Varieties generated by completions

Andréka, H. and Németi, I.

Abstract

We prove that persistently finite algebras are not created by completions of algebras, in any ordered discriminator variety. A persistently finite algebra is one without infinite simple extensions. We prove that finite measurable relation algebras are all persistently finite. An application of these theorems is that the variety generated by the completions of representable relation algebras does not contain all relation algebras. This answers Problem 1.1(1) from R. Maddux [21] in the negative. At the same time, we confirm the suggestion in that paper that the finite maximal relation algebras constructed in M. Frias and R. Maddux [5] are not in the variety generated by the completions of representable relation algebras. We prove that there are continuum many varieties between the variety generated by the completions of representable relation algebras and the variety of relation algebras.

1 Introduction

Completions of partially ordered sets are obtained, intuitively, by filling-in non-existent suprema. Various kinds of completions are in use, for example in a join-completion we fill-in all suprema, and in an ideal-completion we fill-in the suprema of nonempty directed sets only. For Boolean algebras, most of these various kinds of completions coincide with the so-called Dedekind-MacNeille completion that we will simply call completion. Completions were generalized from Boolean algebras to Boolean algebras with operators by J. D. Monk [22], where he also showed that the completion of a relation algebra is again a relation algebra. It was natural to expect that the completion of a representable relation algebra is again representable. To a great surprise, this was disproved by I. M. Hodkinson [12]. Maddux [21] began to investigate

what kind of structures can a non-representable completion bring in. He exhibits infinitely many finite non-representable relation algebras that can be embedded into a completion of a representable relation algebra. He then asks if the variety $\mathsf{Var}(\mathsf{RRA}^c)$ generated by all the completions of representable relation algebras contains all relation algebras [21, Problem1.1(1)].

In this paper, we deal with structures that cannot be created by completions. We prove that in ordered discriminator varieties, completions cannot create persistently finite algebras (Theorem 2.1). In Frias-Maddux [5], an infinity of maximal, finite, integral relation algebras are constructed. Since these are all persistently finite, they are all outside of $\mathsf{Var}(\mathsf{RRA}^c)$. With this we confirm the suggestion in [21] that these might be outside of $\mathsf{Var}(\mathsf{RRA}^c)$. In this paper, we present another sequence of persistently finite relation algebras that are not integral. Namely, we prove that being finite measurable is a persistent property (Theorem 3.1). All this gives a negative answer to Problem 1.1(1) in [21]. More can be proved: there are continuum many varieties between $\mathsf{Var}(\mathsf{RRA}^c)$ and the variety RA of all relation algebras (Theorem 4.1).

Section 2 deals with discriminator varieties. We define a general notion of density, and we prove that in ordered discriminator varieties, dense extensions do not create new persistently finite algebras. Sections 3 and 4 deal with relation algebras. In section 3 we prove that finite measurable relation algebras are all persistently finite, and we list some consequences of this. In section 4, we use the theorems proved in the previous sections to show that, in a sense, there is much room between $\mathsf{Var}(\mathsf{RRA}^c)$ and RA: there are continuum many varieties between the two. However, this leaves open the question of how far these varieties are from each other, that is whether $\mathsf{Var}(\mathsf{RRA}^c)$ is finitely axiomatizable over RA (this is [21, Problem 1.1(2)]).

2 Dense extensions in discriminator varieties

We recall the notion of a discriminator variety from [3]. All what we need in the present paper from this fascinating branch of universal algebra is contained in [3, section 1] (a summary which is taken from Werner [24]). A variety V is called a *discriminator variety* if it is generated by a class K of algebras as a variety, and there is a term γ in the language of the variety that represents the *quaternary discriminator* function g on each member of

K, where the function g on a set A is defined by the property that

$$g(x,y,u,v) = u \text{ when } x = y \quad \text{and} \quad g(x,y,u,v) = v \text{ when } x \neq y$$

for all $x, y, u, v \in A$. We will not distinguish the function g from the term γ that defines it.

In this section we deal with ordered discriminator varieties as defined in [3, section 4]. This is a discriminator variety where a partial order $\leq$ is defined by a finite set of equations:

$$x \leq y \qquad \text{iff} \qquad \tau_i(x,y) = \sigma_i(x,y) \text{ for } i \leq n$$

for some number n and terms $\tau_i, \sigma_i,\ i \leq n$ in the language of the variety. Discriminator varieties of Boolean algebras with operators are ordered discriminator varieties where $\leq$ is defined as the Boolean ordering, in particular all varieties of relation algebras or all varieties of finite dimensional cylindric algebras are ordered discriminator varieties.

