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SIGN-CONSTANCY OF GREEN'S FUNCTIONS FOR TWO-POINT IMPULSIVE BOUNDARY VALUE PROBLEMS

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Abstract. We consider the following second order impulsive differential equation with delays

$$\begin{cases}(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t),\\ \quad t \in [0,\omega],\\ x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,...,r.\end{cases}$$

In this paper we obtain sufficient conditions of nonpositivity of Green's functions for impulsive differential equation. All results are formulated in the form of theorems about differential inequalities. It should be noted that the sign-constancy of the coefficients $b_j(t)$ was assumed in all the literature devoted to impulsive functional differential equations. One of the main purposes of this work is to propose a technique allowing us to avoid these assumptions.

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Keywords: second order impulsive differential equations, boundary value problems, sign-constancy of Green's functions

1. INTRODUCTION

Impulsive differential equations have attracted an attention of a number of recognized mathematicians and have applications in many spheres of science from physics, biology, medicine to economical studies. The following well-known books can be noted in this context [14, 17–19]. In the book [3], the concept of the general theory of functional differential equations was presented. On the basis of this concept nonoscillation for the first order functional differential equations was considered in [4], where positivity of the Cauchy and Green's functions of the periodic problem was firstly studied. A concept of nonoscillation for the first order differential equations is also considered in the book [1]. The positivity of Green's function of one- and two-point boundary value problems for functional differential impulsive equations was studied in [2, 5–8, 10–13, 15]. It should be noted that the sign-constancy of the coefficients $b_j(t)$ was assumed in all the results for impulsive functional differential

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equations. One of the main purposes of this work is to propose a technique allowing us to avoid these assumptions.

Let us consider the following impulsive equations:

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega], \tag{1.1}$$

$$\begin{aligned} &x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,\dots,r, \\ &0 = t_0 < t_1 < t_2 < \dots < t_r < t_{r+1} = \omega, \end{aligned} \tag{1.2}$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \tag{1.3}$$

where f, a_j, b_j: $[0,\omega] \to \mathbb{R}$ are summable functions and τ_j, θ_j: $[0,\omega] \to [0,+\infty)$ are measurable functions for $j = 1,2,\dots,p$, p and r are natural numbers, γ_k and δ_k are real positive numbers.

Let $D(t_1,t_2,\dots,t_r)$ be a space of functions x: $[0,\omega] \to \mathbb{R}$ such that their derivative $x'(t)$ is absolutely continuous on every interval $t \in [t_i,t_{i+1})$, $i = 0,1,\dots,r$, $x'' \in L_\infty$, we assume also that there exist the finite limits $x(t_i - 0) = \lim_{t\to t_i^-} x(t)$ and $x'(t_i - 0) = \lim_{t\to t_i^-} x'(t)$ and condition (1.2) is satisfied at points t_i $(i = 0,1,\dots,r)$. As a solution x we understand a function $x \in D(t_1,t_2,\dots,t_r)$ satisfying (1.1)-(1.3).

There is not many works on sign-constancy of Green's functions of second-order impulsive boundary value problems. We can note only the results of [5–8, 16] where these problems are considered. In these papers the coefficients $b_j(t)$ were assumed to be nonpositive. Using approaches of these papers, we obtain the results on sign-constancy of Green's functions to impulsive two-point boundary value problems without sign assumption on the coefficients $b_j(t)$. Our results are presented in the form of algebraic inequalities, establishing the smallness of the coefficients $a_j(t)$, $|b_j(t)|$, $j = 1,\dots,p$, $t \in [0,\omega]$.

2. PRELIMINARIES

For equation (1.1)-(1.3) we consider the following variants of boundary conditions:

$$x(0) = 0, \quad x(\omega) = 0, \tag{2.1}$$

$$x'(0) = 0, \quad x(\omega) = 0, \tag{2.2}$$

$$x(0) = 0, \quad x'(\omega) = 0, \tag{2.3}$$

$$x'(0) = 0, \quad x'(\omega) = 0. \tag{2.4}$$

General solution of the equation (1.1)-(1.3) can be represented in the form [4]:

$$x(t) = \nu_1(t)x(0) + C(t,0)x'(0) + \int_0^t C(t,s)f(s)ds, \tag{2.5}$$

where

- $\nu_1(t)$ is a solution of the homogeneous equation

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = 0, \quad t \in [0,\omega], \tag{2.6}$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,...,r, \\ 0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega, \tag{2.7}$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \tag{2.8}$$

with the initial conditions $x(0) = 1$, $x'(0) = 0$.

- $C(t,s)$, called the Cauchy function of the equation (2.6)-(2.8), is the solution of the equation

$$(L_s x)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = 0, \quad t \in [s,\omega], \tag{2.9}$$

$$x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = m,...,r, \\ 0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega, \tag{2.10}$$

where m is a number, such that $t_{m-1} < s \le t_m$,

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < s, \tag{2.11}$$

satisfying the initial conditions $C(s,s) = 0$, $C_t'(s,s) = 1$ and $C(t,s) = 0$ for $t < s$.

