



## SOME RESULTS ON THE $q$ -ANALOGUES OF THE INCOMPLETE FIBONACCI AND LUCAS POLYNOMIALS

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*Abstract.* In the present paper, we introduce new families of the  $q$ -Fibonacci and  $q$ -Lucas polynomials, which are represented here as the incomplete  $q$ -Fibonacci polynomials  $F_n^k(x, s, q)$  and the incomplete  $q$ -Lucas polynomials  $L_n^k(x, s, q)$ , respectively. These polynomials provide the  $q$ -analogues of the incomplete Fibonacci and Lucas numbers. We give several properties and generating functions of each of these families  $q$ -polynomials. We also point out the fact that the results for the  $q$ -analogues which we consider in this article for  $0 < q < 1$  can easily be translated into the corresponding results for the  $(p, q)$ -analogues (with  $0 < q < p \leq 1$ ) by applying some obvious parametric variations, the additional parameter  $p$  being redundant.

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### 1. INTRODUCTION

The Fibonacci numbers are defined by the following recurrence relation:

$$F_{n+1} = F_n + F_{n-1} \quad (n \in \{0, 1, 2, \dots\})$$

with the initial conditions  $F_0 = 0$  and  $F_1 = 1$  and the Lucas numbers are defined by the same recurrence relation with the different initial conditions  $L_0 = 2$  and  $L_1 = 1$ .

In existing literature, there are many extensions and generalizations of the Fibonacci numbers. For instance, Filipponi [13] defined the incomplete Fibonacci and Lucas numbers as follows:

$$F_n(k) = \sum_{j=0}^k \binom{n-1-j}{j} \quad (0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor) \quad (1.1)$$

and

$$L_n(k) = \sum_{j=0}^k \frac{n}{n-j} \binom{n-j}{j} \quad (0 \leq k \leq \lfloor \frac{n}{2} \rfloor), \quad (1.2)$$

where  $n = 1, 2, 3, \dots$ . We note that

$$F_n(\lfloor \frac{n-1}{2} \rfloor) = F_n$$

and

$$L_n(\lfloor \frac{n}{2} \rfloor) = L_n.$$

The generating functions of these numbers were studied by Pintér and Srivastava [17]. Many other authors have also studied this topic (see, for example, [8–12, 18, 19, 21, 22]).

For  $0 < q < 1$ , the  $q$ -integer is defined by

$$[n] := [n]_q = \frac{1 - q^n}{1 - q} \quad (1.3)$$

and the  $q$ -factorial is defined by

$$[n]! := \begin{cases} [n] \cdot [n-1] \cdots [1] & (n = 1, 2, 3, \dots) \\ 1 & (n = 0). \end{cases} \quad (1.4)$$

The  $q$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]! [k]!} \quad (0 \leq k \leq n) \quad (1.5)$$

with (see [2] and [20])

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = 0 \quad (n < k).$$

The Heine's binomial formula is recalled here as follows (see [2, p. 2]):

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{\infty} \frac{[n]_q [n+1]_q \cdots [n+j-1]_q}{[j]_q!} x^j.$$

The  $q$ -difference operator  $D_q$  is defined as follows:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

if  $x \neq 0$ .

Cigler [6] introduced the  $q$ -Fibonacci polynomials which are defined below:

$$F_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^j x^{n-1-2j} \quad (n \in \{0, 1, 2, \dots\}). \quad (1.6)$$

Also, in [4], we have the following explicit formula for the  $q$ -Lucas polynomials:

$$L_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \quad (n \in \{0, 1, 2, \dots\}). \quad (1.7)$$

A recurrence relation for the  $q$ -Fibonacci polynomials is given by

$$F_n(x, s, q) = xF_{n-1}(x, s, q) + (q-1)sD_q F_{n-1}(x, s, q) + sF_{n-2}(x, s, q),$$

with the initial values  $F_0(x, s, q) = 0$  and  $F_1(x, s, q) = 1$ . The  $q$ -Lucas polynomials satisfy the same recurrence relation as above:

$$L_n(x, s, q) = xL_{n-1}(x, s, q) + (q-1)sD_q L_{n-1}(x, s, q) + sL_{n-2}(x, s, q),$$

but with the initial values given by (see, for details, [4])

$$L_0(x, s, q) = 2 \quad \text{and} \quad L_1(x, s, q) = x.$$

The following formulas provide relationship between these polynomials:

$$L_n(x, s, q) = F_{n+1}(x, s, q) + sF_{n-1}(x, s, q)$$

and

$$L_n(x, qs, q) = F_{n+1}(x, s, q) + q^n s F_{n-1}(x, s, q).$$

For more details about the  $q$ -analogues of the Fibonacci polynomials, see [1, 3–5, 7, 14]. There are only a few studies for the  $q$ -Fibonacci and  $q$ -Lucas polynomials and for the extensions and generalizations of these polynomials.