Since in ordered discriminator varieties the ordering can be an arbitrary partial order, there is a great variety for the notion of a completion (see, e.g., [23]). We will use a general notion of density that covers most of completions.

Definition 2.1 *We say that $X \subseteq A$ is* dense in $\langle A, \leq \rangle$ *when for all $a \in A$ there is $x \in X$ with $x \leq a$, and for all $x \in X, x < a$ there is $y \in X$ with $x < y \leq a$.*

This density property is stronger than join-density (which means that each element of A is the supremum of the elements of X below it), but weaker than ideal-density (which means that each element of A is the supremum of its downset intersected with X that is in turn an ideal, i.e., a nonempty directed downward closed set). In particular, a Boolean algebra is dense in the above sense in its completion, see [8, Lemma 15.1] or Lemma 4.1.

Usually we denote the universes of algebras $\mathfrak{A}, \mathfrak{B}, \ldots$ with $A, B, \ldots$.

We say that an ordered algebra $\mathfrak{B}$ is a *dense extension* of $\mathfrak{A}$, or that $\mathfrak{A}$ is a *dense subalgebra* of $\mathfrak{B}$, when A is dense in $\mathfrak{B}$. For a class K of similar algebras, K^d denotes the class of dense extensions of members of K and $\mathsf{Var}\mathsf{K}$ denotes the variety generated by K.

Let $\mathfrak{A}$ be an algebra and K a class of similar algebras. We say that $\mathfrak{A}$ is *persistently finite in* K if $\mathfrak{A}$ is finite, and all its simple extensions that are in K are finite.

The following theorem says that dense extensions do not generate new persistently finite algebras, in an ordered discriminator variety.

Theorem 2.1 *Assume that* V *is an ordered discriminator variety of finite similarity type. Assume that* $\mathfrak{A}$ *is simple and persistently finite in* V*. Then* $\mathfrak{A} \in \mathsf{Var}(\mathsf{K}^d \cap \mathsf{V})$ *implies that* $\mathfrak{A} \in \mathsf{VarK}$*, for any subclass* K *of* V*.*

Proof. Let V, K and $\mathfrak{A}$ be as in the statement of the theorem. Assume that $\mathfrak{A} \in \mathsf{Var}(\mathsf{K}^d \cap \mathsf{V})$, we will show that $\mathfrak{A} \in \mathsf{VarK}$. Let $\mathbf{H}\mathsf{L}$, $\mathbf{S}\mathsf{L}$ and $\mathbf{P}\mathsf{L}$ denote the classes of all homomorphic images, all subalgebras and all direct products, respectively, of members of L, for L a class of similar algebras. Thus, $\mathsf{VarL} = \mathbf{HSP}\,\mathsf{L}$, for any class L. If $\mathfrak{D}_i$ is a dense subalgebra of $\mathfrak{E}_i$ for all $i \in I$ then the direct product of the $\mathfrak{D}_i, i \in I$ is a dense subalgebra of the direct product of the $\mathfrak{E}_i, i \in I$, this is straightforward to check. Thus we have

$$\mathbf{P}(\mathsf{K}^d \cap \mathsf{V}) \subseteq (\mathbf{P}\mathsf{K})^d \cap \mathsf{V}. \tag{1}$$

By $\mathfrak{A} \in \mathsf{Var}(\mathsf{K}^d \cap \mathsf{V})$ and (1) we have that $\mathfrak{A} \in \mathbf{HS}((\mathbf{P}\mathsf{K})^d \cap \mathsf{V})$, so there are algebras $\mathfrak{B}, \mathfrak{E}, \mathfrak{D}$ and a homomorphism h such that

- $h : \mathfrak{B} \to \mathfrak{A}$ is a surjective homomorphism,
- $\mathfrak{B}$ is a subalgebra of $\mathfrak{E} \in \mathsf{V}$,
- $\mathfrak{D}$ is a dense subalgebra of $\mathfrak{E}$ and $\mathfrak{D} \in \mathbf{P}\mathsf{K}$.