If the boundary value problem (1.1)-(1.3), (2.i), $i = \overline{1,4}$ is uniquely solvable, then its solution can be represented as

$$x(t) = \int_0^{\omega} G_i(t,s) f(s) ds, \quad i = \overline{1,4}, \tag{2.12}$$

where $G_i(t,s)$ is Green's function of the problem (1.1)-(1.3), (2.i) respectively [5].

Using general representation of the solution (2.5), the following formulas for Green's functions can be obtained:

$$G_1(t,s) = C(t,s) - C(t,0)\frac{C(\omega,s)}{C(\omega,0)}, \tag{2.13}$$

$$G_2(t,s) = C(t,s) - C(\omega,s)\frac{\nu_1(t)}{\nu_1(\omega)}, \tag{2.14}$$

$$G_3(t,s) = C(t,s) - C(t,0)\frac{C_t'(\omega,s)}{C_t'(\omega,0)}, \tag{2.15}$$

$$G_4(t,s) = C(t,s) - C_t'(\omega,s)\frac{\nu_1(t)}{\nu_1'(\omega)}. \tag{2.16}$$

Below the following definition will be used.

Definition 1. We call $[0,\omega]$ a semi-nonoscillation interval of (2.6)-(2.8), if every nontrivial solution having zero of derivative does not have zero on this interval.

3. Sign-constancy of Green's functions for $b_j(t) \le 0$

Denote $G^{\xi}(t,s)$ the Green's function of the problem (1.1)-(1.3) with boundary conditions

$$x(\xi) = 0, \quad x'(\xi) = 0. \tag{3.1}$$

In the paper [5], the following theorem has been proven for the problems (1.1)-(1.3), (2.i).

Lemma 1. *Assume that the following conditions are fulfilled:*

(1) $b_j(t) \le 0,\ j = 1,\dots,p,\ t \in [0,\omega]$.

(2) *The Cauchy function $C_1(t,s)$ of the first order impulsive equation*

$$\begin{aligned} &y'(t) + \sum_{j=1}^{p} a_j(t) y(t - \tau_j(t)) = 0, \quad t \in [0,\omega], \\ &y(t_k) = \delta_k y(t_k - 0), \quad k = 1,2,\dots,r, \\ &y(\zeta) = 0, \quad \zeta < 0, \end{aligned} \tag{3.2}$$

is positive for $0 \le s \le t \le \omega$.

(3) *Green's function $G^{\xi}(t,s)$ of the problem (1.1)-(1.3),(3.1) is nonnegative for $t,s \in [0,\xi]$ for every $0 < \xi < \omega$.*

(4) $[0,\omega]$ *is a semi-nonoscilation interval of* $(Lx)(t) = 0$.

Then Green's functions $G_i(t,s)$, $i = \overline{1,3}$ are nonpositive for $t,s \in [0,\omega]$ and under the additional condition $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t - \theta_j(t)) \not\equiv 0$, $t \in [0,\omega]$, where

$$\chi_{[0,\omega]}(t) = \begin{cases} 1, & t \in [0,\omega], \\ 0, & t \notin [0,\omega], \end{cases} \tag{3.3}$$

$G_4(t,s) \le 0$ for $t,s \in [0,\omega]$.

Using the results of [9] (see Theorems 4.1 and 4.2 from [9]), it is easy to see, that if $a_j(t) \ge 0,\ j = 1,\dots,p,\ t \in [0,\omega]$, then the assumption 4) follows from the rest 3 assumptions of Lemma 1, so we can exclude it and rewrite Theorem 1 in the following form.

Theorem 1. *Assume that the following conditions are fulfilled:*

(1) $a_j(t) \ge 0,\ b_j(t) \le 0,\ j = 1,\dots,p,\ t \in [0,\omega]$.

(2) *The Cauchy function $C_1(t,s)$ of the first order equation (3.2) is positive for $0 \le s \le t \le \omega$.*

(3) *Green's function $G^{\xi}(t,s)$ of the problem (1.1)-(1.3),(3.1) is nonnegative for $t,s \in [0,\xi]$ for every $0 < \xi < \omega$.*

Then Green's functions $G_i(t,s)$, $i=\overline{1,3}$ *are nonpositive for* $t,s\in[0,\omega]$ *and under the additional condition* $\sum_{j=1}^{p} b_j(t)\chi_{[0,\omega]}(t-\theta_j(t))\not\equiv 0$, $t\in[0,\omega]$, *Green's function* $G_4(t,s)\le 0$ *for* $t,s\in[0,\omega]$.

In Theorem 1 we have assumed that the Cauchy function $C_1(t,s)$ of the first order impulsive equation (3.2) is positive. In the lemma below, we will formulate the results of [4] on the conditions of positivity of Cauchy function of the first order impulsive differential equation

$$\begin{aligned} &y'(t)+a_1(t)y(t-\tau_1(t))=0, \quad t\in[0,\omega],\\ &y(t_k)=\delta_k y(t_k-0), \quad k=1,2,...,r,\\ &y(\zeta)=0, \quad \zeta<0. \end{aligned} \tag{3.4}$$

Lemma 2. *Let* $0<\delta_j<1$ *for* $j=1,...,r$ *and the following condition be fulfilled*

$$\frac{1+\ln B(t)}{e}\ge\int_{m(t)}^{t}a_+(s)ds, \tag{3.5}$$

where $B(t)=\prod_{j\in D_t}\delta_j$, $D_t=\{i: t_i\in[t-\tau_1(t),t]\}$, $a_+(t)=\max\{a_1(t),0\}$ *and* $m(t)=\max\{t-\tau_1(t),0\}$. *Then Cauchy function of the first order impulsive differential equation (3.4) is positive.*

In the case of $p>1$ the following sufficient condition proven in [4] can be used.

