, u_2 \in H_2$, we have

$$
\begin{aligned}
&\langle \varphi'(u), (u_2 - u_1) \rangle \\
&= \int_0^T (|u_2^{(p)}(t)|^2 - |u_1^{(p)}(t)|^2)\,dt + \int_0^T A(t,u(t))u(t) \cdot (u_2(t) - u_1(t))\,dt \\
&\quad - \int_0^T (g(t,u(t)), (u_2(t) - u_1(t)))\,dt + \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))(u_2 - u_1)^k(t_l) \\
&= \int_0^T |u_2^{(p)}(t)|^2\,dt - \int_0^T (A(t,u(t))u_2(t), u_2(t))\,dt - \int_0^T (g(t,u(t)), u_2(t))\,dt \\
&\quad - \int_0^T |u_1^{(p)}(t)|^2\,dt + \int_0^T (A(t,u(t))u_1(t), u_1(t))\,dt + \int_0^T (g(t,u(t)), u_1(t))\,dt \\
&\quad + \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))u_2^k(t_l) - \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))u_1^k(t_l) \\
&\geq \int_0^T |u_2^{(p)}(t)|^2\,dt - \int_0^T (A_4 u_2(t), u_2(t))\,dt - \int_0^T (g(t,u(t)), u_2(t))\,dt \\
&\quad - \int_0^T |u_1^{(p)}(t)|^2\,dt + \int_0^1 (A_3 u_1(t), u_1(t))\,dt + \int_0^T (g(t,u(t)), u_1(t))\,dt \\
&\quad + \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))u_2^k(t_l) - \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))u_1^k(t_l) \\
&= \|u_2\|_2^2 - \int_0^T (g(t,u(t)), u_2(t))\,dt + \|u_1\|_1^2 + \int_0^T (g(t,u(t)), u_1(t))\,dt \\
&\quad + \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))u_2^k(t_l) - \sum_{l=1}^{M}\sum_{k=1}^{N} I_{kl}(u^k(t_l))u_1^k(t_l) \\
&\geq C_1\|u_2\|^2 + C_2\|u_1\|^2 - C_3\|u_2\| - C_4\|u_1\| + C_5 \qquad (3.13)
\end{aligned}
$$

where $C_i \in \mathbf{R}^+ (i = 1,2)$, $\mathbf{R}^+$ is a set of positive constants, $C_i \in \mathbf{R} (i = 3,4,5)$. It follows that there exists $R_0 > 0$ such that

$$\langle \varphi'(u), u_2 - u_1 \rangle > 1, \qquad \forall u \in H_1^p$$

with $\|u_2\| > R_0$ or $\|u_1\| > R_0$. Denote by $\mathcal{M} = (H_2 \cap \bar{U}_{R_0}) \oplus H_1$, where $\bar{U}_{R_0}$ is a closed ball with center 0 and radius R_0. Since φ is decreasing along vector field $V(u) = -u_2 + u_1$ for every $u = u_2 + u_1 \notin \mathcal{M}$, we can define the flow $\sigma(t,u) =$

$e^{-t}u_2+e^tu_1$, and the time T_u arriving at $\mathcal{M}$ satisfies $e^{-T_u}\|u_2\|=R_0$. Set

$$\eta(t,u_2+u_1)=\begin{cases}u_2+u_1, & \text{for } \|u\|\le R_0,\\ \sigma(T_ut,u), & \text{for } \|u\|>R_0.\end{cases}$$

We can verify that for any $-a>-\varphi(0)$ large enough, $\eta(t,u)$ is a deformation retract from (H_1^p,φ_a) to $(\mathcal{M},\mathcal{M}\cap\varphi_a)$. Hence,

$$H_q(H_1^p,\varphi_a;\mathbf{R})\cong H_q(\mathcal{M},\mathcal{M}\cap\varphi_a;\mathbf{R}). \tag{3.14}$$