Note that $\mathfrak{E} \notin \mathsf{VarK}$ may be the case. We will show that $\mathfrak{A} \in \mathbf{SH}\{\mathfrak{D}\} \subseteq \mathsf{VarK}$.

Let $a_1, \ldots, a_n$ be a repetition-free listing of the elements of A. There is such a listing because $\mathfrak{A}$ is finite. Since $\mathfrak{A}$ is simple, it has at least two elements, so $n \geq 2$. Let $b_1, \ldots, b_n \in B$ be such that $h(b_j) = a_j$ for all $1 \leq j \leq n$. There are such elements because h is surjective. We may assume that $\mathfrak{B}$ is generated by $\{b_1, \ldots, b_n\}$, because of the following. $\mathfrak{A}$ is a homomorphic image of the subalgebra $\mathfrak{B}'$ of $\mathfrak{B}$ generated by $b_1, \ldots, b_n$ and $\mathfrak{B}'$ is a subalgebra of $\mathfrak{E}$ by $\mathfrak{B}' \subseteq \mathfrak{B} \subseteq \mathfrak{E}$, so we can choose this $\mathfrak{B}'$ in the first step. In the following, assume that $\mathfrak{B}$ is generated by $b_1, \ldots, b_n$.

By [3, Lemma 2.1], the kernel $\ker(h)$ of h is a compact congruence of $\mathfrak{B}$. Since V is a discriminator variety, compactness implies that $\ker(h)$ is principal, see [3, Thm.1.1.III.(iii)]. Let $r, s \in B$ be such that

$$\ker(h) \text{ is generated by the pair } \{\langle r, s\rangle\} \text{ as a congruence in } \mathfrak{B}. \tag{3}$$

We want to use the property of $\mathfrak{A}$ that all its simple extensions in V are finite. To this end, let us take a subdirect decomposition of $\mathfrak{E}$ to simple factors, say,

$\mathfrak{E}$ is a subdirect product of $\langle \mathfrak{C}_i : i \in I \rangle$ with projections $\pi_i : \mathfrak{E} \to \mathfrak{C}_i$ where each $\mathfrak{C}_i$ is simple. There is such a decomposition by Birkhoff's subdirect decomposition theorem and because in discriminator varieties the subdirectly irreducible algebras are exactly the simple ones, see [3, Thm.1.1.II]. Let

$$I_0 = \{i \in I : \pi_i(r) = \pi_i(s) \text{ and } \pi_i(b_1) \neq \pi_i(b_2)\}.$$

We show that

(4) $h_i = \{\langle h(b), \pi_i(b) \rangle : b \in B\}$ embeds $\mathfrak{A}$ into $\pi_i[\mathfrak{B}] \subseteq \mathfrak{C}_i$, for all $i \in I_0$.

Indeed, let $i \in I_0$. Then $\langle r, s \rangle \in \ker(\pi_i)$ by the definition of I_0. Thus $\ker(h) \subseteq \ker(\pi_i)$ because $\ker(h)$ is generated by this pair as a congruence in $\mathfrak{B} \subseteq \mathfrak{E}$. Now, $\pi_i : \mathfrak{B} \to \mathfrak{C}_i$ since $\mathfrak{B} \subseteq \mathfrak{E}$ and $\pi_i : \mathfrak{E} \to \mathfrak{C}_i$. Therefore $h_i : \mathfrak{A} \to \pi_i[\mathfrak{B}]$, by $\ker(h) \subseteq \ker \pi_i$ and $h : \mathfrak{B} \to \mathfrak{A}$. Since $\mathfrak{A}$ is simple, then either h_i is an embedding, or h_i maps $\mathfrak{A}$ onto the one-element algebra. However, this latter case would entail $\pi_i(b_1) = \pi_i(b_2)$, which is not the case by the definition of I_0 and $i \in I_0$. Thus h_i is an embedding, and the proof of (4) is complete.

Now, by using persistently finiteness of $\mathfrak{A}$, we show that there is a natural number k such that

(5) the cardinality of $\mathfrak{C}_i$ is smaller than k, for all $i \in I_0$.