Lemma 3. *Let* $a_j(t)\ge 0$ *for* $j=1,...,p$, $0<\delta_k\le 1$ *for* $k=1,...,r$ *and*

$$\int_0^{\omega}\sum_{j=1}^{p}a_j(s)ds<\prod_{k=1}^{r}\delta_k, \tag{3.6}$$

then Cauchy function of the first order impulsive differential equation (3.2) is positive.

In the next section we will consider an auxiliary impulsive boundary value problem, which will provide the conditions of nonnegativity of Green's function $G^{\xi}(t,s)$.

4. AUXILIARY BOUNDARY VALUE PROBLEM $x''(t)=z(t)$

Let us consider the following equation:

$$x''(t)=z(t), \quad t\in[0,\omega], \tag{4.1}$$

$$x(t_k)=\gamma_k x(t_k-0), \quad x'(t_k)=\delta_k x'(t_k-0), \quad k=1,2,...,r, \tag{4.2}$$

$$x(\zeta)=0, \quad x'(\zeta)=0, \quad \zeta<0. \tag{4.3}$$

Denote by $\bar{C}(t,s)$ the Cauchy function of the equation (4.1)-(4.3). According to [6], Cauchy function for this impulsive equation has the form, represented on the

Figure 1. This corresponds to the following equation:

$$\begin{aligned}\bar{C}(t,s) = \chi_{[0,t]}(s)\Big[&\sum_{i=0}^{r}(t-s)\chi_{[t_i,t_{i+1})}(t)\chi_{[t_i,t_{i+1})}(s)\\ &+\sum_{i=1}^{r}(\gamma_i(t_i-s)+\delta_i(t-t_i))\chi_{[t_i,t_{i+1})}(t)\chi_{[t_{i-1},t_i)}(s)\\ &+\sum_{i=2}^{r}\sum_{j=0}^{i-2}\Big[\prod_{k=j+1}^{i}\gamma_k(t_{j+1}-s)+\sum_{l=j+2}^{i}\prod_{k=l}^{i}\gamma_k\prod_{k=j+1}^{l-1}\delta_k(t_l-t_{l-1})\\ &+\prod_{k=j+1}^{i}\delta_k(t-t_i)\Big]\chi_{[t_i,t_{i+1})}(t)\chi_{[t_j,t_{j+1})}(s)\Big].\end{aligned} \tag{4.4}$$

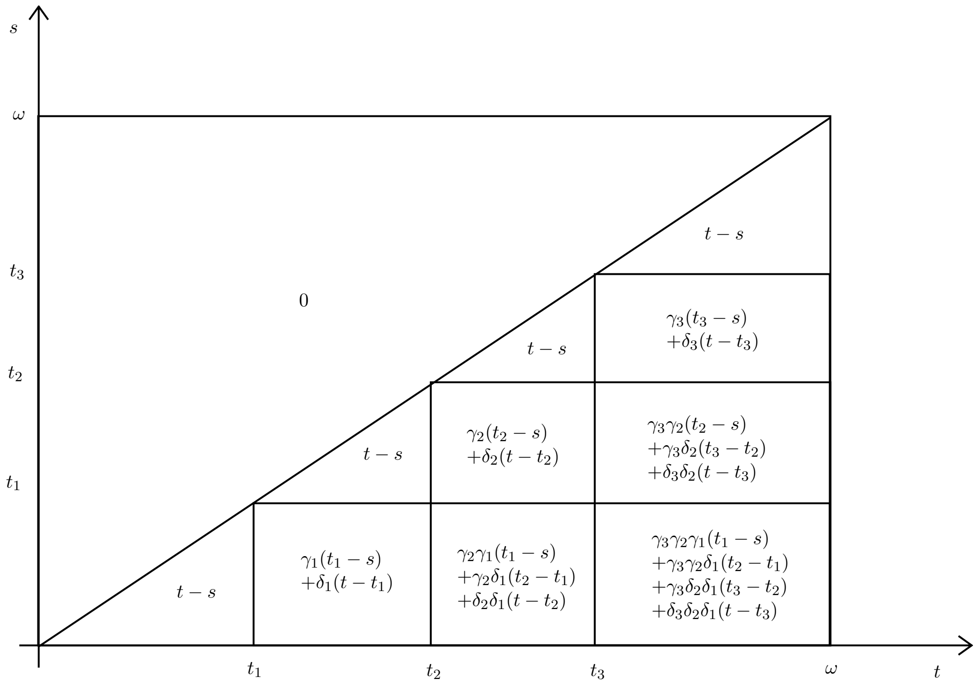

FIGURE 1. The Cauchy function of impulsive equation (4.1) - (4.3).