We choose to remark in passing that several authors (see, for example, [23] and [24]) studied the so-called  $(p, q)$ -Fibonacci and  $(p, q)$ -Lucas polynomials by introducing a seemingly redundant parameter  $p$ , since the so-called  $(p, q)$ -number  $[n]_{p,q}$  is given (for  $0 < q < p \leq 1$ ) by

$$\begin{aligned} [n]_{p,q} &:= \begin{cases} \frac{p^n - q^n}{p - q} & (n \in \{1, 2, 3, \dots\}) \\ 0 & (n = 0) \end{cases} \\ &= p^{n-1} \left( \frac{q}{p} \right)_n \\ &=: p^{n-1} [n]_{\frac{q}{p}} \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} [n]_q &:= \frac{1 - q^n}{1 - q} \\ &= p^{1-n} \left( \frac{p^n - (pq)^n}{p - (pq)} \right) \end{aligned}$$

$$= p^{1-n} [n]_{p,pq}, \tag{1.9}$$

which do exhibits the fact that the results for the  $q$ -analogues which we consider in this article for  $0 < q < 1$  can easily be translated into the corresponding results for the  $(p, q)$ -analogues (with  $0 < q < p \leq 1$ ) by applying some obvious parametric variations, the additional parameter  $p$  being redundant.

The aim of this paper is to introduce and study  $q$ -analogues of (1.1) and (1.2). We thus investigate the incomplete  $q$ -Fibonacci and  $q$ -Lucas polynomials and derive some of their properties including their generating functions.

### 2. MAIN RESULTS

We define the incomplete  $q$ -Fibonacci and  $q$ -Lucas polynomials by using the explicit formulas as follows.

**Definition 1.** The incomplete  $q$ -Fibonacci polynomials are defined by

$$\mathcal{F}_n^k := F_n^k(x, s, q) = \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^j x^{n-1-2j} \tag{2.1}$$

for  $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ .

If we take  $q \rightarrow 1-$  in (2.1), we get incomplete bivariate Fibonacci numbers in [22]. On the other hand, if we take  $q \rightarrow 1-$  and  $s = x = 1$  in (2.1), we get incomplete Fibonacci numbers studied in [13].

TABLE 1. Incomplete  $q$ -Fibonacci polynomials for  $n = 1, \dots, 8$  and  $k = 0, 1, 2, 3, 4$ .

$n/k$	0	1	2	$F_n^k(x, s, q) = \mathcal{F}_n^k$		
				3		
1	1					
2	$x$					
3	$x^2$	$x^2 + qs$				
4	$x^3$	$x^3 + [2]qsx$				
5	$x^4$	$x^4 + [3]qsx^2$	$x^4 + [3]qsx^2 + q^3s^2$			
6	$x^5$	$x^5 + [4]qsx^3$	$x^5 + [4]qsx^3 + [3]q^3s^2x$			
7	$x^6$	$x^6 + [5]qsx^4$	$x^6 + [5]qsx^4 + \frac{[3][4]}{[2]}q^3s^2x^4$	$x^6 + [5]qsx^4 + \frac{[3][4]}{[2]}q^3s^2x^4 + q^6s^3$		
8	$x^7$	$x^7 + [6]qsx^5$	$x^7 + [6]qsx^5 + \frac{[4][5]}{[2]}q^3s^2x^3$	$x^7 + [6]qsx^5 + \frac{[4][5]}{[2]}q^3s^2x^3 + [4]q^6s^3x$		

We now define incomplete  $q$ -Lucas polynomials.

**Definition 2.** The incomplete  $q$ -Lucas polynomials are defined by

$$\mathcal{L}_n^k := L_n^k(x, s, q) = \sum_{j=0}^k q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \tag{2.2}$$

for  $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ .

TABLE 2. Incomplete  $q$ -Lucas polynomials for  $n = 1, \dots, 7$  and  $k = 0, 1, 2, 3$ .