Indeed, $\mathfrak{C}_i$ is simple and $\mathfrak{C}_i \in \mathsf{V}$ for all $i \in I$ by our assumption $\mathfrak{E} \in \mathsf{V}$. Then $h_i : \mathfrak{A} \to \mathfrak{C}_i$ implies that $\mathfrak{C}_i$ is finite. However, we have to show more, we have to show the existence of the finite upper bound k. Proving by contradiction, assume that $\mathfrak{A}$ has ever larger simple extensions $\mathfrak{M}_n$ in V. Then $\mathfrak{A}$ is embeddable into an infinite ultraproduct $\mathfrak{M}$ of these. Now, $\mathfrak{M} \in \mathsf{V}$ because V is a variety, so it is closed under taking ultraproducts. Since V is a discriminator variety, the class of its simple members is closed under taking ultraproducts ([3, Thm.1.1.II.(iv)]), and hence $\mathfrak{M}$ is simple. This contradicts persistently finiteness of $\mathfrak{A}$, and so there is a finite upper bound k for the simple V-extensions of $\mathfrak{A}$. This proves (5).

Next we show that

(6) $$I_0 \neq \emptyset.$$

Indeed, let $b^* = g(r, s, b_1, b_2)$, where g is the quaternary discriminator term. Now, $b^* \in B$ by its definition, and $h(b^*) = g(h(r), h(s), h(b_1), h(b_2))$ because

h is a homomorphism and g is a term in the language. Since $\ker(h)$ is generated by $\langle r, s\rangle$, we have that $h(r) = h(s)$. Thus, $h(b^*) = h(b_1) = a_1$, by the definition of g. Assume that $i \notin I_0$. Then either $\pi_i(r) \neq \pi_i(s)$ or $\pi_i(b_1) = \pi_i(b_2)$. In either case, $\pi_i(b^*) = \pi_i(b_2)$. Assume that $I_0 = \emptyset$, then $\pi_i(b^*) = \pi_i(b_2)$ for all $i \in I$, which implies $b^* = b_2$. This is a contradiction, since $h(b^*) = a_1$ while $h(b_2) = a_2 \neq a_1$. Thus, $I_0 \neq \emptyset$ and (6) has been proved.

We begin a "transit" from $\mathfrak{E}$ to $\mathfrak{D}$. So far, (4) implies that $h_i[\mathfrak{A}] \subseteq \pi_i[\mathfrak{B}] \subseteq \pi_i[\mathfrak{E}]$. Note that $\mathfrak{D}$ is a subalgebra of $\mathfrak{E}$ and $\mathfrak{D} \in \mathbf{P}\mathsf{K}$. Our plan is to find an $i \in I_0$ such that $h_i[\mathfrak{A}] \subseteq \pi_i[\mathfrak{D}]$. This would show that $\mathfrak{A} \in \mathbf{SH}\{\mathfrak{D}\} \subseteq \mathsf{VarK}$. To this end, we want to find $d_1, \ldots, d_n \in D$ such that there is an $i \in I_0$ such that $\pi_i(b_1) = \pi_i(d_1), \ldots, \pi_i(b_n) = \pi_i(d_n)$. This would imply that $\pi_i[\mathfrak{B}] \subseteq \pi_i[\mathfrak{D}]$ because $\mathfrak{B}$ is generated by $b_1, \ldots, b_n$, and then (4) implies that $h_i[\mathfrak{A}] \subseteq \pi_i[\mathfrak{D}]$.

From now on, the following notation will be convenient to use. Let $\alpha(e_1, \ldots, e_m)$ be a conjunction of equations and non-equations of terms of elements from E. These are all open Horn formulas. $I(\alpha)$ denotes those factors from I where α is true under taking the projections:

$$I(\alpha(e_1, \ldots, e_m)) = \{i \in I : \mathfrak{C}_i \models \alpha(\pi_i(e_1), \ldots, \pi_i(e_m))\}.$$