Let us build now the Green's function $\bar{G}(t,s)$ of the problem (4.1)-(4.3) with boundary conditions

$$x(\omega) = 0, \quad x'(\omega) = 0. \tag{4.5}$$

The solution of this boundary value problem will be of the following form:

$$x(t) = \int_0^{\omega} \Big[\bar{C}(t,s) + \frac{\nu_1(t)}{\nu_1(\omega)} \Big[\frac{\bar{C}'_t(\omega,s)}{\bar{C}'_t(\omega,0)} \bar{C}(\omega,0) - \bar{C}(\omega,s) \Big] - \frac{\bar{C}(t,0)}{\bar{C}'_t(\omega,0)} \Big] z(s)ds, \tag{4.6}$$

with the corresponding Green's function

$$\bar{G}(t,s) = \bar{C}(t,s) + \frac{\nu_1(t)}{\nu_1(\omega)} \Big[\frac{\bar{C}'_t(\omega,s)}{\bar{C}'_t(\omega,0)} \bar{C}(\omega,0) - \bar{C}(\omega,s) \Big] - \frac{\bar{C}(t,0)}{\bar{C}'_t(\omega,0)}, \tag{4.7}$$

where

$$\nu_1(t) = \begin{cases} 1, & t \in [0,t_1), \\ \prod_{k=1}^{r} \gamma_k, & t \in [t_k, t_{k+1}). \end{cases} \tag{4.8}$$

See the graphical representation of Green's function $\bar{G}(t,s)$ on the Figure 2.

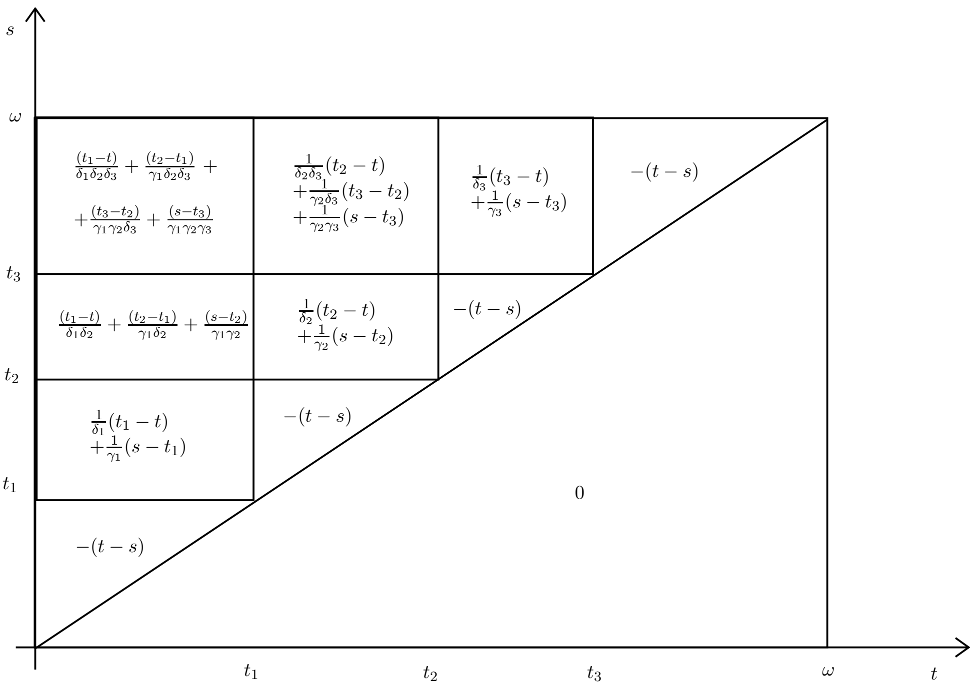

FIGURE 2. The Green's function of impulsive equation (4.1) - (4.3), (4.5).

Theorem 2. *If* $a_j(t) \geq 0,\ b_j(t) \leq 0$ *for* $j = \overline{1,p},\ 0 < \gamma_k \leq 1,\ 0 < \delta_k \leq 1$ *for* $k = \overline{1,r}$, *and if the condition*

$$
\begin{aligned}
&\frac{\omega}{\prod_{k=1}^{r}\delta_k}\operatorname*{ess\,sup}_{t\in[0,\omega]}\sum_{j=1}^{p}|a_j(t)| \\
&+\omega\left(\sum_{i=1}^{r}\frac{t_i-t_{i-1}}{\prod_{k=i}^{r}\delta_k\prod_{k=1}^{i}\gamma_k}\gamma_i+\frac{\omega-t_r}{\prod_{k=1}^{r}\gamma_k}\right)\operatorname*{ess\,sup}_{t\in[0,\omega]}\sum_{j=1}^{p}|b_j(t)|<1
\end{aligned}
\tag{4.9}
$$

is satisfied, then Green's function $G(t,s)$ *of the (1.1)-(1.3),(4.5) is nonnegative.*