On the other hand, by (3.11), we have

$$\begin{aligned}V(t,u)&=\int_0^T(\nabla V(t,su),u)\,ds+V(t,0)\\&=\int_0^T(A(t,su)su,u)\,ds+\int_0^T(g(t,su),u)\,ds+V(t,0).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^T V(t,u)\,dt\\&=\int_0^T\left(\int_0^T(A(t,su)su,u)\,ds+\int_0^T(g(t,su),u)\,ds+V(t,0)\right)dt\\&=\int_0^T\int_0^T(A(t,su)su,u)\,ds\,dt+\int_0^T\int_0^T(g(t,su),u)\,ds\,dt+\int_0^T V(t,0)\,dt\\&\ge\int_0^T\int_0^T(A_3su,u)\,ds\,dt+\int_0^T\int_0^T(g(t,su),u)\,ds\,dt+\int_0^T V(t,0)\,dt\\&=\frac12\int_0^T(A_3u,u)\,dt+\int_0^T\int_0^T(g(t,su),u)\,ds\,dt+\int_0^T V(t,0)\,dt\end{aligned}$$

because of (3.12). Hence, for any $u=u_1+u_2\in\mathcal{M}$, we have

$$\begin{aligned}\varphi(u)\le&\frac12\int_0^T|u^{(p)}|^2\,dt-\frac12\int_0^T(A_3u,u)\,dt-\int_0^T\int_0^T(g(t,su),u)\,ds\,dt\\&-\int_0^T V(t,0)\,dt+\sum_{i=1}^N\sum_{j=1}^p\sum_{k=1}^q\int_0^{u_i^{(j-1)}(t_k)}I_{ijk}(s)\,ds\\\le&\frac12\int_0^T\left((|u_1^{(p)}|^2+|u_2^{(p)}|^2+2(u_1^{(p)},u_2^{(p)})\right)dt-\frac12\int_0^T(A_3u_1,u_1)\,dt\\&-\frac12\int_0^T(A_3u_2,u_2)\,dt-\int_0^T(A_3u_1,u_2)\,dt-\int_0^T\int_0^T(g(t,su),u_1)\,ds\,dt\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_0^T (g(t,su),u_2)\,ds\,dt + C_6\\
&= \frac{1}{2}\int_0^T \left(|u_1^{(p)}|^2-(A_3u_1,u_1)\right)dt + \frac{1}{2}\int_0^T \left(|u_2^{(p)}|^2-(A_3u_2,u_2)\right)dt\\
&\quad+\int_0^T \left((u_1^{(p)},u_2^{(p)})-(A_3u_1,u_2)\right)dt - \int_0^T\int_0^T (g(t,su),u_1)\,ds\,dt\\
&\quad-\int_0^T\int_0^T (g(t,su),u_2)\,ds\,dt + C_6\\
&= \phi_{A_3}(u_1,u_1)+\phi_{A_3}(u_2,u_2) - \int_0^T\int_0^T (g(t,su),u_1)\,ds\,dt\\
&\quad-\int_0^T\int_0^T (g(t,su),u_2)\,ds\,dt + C_6\\
&\le -C_7\|u_1\|^2 + C_8\|u_1\| + C_9, \qquad (3.15)
\end{aligned}
$$

where $C_7 \in \mathbf{R}^+, C_i \in \mathbf{R}$ $(i = 6,8,9)$.

By (3.12), we get $0 < -\phi_{A_3}(u,u) \le -\phi_{A_4}(u,u)$. Since $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ on H_1 and H_1 is finitely dimensional, $(-\phi_{A_4}(\cdot,\cdot))^{\frac{1}{2}}$ is also the norm of H_1. Similar to (3.15), we have

$$
\varphi(u) \ge -C_{10}\|u_1\|^2 + C_{11}\|u_1\| + C_{12}, \qquad (3.16)
$$

where $C_{10} \in \mathbf{R}^+, C_i \in \mathbf{R} (i = 11,12)$.

(3.15) and (3.16) show that for any $u = u_1 + u_2 \in \mathcal{M}$,

$$
\varphi(u) \to -\infty \qquad \textit{if and only if} \qquad \|u_1\| \to +\infty
$$

uniformly in $u_2 \in H_2 \cap \bar{U}_{R_0}$. Thus, there exist $T > 0, a_1 < a_2 < -T, R_1 > R_2 > R_0$ such that

$$
(H_2 \cap \bar{U}_{R_0}) \oplus (H_1 \backslash U_{R_1}) \subset \varphi_{a_1} \cap \mathcal{M} \subset (H_2 \cap \bar{U}_{R_0}) \oplus (H_1 \backslash U_{R_2}) \subset \varphi_{a_2} \cap \mathcal{M}.
$$