With this notation, $I_0 = I(r = s \wedge b_1 \neq b_2)$. We say that $J \subseteq I$ is *definable* if J is $I(\alpha(e_1, \ldots, e_m))$ for some open Horn formula α and elements $e_1, \ldots, e_m$ of E. In discriminator varieties, to each open Horn formula α there is an equation e such that in each simple member of the variety, α and e have the same truth-evaluations, see [3, Thm.1.1.V.]. Therefore, there are terms ρ, δ such that $I(\alpha(e_1, \ldots, e_m)) = I(\rho(e_1, \ldots, e_m) = \delta(e_1, \ldots, e_m))$. Since $\rho(e_1, \ldots, e_m), \delta(e_1, \ldots, e_m)$ are elements of E, we have that J is definable exactly when there are $p, q \in E$ such that $J = \{i \in I : \pi_i(p) = \pi_i(q)\}$. In this case, we say that J is defined by $p = q$. Assume that $J \subseteq I$. By "$\alpha(e_1, \ldots, e_m)$ holds on J" we understand that $\alpha(\pi_i(e_1), \ldots, \pi_i(e_m))$ holds for all $i \in J$. We note that in ordered discriminator varieties, $\tau < \sigma$ counts as an equation, because $<$ is defined by a finite set of equations which is an open Horn formula.

We say that $J \subseteq I$ is *bounded* if there is a natural number k such that $|C_i| \leq k$ for all $i \in J$. Finally, when $a, b \in E$ and $J \subseteq I$, we say that a *approximates* b in J if there is $j \in J$ such that $a_j = b_j$. To find our $d_1, \ldots, d_n$, we will use the following statement (7) repeatedly. We note that this statement is the "heart" of the proof of Theorem 2.1.

(7) Each element of E can be approximated by an element of D in any nonempty definable bounded subset of I.

Indeed, to prove (7), let $b \in E$ and let J be a nonempty definable bounded subset of I. By density of D in E, there is an $a \in D$ such that $a \leq b$. If a approximates b in J then we are done. Assume that a does not approximate b in J, then

$$a < b \text{ on } J. \tag{8}$$

Define $b_1 = g(p, q, b, a)$, where J is defined by $p = q$. Then b_1 is b on J and b_1 is a outside J and $a < b_1$ by (8). Choose $a_1 \in D$ such that $a < a_1 \leq b_1$, there is such an a_1 by Definition 2.1. If a_1 approximates b on J, then we are done. So, assume that a_1 does not approximate b on J and let $J_1 = I(a < a_1)$. Then J_1 is a subset of J because $a < a_1 \leq b_1$ but a and b_1 agree outside of J. (We note that the purpose of using b_1 in place of b was to achieve $J_1 \subseteq J$.) By $a < a_1$ we have that J_1 is nonempty. J_1 still has the bound k because it is a subset of J. Also, J_1 is definable by $J_1 = I(a < a_1)$. Thus, J_1 is a nonempty, definable bounded subset of J and

$$a < a_1 < b \text{ on } J_1.$$

We proceed this way: assume that for m we already defined $a_1, \ldots, a_m \in D$ and a nonempty definable subset J_m of J such that

$$a < a_1 < \cdots < a_m < b \text{ on } J_m.$$

Having this, by $a_m < b$ and Definition 2.1 we can find an $a_{m+1} \in D$ such that $a_m < a_{m+1} < b$. If a_{m+1} approximates b on J_m then we are done. If not, then let $b_{m+1} = g(p_m, q_m, b, a_{m+1})$ where J_m is defined by $p_m = q_m$. Let $J_{m+1} = I(a_m < a_{m+1})$, this is a nonempty definable subset of J_m such that

$$a < a_1 < \cdots < a_m < a_{m+1} < b \text{ on } J_{m+1}. \tag{9}$$

In particular, (9) implies that $|C_i| \geq m + 3$ for all $i \in J_{m+1}$. Because J is bounded, this process cannot be continued ad infinitum, so at one of the steps we have to find an approximation of b as desired. This proves (7).

We turn to finding $d_1, \ldots, d_n \in D$ such that $I(b_1 = d_1 \wedge \cdots \wedge b_n = d_n) \cap I_0 \neq \emptyset$. We have seen that I_0 is a nonempty definable bounded subset of I, see (5), (6). Assume that $p = q$ defines I_0. We begin with b_1. By (7), there is