Proof. We can rewrite equation (1.1)-(1.3) in the form

$$
\begin{aligned}
&z(t)+\sum_{j=1}^{p}a_j(t)\int_0^{\omega}\bar{G}_t'\left(t-\tau_j(t),s\right)\chi_{[0,\omega]}\left(t-\tau_j(t)\right)z(s)ds \\
&\quad+\sum_{j=1}^{p}b_j(t)\int_0^{\omega}\bar{G}\left(t-\theta_j(t),s\right)\chi_{[0,\omega]}\left(t-\theta_j(t)\right)z(s)ds=f(t).
\end{aligned}
\tag{4.10}
$$

Denote the operator $K\colon L_\infty \to L_\infty$ as follows:

$$
\begin{aligned}
(Kz)(s)=\int_0^{\omega}\Big[&-\sum_{j=1}^{p}a_j(t)\bar{G}_t'\left(t-\tau_j(t),s\right)\chi_{[0,\omega]}\left(t-\tau_j(t)\right) \\
&-\sum_{j=1}^{p}b_j(t)\bar{G}\left(t-\theta_j(t),s\right)\chi_{[0,\omega]}\left(t-\theta_j(t)\right)\Big]z(s)ds.
\end{aligned}
\tag{4.11}
$$

For $a_j(t) \geq 0,\ b_j(t) \leq 0,\ j = 1,\ldots,p$, the operator $K\colon L_\infty \to L_\infty$ is positive. If the spectral radius $\rho(K) < 1$, then the solution of (4.10) can be represented in the form:

$$
z=(I-K)^{-1}f=\left[\sum_{j=0}^{\infty}K^j\right]f. \tag{4.12}
$$

The solution $x(t)$ of the boundary value problem (1.1)-(1.3),(4.5) can be written in the form:

$$
x=\left(\bar{G}\sum_{j=0}^{\infty}K^j\right)f, \tag{4.13}
$$

where $\bar{G}\sum_{j=0}^{\infty}K^j$ is Green's operator for (1.1)-(1.3) with boundary conditions (4.5). From nonnegativity of Green's function $\bar{G}(t,s)$ and positivity of operator $K\colon L_\infty \to L_\infty$ it follows that Green's function $G(t,s)$ for (1.1)-(1.3),(4.5) is nonnegative.

Now let us find the conditions that the spectral radius of the operator $K\colon L_\infty \to L_\infty$ is less than one. It is satisfied, when

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in[0,\omega]}\sum_{j=1}^{p}|a_j(t)|\operatorname*{ess\,sup}_{t\in[0,\omega]}\int_0^\omega|\bar{G}_t'(t,s)|ds\\ +\operatorname*{ess\,sup}_{t\in[0,\omega]}\sum_{j=1}^{p}|b_j(t)|\operatorname*{ess\,sup}_{t\in[0,\omega]}\int_0^\omega|\bar{G}(t,s)|ds<1.\end{aligned}\tag{4.14}$$

If $0<\gamma_k\le 1$, $0<\delta_k\le 1$ for $k=\overline{1,r}$, then:

$$\operatorname*{ess\,sup}_{t,s\in[0,\omega]\times[0,\omega]}|\bar{G}(t,s)|=\sum_{i=1}^{r}\frac{t_i-t_{i-1}}{\prod_{k=i}^{r}\delta_k\prod_{k=1}^{i}\gamma_k}\gamma_i+\frac{\omega-t_r}{\prod_{k=1}^{r}\gamma_k},\tag{4.15}$$

$$\operatorname*{ess\,sup}_{t,s\in[0,\omega]\times[0,\omega]}|\bar{G}_t'(t,s)|=\frac{1}{\prod_{k=1}^{r}\delta_k}.\tag{4.16}$$

Thus, the spectral radius $\rho(K)<1$, if $0<\gamma_k\le 1$, $0<\delta_k\le 1$ for $k=\overline{1,r}$ and

$$\begin{aligned}\frac{\omega}{\prod_{k=1}^{r}\delta_k}\operatorname*{ess\,sup}_{t\in[0,\omega]}\sum_{j=1}^{p}|a_j(t)|\\ +\omega\left(\sum_{i=1}^{r}\frac{t_i-t_{i-1}}{\prod_{k=i}^{r}\delta_k\prod_{k=1}^{i}\gamma_k}\gamma_i+\frac{\omega-t_r}{\prod_{k=1}^{r}\gamma_k}\right)\operatorname*{ess\,sup}_{t\in[0,\omega]}\sum_{j=1}^{p}|b_j(t)|<1.\end{aligned}\tag{4.17}$$

□

Example 1. Let us consider the following differential equation

$$x''(t)+a_1(t)x'(t-\tau_1(t))+b_1(t)x(t-\theta_1(t))=f(t),\quad t\in[0,3],\tag{4.18}$$

with impulses (1.2), where

$$\begin{aligned}t_1&=1, & \gamma_1&=0.9, & \delta_1&=0.5,\\ t_2&=1.5, & \gamma_2&=0.6, & \delta_2&=0.7,\\ t_3&=2.5, & \gamma_3&=0.8, & \delta_3&=0.4,\end{aligned}\tag{4.19}$$

and $a_1(t)\ge 0$, $b_1(t)\le 0$.