Let $\sigma(t,u) = e^{-t}u_2 + e^t u_1$, for every $u \in \mathcal{M} \cap (\varphi_{a_2} \backslash \varphi_{a_1})$, $\varphi(\sigma(t,u))$ is continuous with respect to t, $\varphi(\sigma(0,u)) = \varphi(u) > a_1$ and $\varphi(\sigma(t,u)) \to -\infty (t \to +\infty)$, so there exists uniquely $t = T_1(u)$ such that $\sigma(t,u) \in \mathcal{M} \cap \varphi_{a_1}$ and $\varphi(\sigma(t,u)) = a_1$. Define

$$
\eta_1(t,u) = \begin{cases} u, & \text{for } u \in \mathcal{M} \cap \varphi_{a_1},\\ \sigma(T_1(u)t,u), & \text{for } u \in \mathcal{M} \cap (\varphi_{a_2} \backslash \varphi_{a_1}) \end{cases}
$$

and

$$
\eta_2(t,u) = \begin{cases} u, & \text{for } \|u_1\| \ge R_1,\\ u_2 + tu_1 + (1-t)\frac{R_1}{\|u_1\|}u_1, & \text{for } \|u_1\| < R_1. \end{cases}
$$

By the map $\eta(t,u) = \eta_2(t,\eta_1(t,u))$ we can verify that $(H_2 \cap \bar{U}_{R_0} \oplus (H_1 \backslash U_{R_1}); \mathbf{R})$ is a strong deformation retract of $(\mathcal{M} \cap \varphi_{a_2}; \mathbf{R})$. Hence,

$$\begin{aligned} H_q(\mathcal{M}, \mathcal{M} \cap \varphi_{a_2}; \mathbf{R}) &\cong H_q((H_2 \cap \bar{U}_{R_0}) \oplus H_1, (H_2 \cap \bar{U}_{R_0}) \oplus (H_1 \backslash U_{R_1}); \mathbf{R}) \\ &\cong H_q(H_1 \cap \bar{U}_{R_1}, \partial(H_1 \cap U_{R_1}); \mathbf{R}) \\ &\cong \delta_{q\gamma} \mathbf{R}. \end{aligned} \tag{3.17}$$

(3.14) and (3.17) imply that (3.10) holds.

Now all the conditions of Theorem 5 are satisfied, and the corresponding conclusions hold. The proof is complete. □

Proof of Theorem 2. By the proof of Theorem 1, we know that $\varphi \in C^1(H_1^p, \mathbf{R})$, and φ satisfies PS-condition when $(I_1), (V_1) - (V_3)$ with $i_p(A_1) = i_p(A_2) > 0$, $\nu_p(A_2) = 0$ hold. Moreover, we find that φ is even and $\varphi(0) = 0$ when $(I_1), (I_2), (V_4)$ and (V_5) hold. In fact,

$$\begin{aligned} &\varphi(-u) \\ &= \frac{1}{2}\int_0^T |-u^{(p)}|^2\,dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{-u_i^{(j-1)}(t_k)} I_{ijk}(s)\,ds - \int_0^T V(t,-u)\,dt. \\ &= \frac{1}{2}\int_0^T |u^{(p)}|^2\,dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{u_i^{(j-1)}(t_k)} -I_{ijk}(-s_1)\,ds_1 - \int_0^T V(t,u)\,dt. \\ &= \frac{1}{2}\int_0^T |u^{(p)}|^2\,dt + \sum_{i=1}^N \sum_{j=1}^p \sum_{k=1}^q \int_0^{u_i^{(j-1)}(t_k)} I_{ijk}(s_1)\,ds_1 - \int_0^T V(t,u)\,dt \\ &= \varphi(u). \end{aligned}$$