$n/k$	0	1	2	$U_n^k(x, s, q) = \mathcal{L}_n^k$		3
1	$x$					
2	$x^2$	$x^2 + [2]s$				
3	$x^3$	$x^3 + [3]sx$				
4	$x^4$	$x^4 + [4]sx^2$	$x^4 + [4]sx^2 + \frac{[4]}{[2]}qs^2$			
5	$x^5$	$x^5 + [5]sx^3$	$x^5 + [5]sx^3 + \frac{[5]}{[2]}qs^2x$			
6	$x^6$	$x^6 + [6]sx^4$	$x^6 + [6]sx^4 + \frac{[3][6]}{[2]}qs^2x^2$	$x^6 + [6]sx^4 + \frac{[3][6]}{[2]}qs^2x^2 + [6]q^3s$		
7	$x^7$	$x^7 + [7]sx^5$	$x^7 + [7]sx^5 + \frac{[4][7]}{[2]}qs^2x^3$	$x^7 + [7]sx^5 + \frac{[4][7]}{[2]}qs^2x^3$	$x^7 + [7]sx^5 + \frac{[4][7]}{[2]}qs^2x^3 + [7]q^3s^3x$	

In particular, if we take  $q \rightarrow 1-$  in (2.2), then we get incomplete bivariate Lucas numbers studied in [22]. Moreover, if we take  $q \rightarrow 1-$  and  $s = x = 1$  in (2.2), then we get incomplete Lucas numbers studied in [13].

2.1. Recurrence relations

**Theorem 1.** The recurrence relation of the incomplete  $q$ -Fibonacci polynomials is given by

$$\mathcal{F}_{n+2}^{k+1} = x\mathcal{F}_{n+1}^{k+1} + (q-1)sD_q\mathcal{F}_{n+1}^k + s\mathcal{F}_n^k \tag{2.3}$$

for  $0 \leq k \leq \frac{n-2}{2}$ .

*Proof.* We find from (2.1) that

$$\begin{aligned} & x\mathcal{F}_{n+1}^{k+1} + (q-1)sD_q\mathcal{F}_{n+1}^k + s\mathcal{F}_n^k \\ &= \sum_{j=0}^{k+1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n+1-2j} \\ & \quad + (q-1) \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} [n-2j] s^{j+1} x^{n-1-2j} \\ & \quad + \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j-1 \\ j \end{bmatrix} s^{j+1} x^{n-1-2j} \\ &= \sum_{j=0}^{k+1} \left\{ q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + (q-1)q^{\binom{j}{2}} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} [n-2j+2] \right. \\ & \quad \left. + q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j-1 \end{bmatrix} \right\} s^j x^{n+1-2j}. \end{aligned}$$

It is known that (see [5])

$$q^{\binom{j+1}{2}} \begin{bmatrix} n-j+1 \\ j \end{bmatrix} =$$

$$= q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + (q-1)q^{\binom{j}{2}} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} [n-2j+2] + q^{\binom{j}{2}} \begin{bmatrix} n-j \\ j-1 \end{bmatrix}.$$

Therefore, the recurrence relation (2.3) is seen to hold true.  $\square$

In Theorem 1, by taking  $q \rightarrow 1-$  and  $x = s = 1$ , we get the following result:

$$F_{n+2}(k+1) = F_{n+1}(k+1) + F_n(k) \quad (0 \leq k \leq \frac{n-2}{2}),$$

where  $F_n(k)$  are the incomplete Fibonacci numbers studied in [13].

The following theorem results, in part, from the recurrence relation (2.3).

**Theorem 2.** *The following non-homogeneous recurrence relation of the incomplete  $q$ -Fibonacci polynomials holds true:*

$$\mathcal{F}_{n+2}^k = (x + (q-1)sD_q)\mathcal{F}_{n+1}^k + s\mathcal{F}_n^k - q^{\binom{k}{2}}q^n \begin{bmatrix} n-k-1 \\ k \end{bmatrix} s^{k+1}x^{n-2k-1}. \quad (2.4)$$

*Proof.* The proof of the non-homogeneous recurrence relation (2.4) follows from Definition 1 and the equation (2.3).  $\square$

**Theorem 3.** *It is asserted that*

$$\mathcal{L}_n^k = \mathcal{F}_{n+1}^k + s\mathcal{F}_{n-1}^{k-1} \quad (0 \leq k \leq \frac{n}{2}). \quad (2.5)$$