For an auxiliary differential equation

$$x''(t)=z(t),\quad t\in[0,3],\tag{4.20}$$

with impulses (1.2), (4.19) and boundary conditions

$$x(3)=0,\quad x'(3)=0,\tag{4.21}$$

we constructed Green's function $\bar{G}(t,s)$. It is represented on the Figure 3 and its derivative $\bar{G}_t'(t,s)$ - on the Figure 4.

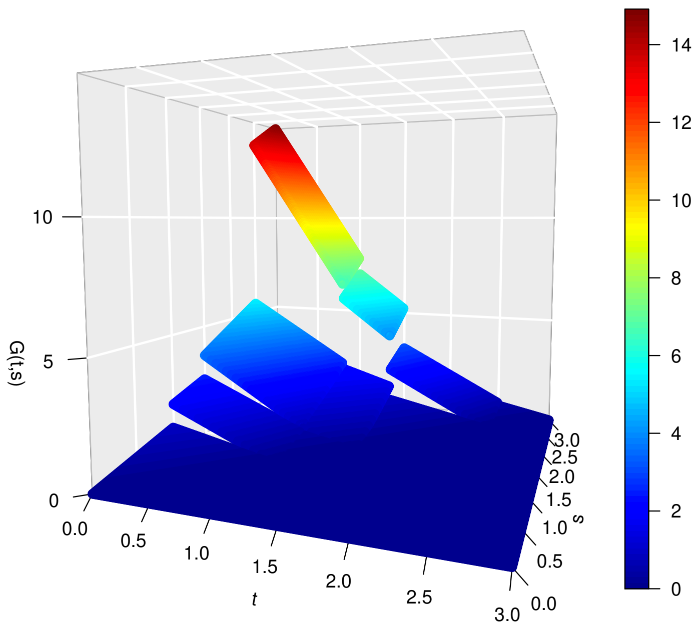

FIGURE 3. $\bar{G}(t,s)$ of (4.20), (4.19), (4.21).

Calculating maximum values of $|\bar{G}(t,s)|$ and $|\bar{G}_t'(t,s)|$ in $t,s \in [0,3]\times[0,3]$, we obtain:

$$\operatorname*{ess\,sup}_{t,s\in[0,3]\times[0,3]} |\bar{G}(t,s)| = 14.914, \tag{4.22}$$

$$\operatorname*{ess\,sup}_{t,s\in[0,3]\times[0,3]} |\bar{G}_t'(t,s)| = 7.143. \tag{4.23}$$

According to Theorem 2, for our example, we obtain the following condition of positivity of Green's function $G(t,s)$:

$$21.429 \operatorname*{ess\,sup}_{t\in[0,3]}|a_1(t)| + 44.742 \operatorname*{ess\,sup}_{t\in[0,3]}|b_1(t)| < 1. \tag{4.24}$$

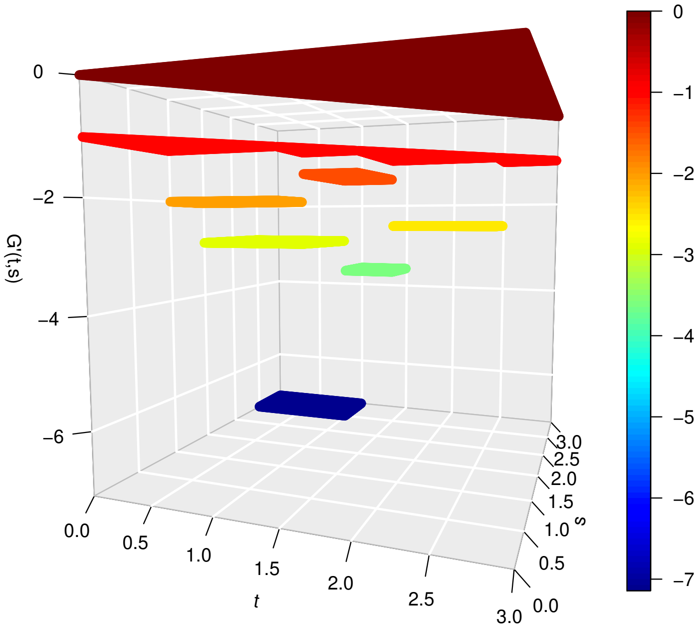

FIGURE 4. $\bar{G}'_t(t,s)$ of (4.20), (4.19), (4.21).

5. SIGN-CONSTANCY OF GREEN'S FUNCTIONS FOR THE CASE WHEN $b_j(t)$ CAN CHANGE SIGN

For the case when $b_j(t)$, $j = 1,...,p$, changes sign, there can be considered an auxiliary differential equation:

$$(L^- x)(t) \equiv x''(t) + \sum_{j=1}^{p} a_j(t)x'(t-\tau_j(t)) + \sum_{j=1}^{p} b_j^-(t)x(t-\theta_j(t)) = z(t), \quad t \in [0,\omega], \tag{5.1}$$

$$\begin{aligned} &x(t_k) = \gamma_k x(t_k - 0), \quad x'(t_k) = \delta_k x'(t_k - 0), \quad k = 1,2,...,r, \\ &0 = t_0 < t_1 < t_2 < ... < t_r < t_{r+1} = \omega, \end{aligned} \tag{5.2}$$

$$x(\zeta) = 0, \quad x'(\zeta) = 0, \quad \zeta < 0, \tag{5.3}$$

where

$$b_j^-(t) = \begin{cases} b_j(t), & b_j(t) \le 0, \\ 0, & b_j(t) > 0. \end{cases} \tag{5.4}$$