$d_1 \in D$ which approximates b_1 on I_0. Let $I_1 = I(b_1 = d_1 \wedge p = q)$. Then I_1 is a nonempty definable bounded subset of I_0. By (7), there is $d_2 \in D$ which approximates b_2 in I_1. Let $I_2 = I(b_1 = d_1 \wedge b_2 = d_2 \wedge p = q)$. Then I_2 is a nonempty definable bounded subset of I_0, and so on. In the last step we get a $d_n \in D$ such that d_n approximates b_n on $I_{n-1} = I(b_1 = d_1 \wedge \cdots \wedge b_{n-1} = d_{n-1} \wedge p = q)$. Let $J = I(b_1 = d_1 \wedge \cdots \wedge b_n = d_n \wedge p = q)$. Then J is nonempty, let $i \in J$ be arbitrary. Then, $\pi_i(b_1), \dots, \pi_i(b_n) \in \pi_i[\mathfrak{D}]$ by $d_1, \dots, d_n \in D$ and $\pi_i(b_1) = \pi_i(d_1) \dots, \pi_i(b_n) = \pi_i(d_n)$. By (4) we have that $h_i(a_j) = \pi_i(b_j)$ for all $1 \leq j \leq n$, thus $h_i[\mathfrak{A}] \subseteq \pi_i[\mathfrak{D}]$ because $A = \{a_1, \dots, a_n\}$. By assumption we have that $\mathfrak{D} \in \mathbf{PK}$, so $\mathfrak{A} \in \mathbf{SHP}\,\mathsf{K} \subseteq \mathsf{VarK}$. With this, the proof of Theorem 2.1 is complete. □

3 Persistently finite non-representable relation algebras

An algebra $\mathfrak{A} = \langle A, +, -, ;, ^\smile, 1' \rangle$ is a *concrete algebra of binary relations* if $\mathfrak{A}$ is a set of binary relations with a biggest one, and the operations $+, -, ;, ^\smile, 1'$ are, respectively, the following natural operations on binary relations: union of two relations, taking the complement of a relation with respect to the biggest relation, relational composition of two relations, converse of a relation, and the identity relation on the domain of the biggest relation. The class of all algebras isomorphic to concrete algebras of binary relations is denoted by RRA. This is a variety which is not definable by a finite set of equations (classic results due to A. Tarski and J. D. Monk, respectively). The variety $\mathsf{RA} \supseteq \mathsf{RRA}$ of relation algebras is a finitely axiomatized variety that approximates RRA surprisingly well: an algebra $\mathfrak{A} = \langle A, +, -, ;, ^\smile, 1' \rangle$ is a *relation algebra* if $\langle A, ;, ^\smile, 1' \rangle$ is an involuted monoid, $\langle A, +, -, ;, ^\smile \rangle$ is a Boolean algebra with normal and additive operators, and one more identity true of concrete algebras of binary relations also holds in it, namely $r^\smile; -(r; s) \leq -s$. We use $\leq, 0, 1, \cdot$ with their usual definitions in a Boolean algebra. The elements of RRA are called representable relation algebras and the elements of $\mathsf{RA} \setminus \mathsf{RRA}$ are called *non-representable* relation algebras. If $\mathfrak{A} \in \mathsf{RA}$, we assume that its operations are as above.

Assume that $\mathfrak{A} \in \mathsf{RA}$ and $y \in A$. We say that y is a *functional element* if $y^\smile; y \leq 1'$. A relation algebra $\mathfrak{A}$ is called *measurable*, if the identity constant $1'$ is the supremum of atoms, and each atom $x \leq 1'$ is *measurable* in the

sense that $x;1;x$ is the supremum of the functional elements below it. The number of the functional elements below $x;1;x$ is called the *measure of* x. These names reflect their meanings in concrete algebras of binary relations. Namely, assume $\mathfrak{A}$ is such. Then $y \in A$ is functional exactly when y is a function as a relation, a subidentity element x corresponds to a subset X of the domain of the biggest relation via $x = \{\langle u, u\rangle : u \in X\}$. When the biggest element of $\mathfrak{A}$ is of form $U \times U$ then $x;1;x$ is just the square $X \times X$ and it can be showed that the measure of a measurable atom x coincides with the size $|X|$ of X.

The following theorem says that the property of being finite and measurable is persistent in RA, i.e., this property is preserved by simple relation algebra extensions.