*Proof.* Using the equation (2.1), we have

$$\begin{aligned} \mathcal{F}_{n+1}^k + s\mathcal{F}_{n-1}^{k-1} &= \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} + s \sum_{j=0}^{k-1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} s^j x^{n-2-2j} \\ &= \sum_{j=0}^k q^{\binom{j}{2}} \left\{ q^{-\binom{j}{2}} q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} + \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\ &= \sum_{j=0}^k q^{\binom{j}{2}} \left\{ q^j \begin{bmatrix} n-j \\ j \end{bmatrix} + \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\ &= \sum_{j=0}^k q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\ &= \mathcal{L}_n^k \end{aligned}$$

for  $0 \leq k \leq \frac{n}{2}$ .  $\square$

**Theorem 4.** *The following recurrence relation holds true:*

$$\mathcal{L}_n^k(x, qs, q) = \mathcal{F}_{n+1}^k + q^n s \mathcal{F}_{n-1}^{k-1} \quad (0 \leq k \leq \frac{n}{2}). \quad (2.6)$$

*Proof.* We consider

$$\begin{aligned}
\mathcal{F}_{n+1}^k + q^n s \mathcal{F}_{n-1}^{k-1} &= \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&\quad + q^n s \sum_{j=0}^{k-1} q^{\binom{j+1}{2}} \begin{bmatrix} n-j-2 \\ j \end{bmatrix} j s^j x^{n-2-2j} \\
&= \sum_{j=0}^k q^{\binom{j+1}{2}} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&\quad + q^n \sum_{j=1}^k q^{\binom{j}{2}} \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} s^j x^{n-2j} \\
&= \sum_{j=1}^k q^{\binom{j}{2}} \left\{ q^j \begin{bmatrix} n-j \\ j \end{bmatrix} + q^n \begin{bmatrix} n-j-1 \\ j-1 \end{bmatrix} \right\} s^j x^{n-2j} \\
&= \sum_{j=1}^k q^{\binom{j}{2}} q^j \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&= L_n(x, qs, q)
\end{aligned}$$

for  $0 \leq k \leq \frac{n}{2}$ .  $\square$

**Theorem 5.** *The recurrence relation of incomplete  $q$ -Lucas polynomials is given by*

$$\mathcal{L}_{n+2}^{k+1} = x \mathcal{L}_{n+1}^{k+1} + (q-1) s D_q \mathcal{L}_{n+1}^k + s \mathcal{L}_n^k \quad \left( 0 \leq k \leq \frac{n-1}{2} \right). \quad (2.7)$$

*Proof.* In view of (2.3) and (2.5), we find that

$$\begin{aligned}
\mathcal{L}_{n+2}^{k+1} &= \mathcal{F}_{n+3}^{k+1} + s \mathcal{F}_{n+1}^k \\
&= \left( x \mathcal{F}_{n+2}^{k+1} + (q-1) s D_q \mathcal{F}_{n+2}^k + s \mathcal{F}_{n+1}^k \right) \\
&\quad + s \left( x \mathcal{F}_n^k + (q-1) s D_q \mathcal{F}_n^{k-1} + s \mathcal{F}_n^{k-1} \right) \\
&= x \left( \mathcal{F}_{n+2}^{k+1} + s \mathcal{F}_n^k \right) + (q-1) s D_q \left( \mathcal{F}_{n+2}^k + \mathcal{F}_n^{k-1} \right) + s \left( \mathcal{F}_{n+1}^k + s \mathcal{F}_{n-1}^{k-1} \right) \\
&= x \mathcal{L}_{n+1}^{k+1} + (q-1) s D_q \mathcal{L}_{n+1}^k + s \mathcal{L}_n^k
\end{aligned}$$

for  $0 \leq k \leq \frac{n-1}{2}$ .  $\square$

**Theorem 6.** *The following non-homogeneous recurrence relation of the incomplete  $q$ -Lucas polynomials holds true:*

$$\begin{aligned} \mathcal{L}_{n+2}^k &= (x + (q-1)sD_q) \mathcal{L}_{n+1}^k \\ &+ s \mathcal{L}_n^k - \frac{q^{\binom{k}{2}} q^n (q^{-k} [n+1] - 1)}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} s^{k+1} x^{n-2k}. \end{aligned} \quad (2.8)$$

*Proof.* It is easy to derive non-homogeneous recurrence relation (2.8) by using (2.2) and (2.7).  $\square$

## 2.2. Summation formulas

**Theorem 7.** *The following summation formula for the incomplete  $q$ -Fibonacci polynomials holds true:*

$$\sum_{j=0}^{h-1} \frac{1}{x^j} (s \mathcal{F}_{n+j}^k + (q-1)sD_q \mathcal{F}_{n+1+j}^k) = \frac{1}{x^{h-1}} \mathcal{F}_{n+1+h}^{k+1} - x \mathcal{F}_{n+1}^{k+1}, \quad (2.9)$$

where  $x \neq 0$  and  $n \geq 2k + 2$ .