The solution for the boundary value problems (5.1)-(5.3), (2.i) can be written in the form

$$x(t) = \left(G_i^- z\right)(t) \equiv \int_0^\omega G_i^-(t,s)z(s)ds, \quad i = \overline{1,4}. \tag{5.5}$$

The given equation (1.1) can be written in the form:

$$(Lx)(t) = (L^- x)(t) + \sum_{j=1}^{p} b_j^+(t)x(t-\theta_j(t)) = f(t), \tag{5.6}$$

where

$$b_j^+(t) = \begin{cases} b_j(t), & b_j(t) > 0, \\ 0, & b_j(t) \le 0. \end{cases} \tag{5.7}$$

After the substitution (5.5) into (5.6) we obtain

$$z(t) + \sum_{j=1}^{p} b_j^+(t) \int_0^\omega G_i^-\left(t-\theta_j(t),s\right) \chi_{[0,\omega]}\left(t-\theta_j(t)\right) z(s)ds = f(t), \quad i = \overline{1,4}. \tag{5.8}$$

Define the integral operators $K_i : L_\infty \to L_\infty$ for each type of boundary conditions ($i = \overline{1,4}$) by the equality:

$$(K_i z)(t) = -\sum_{j=1}^{p} b_j^+(t) \int_0^\omega \left[G_i^-\left(t-\theta_j(t),s\right) \chi_{[0,\omega]}\left(t-\theta_j(t)\right)\right] z(s)ds. \tag{5.9}$$

Its spectral radius can be denoted by $\rho(K_i)$.

We propose an assertion about nonpositivity of Green's function of (1.1)-(1.3), (2.i) without an assumption $b_j(t) \le 0$, $j = 1,...,p$. For this case we can reformulate Theorem 1 as follows.

Theorem 3. *Assume that the following conditions are fulfilled:*

(1) $a_j(t) \ge 0$, $j = 1,...,p$, $t \in [0,\omega]$.
(2) *The Cauchy function* $C_1(t,s)$ *of the first order equation (3.2) is positive for* $0 \le s \le t \le \omega$.
(3) *Green's function* $G^\xi(t,s)$ *of the problem (5.1)-(5.3),(3.1) is nonnegative for* $t,s \in [0,\xi]$ *for every* $0 < \xi < \omega$.
(4) $\rho(K_i) < 1$.

Then Green's functions $G_i(t,s)$, $i = \overline{1,3}$ *are nonpositive for* $t,s \in [0,\omega]$ *and under the additional condition* $\sum_{j=1}^{p} b_j^-(t)\chi_{[0,\omega]}(t-\theta_j(t)) \not\equiv 0$, $t \in [0,\omega]$, *Green's function* $G_4(t,s) \le 0$ *for* $t,s \in [0,\omega]$.

Proof. The conditions (1)-(3) of Theorem 3 correspond to all the conditions of Theorem 1, so they guarantee that Green's functions $G_i^-(t,s)$, $i = \overline{1,4}$, are nonpositive.

We have assumed above, in equation (5.7), that $b_j^+(t) \geq 0$. Together with the fact that $G_i^-(t,s) \leq 0$, according to Corollary 1, it implies that the operators K_i, $i = \overline{1,4}$, are positive.

If the condition $\rho(K_i) < 1$ holds, then the equation (5.8) can be written as following:

$$z = (I - K_i)^{-1} f = \left[\sum_{j=0}^{\infty} K_i^j \right] f. \tag{5.10}$$

It follows from $G_i^-(t,s) \leq 0$ that all operators K_i^j are positive and, consequently, for this case $\sum_{j=0}^{\infty} K_i^j \geq 0$.

The solution $x(t)$ of the given boundary value problem (1.1)-(1.3), (2.i) can be written in the form:

$$x = \left(G_i^- \sum_{j=0}^{\infty} K_i^j \right) f, \tag{5.11}$$

where Green's operator for the problem (1.1)-(1.3), (2.i) is

$$G_i = G_i^- \sum_{j=0}^{\infty} K_i^j, \tag{5.12}$$

which is nonpositive when the condition $\rho(K_i) < 1$ holds.

□

For the case when $b_j(t)$ can change sign, we can also use the following form of Theorem 1.

Corollary 1. *Assume that the following conditions are fulfilled:*

(1) $0 < \gamma_k \leq 1$, $0 < \delta_k \leq 1$ *for* $k = \overline{1,r}$ *and* $a_j(t) \geq 0$, $j = 1, ..., p$, $t \in [0,\omega]$.