Now it suffices to consider the two possibilities: $i_p(A_0) > i_p(A_1)$ and $i_p(A_0) < i_p(A_1)$. In fact, we only consider the second case because the first case can be investigated as before by Theorem 3. For small $\varepsilon > 0$ satisfying $\nu_p(A_0 + \varepsilon I_N) = 0 = \nu_p(A_1 - \varepsilon I_N)$, $i_p(A_0 + \varepsilon I_N) = i_p(A_0)$ and $i_p(A_1 - \varepsilon I_N) = i_p(A_1)$, similar to (3.13) and (3.15), we can verify that for any $x \in H_1^{p+}(A_0 + \varepsilon I_N)$,

$$\varphi(u) \geq C_{13}\|u\|^2 + C_{14}\|u\| + C_{15},$$

and for any $x \in H_1^{p-}(A_1 - \varepsilon I_N)$,

$$\varphi(u) \leq -C_{16}\|u\|^2 + C_{17}\|u\| + C_{18},$$

where $C_i \in \mathbf{R}^+ (i = 13, 16), C_i \in \mathbf{R}(i = 14, 15, 17, 18)$. Let $j = i_p(A_0), m = i_p(A_1)$. Hence, for these two subspaces $H_1^{p+}(A_0 + \varepsilon I_N)$ and $H_1^{p-}(A_1 - \varepsilon I_N)$ of H_1^p, Theorem 4$(i)(ii)$ hold. The proof is complete. □

4. EXAMPLES

For the sake of simplicity, we only consider (1.1) for the case of $N = p = q = T = 1$. In addition, let $t_1 = \frac{1}{2}$ and $I_{111}(s) = \frac{s}{|s|+1}$, $s \in \mathbf{R}$.

Example 1. Consider the following problem

$$\begin{cases} \ddot{u}(t) + \nabla V(t,u) = 0, \\ u(0) = u(1), \dot{u}(0) = \dot{u}(1), \\ \Delta(\dot{u}(t_1)) = \dot{u}(t_1^+) - \dot{u}(t_1^-) = I_{111}(u(t_1)), \end{cases} \tag{4.1}$$

where $V(t,u) = 4\pi^2 e^{-\frac{t}{10}} u^2 + \pi^2 e^{-\frac{t}{10}}[(u-6)\ln(u^2+1) + 12u \arctan u + 2\arctan u - 2u]$.

Clearly, $|I_{111}(s)| \le 1$ and $I_{111}(0) = 0$. $V(t,u)$ is C^2 in u for every $t \in [0,T]$ and $D_u^2 V(t,u) = \pi^2 e^{-\frac{t}{4}}(8 + \frac{2u+12}{u^2+1})$. Then, $7\pi^2 e^{-\frac{t}{10}} < D_u^2 V(t,u) < 9\pi^2 e^{-\frac{t}{10}}$ as $|u| > 5$. In addition, $A_0(t) = 20\pi^2 e^{-\frac{t}{10}}$. By Lemma 3, we obtain that $i_1(7.5\pi^2 e^{-\frac{t}{4}}) = i_1(8.5\pi^2 e^{-\frac{t}{4}}) = 3$, $i_1(A_0) = 5$, $\nu_1(A_0) = 0$ by simple calculations. Hence, (4.1) has two nontrivial weak solutions by Theorem 1.

Example 2. Consider (4.1), where $V(t,u) = \frac{7}{2}\pi^2 e^{-\frac{t}{10}} u^2 - 8\pi^2 e^{-\frac{t}{10}}[-\frac{1}{2}\ln(u^2+1) + u \arctan u]$. By Example 1, I_{111} satisfies (I_1). Meanwhile, I_{111} is odd. V satisfies (V_1)-(V_2) and is even in u for every $t \in [0,T]$. In addition, $V(t,0) \equiv 0$ and $D_u^2 V(t,u) = \pi^2 e^{-\frac{t}{10}}(7 - \frac{8}{u^2+1})$. Then $6.2\pi^2 e^{-\frac{t}{10}} < D_u^2 V(t,u) < 7\pi^2 e^{-\frac{t}{10}}$ as $|u| > 3$. Meanwhile, $A_0(t) = -\pi^2 e^{-\frac{t}{10}}$. $i_1(6.2\pi^2 e^{-\frac{t}{10}}) = i_1(7\pi^2 e^{-\frac{t}{10}}) = 3$, $\nu_1(7\pi^2 e^{-\frac{t}{10}}) = i_1(A_0) = \nu_1(A_0) = 0$. Hence, (4.1) has at least three distinct pairs of weak solutions by Theorem 2.

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