*Proof.* Our proof uses the principle of mathematical induction on  $h$ . For  $h = 1$ , the equation (2.9) holds true. Suppose that the equation (2.9) holds true for some integer  $h > 1$ . Then, by using the equation (2.1), we get

$$\begin{aligned} &\sum_{j=0}^h \frac{1}{x^j} (s \mathcal{F}_{n+j}^k + (q-1)sD_q \mathcal{F}_{n+1+j}^k) \\ &= \sum_{j=0}^{h-1} \frac{1}{x^j} (s \mathcal{F}_{n+j}^k + (q-1)sD_q \mathcal{F}_{n+1+j}^k) \\ &\quad + \frac{1}{x^h} (s \mathcal{F}_{n+h}^k + (q-1)sD_q \mathcal{F}_{n+1+h}^k) \\ &= \frac{1}{x^{h-1}} \mathcal{F}_{n+1+h}^{k+1} - x \mathcal{F}_{n+1}^{k+1} + \frac{1}{x^h} (s \mathcal{F}_{n+h}^k + (q-1)sD_q \mathcal{F}_{n+1+h}^k) \\ &= \frac{1}{x^h} (x \mathcal{F}_{n+1+h}^{k+1} + s \mathcal{F}_{n+h}^k + (q-1)sD_q \mathcal{F}_{n+1+h}^k) - x \mathcal{F}_{n+1}^{k+1} \\ &= \frac{1}{x^h} \mathcal{F}_{n+h+2}^{k+1} - x \mathcal{F}_{n+1}^{k+1}, \end{aligned}$$

which completes the proof of the assertion (2.9) by the principle of mathematical induction on  $h$ .  $\square$



**Theorem 8.** A summation formula for the incomplete  $q$ -Lucas polynomials is given by

$$\sum_{j=0}^{h-1} \frac{1}{x^j} \left( s \mathcal{L}_{n+j}^k + (q-1) s D_q \mathcal{L}_{n+1+j}^k \right) = \frac{1}{x^{h-1}} \mathcal{L}_{n+1+h}^{k+1} - x \mathcal{L}_{n+1}^{k+1}, \quad (2.10)$$

where  $x \neq 0$  and  $n \geq 2k + 1$ .

*Proof.* For using the principle of mathematical induction on  $h$ , we suppose that the assertion (2.10) is true for some integer  $h > 1$ . We thus find from the equation (2.7) that

$$\begin{aligned} & \sum_{j=0}^h \frac{1}{x^j} \left( s \mathcal{L}_{n+j}^k + (q-1) s D_q \mathcal{L}_{n+1+j}^k \right) \\ &= \sum_{j=0}^{h-1} \frac{1}{x^j} \left( s \mathcal{L}_{n+j}^k + (q-1) s D_q \mathcal{L}_{n+1+j}^k \right) \\ & \quad + \frac{1}{x^h} \left( s \mathcal{L}_{n+h}^k + (q-1) s D_q \mathcal{L}_{n+1+h}^k \right) \\ &= \frac{1}{x^{h-1}} \mathcal{L}_{n+1+h}^{k+1} - x \mathcal{L}_{n+1}^{k+1} + \frac{1}{x^h} \left( s \mathcal{L}_{n+h}^k + (q-1) s D_q \mathcal{L}_{n+1+h}^k \right) \\ &= \frac{1}{x^h} \left( x \mathcal{L}_{n+1+h}^{k+1} + s \mathcal{L}_{n+h}^k + (q-1) s D_q \mathcal{L}_{n+1+h}^k \right) - x \mathcal{L}_{n+1}^{k+1} \\ &= \frac{1}{x^h} \mathcal{L}_{n+h+2}^{k+1} - x \mathcal{L}_{n+1}^{k+1}, \end{aligned}$$

which completes the proof of the assertion (2.10) by the principle of mathematical induction on  $h$ .  $\square$

We now derive some summation formulas for the incomplete  $q$ -Lucas polynomials.