(2) *The condition (3.6) is fulfilled.*

(3) *The inequality*

$$-\sum_{j=1}^{p} a_j(t) \bar{G}_t'\left(t - \tau_j(t), s\right) \chi_{[0,\omega]}\left(t - \tau_j(t)\right) \\ - \sum_{j=1}^{p} b_j(t) \bar{G}\left(t - \theta_j(t), s\right) \chi_{[0,\omega]}\left(t - \theta_j(t)\right) \geq 0 \tag{5.13}$$

is fulfilled for $t \in [0,\omega]$.

(4) *The condition (4.17) is fulfilled.*

Then Green's functions $G_i(t,s)$, $i = \overline{1,3}$ *are nonpositive for* $t,s \in [0,\omega]$ *and under the additional condition* $\sum_{j=1}^{p} b_j(t) \chi_{[0,\omega]}(t - \theta_j(t)) \not\equiv 0$, $t \in [0,\omega]$, *Green's function* $G_4(t,s) \leq 0$ *for* $t,s \in [0,\omega]$.

Example 2. Let us consider the same equation as in Example 1, where $a_1(t) = 0.015t + 0.005$, $b_1(t) = 0.005t - 0.005$ (so, $b_1(t)$ changes sign at the point $t = 1$), $\tau_1(t) = 0.5$, $\theta_1(t) = 0.5$.

$$x''(t) + (0.015t + 0.005)x'(t - 0.5) + (0.005t - 0.005)x(t - 0.5) = f(t), \quad t \in [0,3], \tag{5.14}$$

with impulses (1.2), where

$$\begin{aligned} t_1 &= 1, \quad \gamma_1 = 0.9, \quad \delta_1 = 0.5, \\ t_2 &= 1.5, \quad \gamma_2 = 0.6, \quad \delta_2 = 0.7, \\ t_3 &= 2.5, \quad \gamma_3 = 0.8, \quad \delta_3 = 0.4. \end{aligned} \tag{5.15}$$

Let us verify if the condition (5.13) is satisfied. Denote

$$\begin{aligned} M(t,s) = &-(0.015t + 0.005)\bar{G}'_t(t - 0.5, s)\,\chi_{[0,3]}(t - 0.5) \\ &-(0.005t - 0.005)\bar{G}(t - 0.5, s)\,\chi_{[0,3]}(t - 0.5). \end{aligned} \tag{5.16}$$

The form of function $M(t,s)$, for our example, is shown on Figure 5. It is easy to see that the function $M(t,s) \geq 0$ for $t,s \in [0,3] \times [0,3]$. Thus, the condition 3) of Corollary 1 is fulfilled.

Now let us verify the condition (4.17). It is equivalent to the condition:

$$\operatorname*{ess\,sup}_{t,s \in [0,3]\times[0,3]} \int_0^3 M(t,s)ds < 1, \tag{5.17}$$

or

$$\operatorname*{ess\,sup}_{t,s \in [0,3]\times[0,3]} M(t,s) < \frac{1}{3}. \tag{5.18}$$

For our example, we can calculate

$$\operatorname*{ess\,sup}_{t,s \in [0,3]\times[0,3]} M(t,s) = 0.18. \tag{5.19}$$

Substituting this into the equation (5.18), we can obtain: $0.18 < \frac{1}{3}$. Thus, the condition 4) of Corollary 1 is fulfilled.

Let us use Lemma 3 to check the condition 2) of Corollary 1. We can calculate:

$$\int_0^3 (0.015t + 0.005)dt = 0.0825,$$

$$\prod_{k=1}^{3} \delta_k = 0.14.$$

We obtain $0.0825 < 0.14$, thus, the condition 2) of Corollary 1 holds. So, all the conditions of Corollary 1 are fulfilled. It means that, for our example, Green's functions $G_i(t,s)$, $i = \overline{1,4}$ are nonpositive.

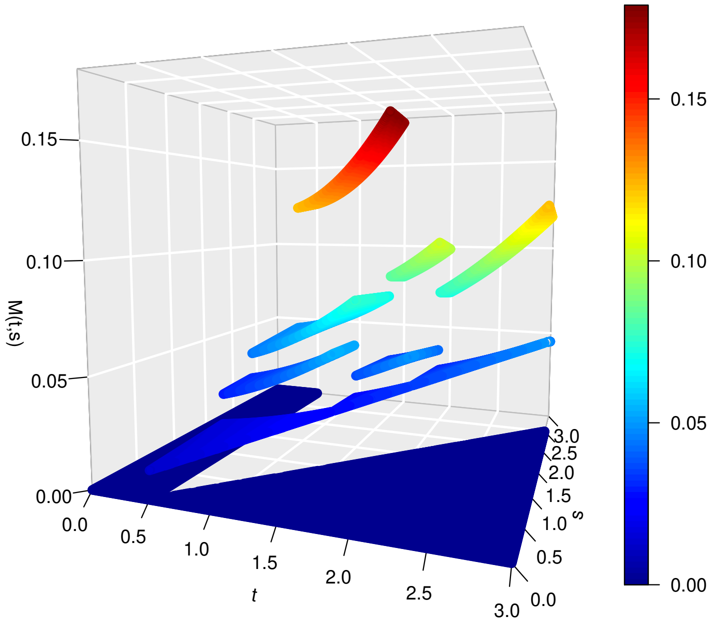

FIGURE 5. $M(t,s)$.

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