**Lemma 1.** It is asserted that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j} = \frac{1}{2} \left( n \mathcal{L}_n - x \frac{d}{dx} \mathcal{L}_n \right). \quad (2.11)$$

*Proof.* We observe that

$$\frac{d}{dx} \mathcal{L}_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j (n-2j) x^{n-2j-1}$$

$$\begin{aligned}
&= n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j-1} \\
&\quad - 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{k}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j-1} \\
&= n x^{-1} \mathcal{L}_n - 2 x^{-1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{k}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} j s^j x^{n-2j},
\end{aligned}$$

which proves the assertion (2.11) of Lemma 1.  $\square$

The following theorem asserts the summation formula of the incomplete  $q$ -Lucas polynomials.

**Theorem 9.** *Let  $\mathcal{L}_n^k$  be the  $n$ th incomplete  $q$ -Lucas polynomial. Then*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left( \lfloor \frac{n}{2} \rfloor - \frac{n}{2} + 1 \right) \mathcal{L}_n + \frac{x}{2} \frac{d}{dx} \mathcal{L}_n. \quad (2.12)$$

*Proof.* By using Definition 2 of the incomplete  $q$ -Lucas polynomials, we get

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left\{ q^{\binom{0}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n \right\} + \left\{ q^{\binom{1}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + q^{\binom{1}{2}} \frac{[n]}{[n-1]} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} s x^{n-2} \right\} \\
&\quad + \left\{ q^{\binom{2}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + q^{\binom{2}{2}} \frac{[n]}{[n-1]} \begin{bmatrix} n-1 \\ 1 \end{bmatrix} s x^{n-2} + q^{\binom{2}{2}} \frac{[n]}{[n-2]} \begin{bmatrix} n-2 \\ 2 \end{bmatrix} s^2 x^{n-4} \right\} \\
&\quad + \dots + \left\{ q^{\binom{j}{2}} \begin{bmatrix} n \\ 0 \end{bmatrix} s^0 x^n + \dots + q^{\binom{\lfloor \frac{n}{2} \rfloor}{2}} \frac{[n]}{[n-\lfloor \frac{n}{2} \rfloor]} \begin{bmatrix} n-\lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right\} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \lfloor \frac{n}{2} \rfloor + 1 - j \right) q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left( \lfloor \frac{n}{2} \rfloor + 1 \right) q^{\binom{j}{2}} \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{j}{2}} j \frac{[n]}{[n-j]} \begin{bmatrix} n-j \\ j \end{bmatrix} s^j x^{n-2j},
\end{aligned}$$

which, in view of Lemma 1, yields

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{L}_n^k = \left( \lfloor \frac{n}{2} \rfloor - \frac{n}{2} + 1 \right) \mathcal{L}_n + \frac{x}{2} \frac{d}{dx} \mathcal{L}_n.$$

□

### 3. GENERATING FUNCTIONS

In this section, we obtain the generating functions of the incomplete  $q$ -Fibonacci and the  $q$ -Lucas polynomials.

**Lemma 2** (see [17]). *Let  $\{s_n\}_{n=0}^\infty$  be a complex sequence satisfying the following non-homogeneous recurrence relation:*

$$s_n = xs_{n-1} + ys_{n-2} + r_n \quad (n > p), \tag{3.1}$$

where  $\{r_n\}$  is a given complex sequence. Then the generating function  $S^k(x, y; t)$  of the sequence  $\{s_n\}$  is given by

$$S^k(x, y; t) = (s_0 - r_0 + (s_1 - xs_0 - r_0)t + G(t))(1 - xt - yt^2)^{-1}, \tag{3.2}$$

where  $G(t)$  denotes the generating function of the given sequence  $\{r_n\}$ .

**Theorem 10.** *The generating function of the incomplete  $q$ -Fibonacci polynomials is given by*

$$U^k(x, s, q; t) = t^{2k+1} \left( \mathcal{F}_{2k+1} + (\mathcal{F}_{2k+2} - (x + (q-1)sD_q)\mathcal{F}_{2k+1})t + q^{\binom{k+2}{2}} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}} \right) (1 - (x + (q-1)sD_q)t - st^2)^{-1}, \tag{3.3}$$

where  $\mathcal{F}_k = F_k(x, s, q)$  are the  $q$ -Fibonacci polynomials.

*Proof.* From (2.1) and (2.4), we get

$$\mathcal{F}_n^k = 0 \quad (0 \leq n \leq 2k + 1)$$

and

$$\mathcal{F}_{2k+1}^k = \mathcal{F}_{2k+1} \quad \text{and} \quad \mathcal{F}_{2k+2}^k = \mathcal{F}_{2k+2}.$$

Also, for  $n \geq 2k + 3$ , we have

$$\begin{aligned} \mathcal{F}_{n+2k+1}^k &= (x + (q-1)sD_q)\mathcal{F}_{n+2k}^k \\ &\quad + s\mathcal{F}_{n+2k-1}^k - q^{\binom{k}{2}} q^{n+2k-1} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} s^{k+1} x^{n-2}. \end{aligned}$$

Let

$$s_n = \mathcal{F}_{n+2k+1}^k$$

and

$$r_0 = r_1 = 0 \quad \text{and} \quad r_n = q^{\binom{k}{2}} q^{n+2k-1} \begin{bmatrix} n+k-2 \\ k \end{bmatrix} s^{k+1} x^{n-2}.$$

We then find that

$$G(t) = q^{\binom{k}{2}} q^{2k+1} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}}$$

and

$$\begin{aligned} S^k(x, s, q; t) &= \left( \mathcal{F}_{2k+1}^k + \left( \mathcal{F}_{2k+2}^k - (x + (q-1)sD_q) \mathcal{F}_{2k+1}^k \right) t \right. \\ &\quad \left. + q^{\binom{k}{2}} q^{2k+1} s^{k+1} t^2 \frac{1}{(1-xtq)_q^{k+1}} \right) \\ &\quad \cdot (1 - (x + (q-1)sD_q)t - st^2)^{-1}. \end{aligned}$$

Therefore, we have

$$U^k(x, s, q; t) = t^{2k+1} S^k(x, s, q; t).$$

□

When  $q \rightarrow 1-$  in Theorem 10, then we obtain the generating function of the incomplete bivariate Fibonacci polynomials (see [22]).

**Theorem 11.** *The generating function of the incomplete  $q$ -Lucas polynomials is given by*

$$\begin{aligned} V^k(x, s, q; t) &= t^{2k} \left( \mathcal{L}_{2k} + (\mathcal{L}_{2k+1} - (x + (q-1)sD_q) \mathcal{L}_{2k}) t \right. \\ &\quad \left. + q^{\binom{k+1}{2}} s^{k+1} t^2 (q^{k+1} (1-xt) + 1) \frac{1}{(1-xtq)_q^{k+1}} \right) \\ &\quad \cdot (1 - (x + (q-1)sD_q)t - st^2)^{-1}, \end{aligned} \quad (3.4)$$

where  $\mathcal{L}_k = L_k(x, s, q)$  are the  $q$ -Lucas polynomials.

*Proof.* Considering the relation between the incomplete  $q$ -Fibonacci and  $q$ -Lucas polynomials, we get

$$V^k(x, s, q; t) = \sum_{n=0}^{\infty} \mathcal{L}_n^k t^n$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \mathcal{F}_{n+1}^k t^n + s \sum_{n=0}^{\infty} \mathcal{F}_{n+1}^{k-1} t^n \\
&= t^{-1} U^k(x, s, q; t) + st U^{k-1}(x, s, q; t).
\end{aligned}$$

□

#### 4. CONCLUSION

In the present study, we have introduced the  $q$ -analogues of the incomplete Fibonacci and Lucas polynomials which satisfy essentially analogous recursion formulas and recurrence relations with the familiar  $q$ -Fibonacci and  $q$ -Lucas polynomials. Applications and some generalizations of the  $q$ -Fibonacci polynomials are given earlier in [15, 16], which contain some nice results for the  $q$ -Fibonacci polynomials. Also, Erkuş-Duman and Tuglu [12] studied various families of multilinear and multilateral generating functions for the generalized bivariate Fibonacci and Lucas polynomials. These works motivate the derivations of similar results for the incomplete  $q$ -Fibonacci and  $q$ -Lucas polynomials which we have investigated in this paper. By means of the relationships (1.8) and (1.9), we have exhibited the fact that the results for the  $q$ -analogues which we consider in this article for  $0 < q < 1$  and the corresponding results for the  $(p, q)$ -analogues (with  $0 < q < p \leq 1$ ) are essentially equivalent, requiring only some obvious parametric variations, the additional parameter  $p$  being redundant.

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