Theorem 3.1 *Assume that $\mathfrak{M}$ is a finite measurable relation algebra. If $\mathfrak{M}$ can be embedded into a simple relation algebra $\mathfrak{A}$, then $\mathfrak{A}$ itself is finite and measurable.*

Proof. Assume that $\mathfrak{M}$ is a finite measurable relation algebra and $\mathfrak{M} \subseteq \mathfrak{A} \in \mathsf{RA}$ where $\mathfrak{A}$ is simple. Let I denote the set of the subidentity atoms—atoms below the identity 1'— of $\mathfrak{M}$, and for all $x \in I$, let F_x denote the set of functional elements below $x;1;x$. Then I is finite, and for all $x \in I$, the set F_x is finite, too, because $\mathfrak{M}$ is finite. Further, $x;1;x$ is the sum of F_x. Temporarily, let us fix $x \in I$.

By $\mathfrak{M} \subseteq \mathfrak{A}$, we have that $e = x;1;x$ is an element of $\mathfrak{A}$, too, and it is an equivalence element, that is to say, $e^\smile = e = e;e$ because $\mathfrak{A} \in \mathsf{RA}$, see [7, Lemma 5.64]. Then the relativization $\mathfrak{A}(e)$ of $\mathfrak{A}$ to e is also a relation algebra [7, Theorem 10.1]. It is simple because $\mathfrak{A}$ is simple and $x;1;x$ is a nonzero square [7, Theorem 10.8]. We note that the universe of $\mathfrak{A}(e)$ is the downset of e in $\mathfrak{A}$, and the operations $+, ;, ^\smile$ of $\mathfrak{A}(e)$ are the same as those of $\mathfrak{A}$ while $-, 1'$ in $\mathfrak{A}(e)$ are the relativized versions of those in $\mathfrak{A}$ which means that $-(a) = e - a$ and the identity of $\mathfrak{A}(e)$ is $1' \cdot e$.

Each $g \in F_x$ is functional in $\mathfrak{M}$, so g is functional in $\mathfrak{A}$, and functional in $\mathfrak{A}(e)$, too. In $\mathfrak{M}$ we have that $e = \sum F_x$, so this same is true in $\mathfrak{A}$ and in $\mathfrak{A}(e)$, since F_x is finite. Thus, in $\mathfrak{A}(e)$ the unit, e is a sum of finitely many functional elements. Therefore, $\mathfrak{A}(e)$ is representable on a finite set because it is simple, by [16, Theorem 4.32]. This means that $\mathfrak{A}(e)$ can be represented such that the unit has the form of $U \times U$ for some finite set U. A finite relation has only finitely many subsets, so, $\mathfrak{A}(e)$ is finite and so atomic.

Since subsets of functions are functions and the unit in $\mathfrak{A}(e)$ is the sum of finitely many functional elements, we have that each element in $\mathfrak{A}(e)$ is the sum of functional elements. In particular, $\mathfrak{A}(e)$ is measurable with finitely many subidentity atoms of finite measure. Since the universe of $\mathfrak{A}(e)$ is the set of all elements of A that are below e, we get that in $\mathfrak{A}$, too, the identity element x of $\mathfrak{A}(e)$ is the sum of finitely many subidentity atoms of finite measure.

A relation algebra is called *finitely measurable* if it is measurable and each subidentity atom has finite measure in it. The identity element of $\mathfrak{A}$ is the sum of I, the set of subidentity atoms of $\mathfrak{M}$, since this is true of $\mathfrak{M} \subseteq \mathfrak{A}$. We have seen that each $x \in I$ in $\mathfrak{A}$ is the finite sum of measurable atoms of finite measure. So in $\mathfrak{A}$, too, the identity element 1' is the sum of finitely many atoms of finite measure. Thus, $\mathfrak{A}$ is finitely measurable. Then $\mathfrak{A}$ is atomic by [10, Theorem 8.3]. We do not know yet whether it is complete or not because we do not know yet that it is finite, since in principle, there might be infinitely many atoms below $x; 1; y$ for distinct subidentity atoms x, y.

We now use the representation theorem Theorem 7.4(ii) from [10]. It says that the completion $\mathfrak{C}$ of $\mathfrak{A}$ is a coset relation algebra that is determined by a group coset frame $(\mathcal{G}, \varphi, S)$ distilled from $\mathfrak{A}$. We are going to show that $\mathfrak{C}$ is finite. The system $\mathcal{G}$ of groups is $\langle \mathfrak{G}_x : x \in J \rangle$ where J is the set of subidentity atoms of $\mathfrak{A}$ and $\mathfrak{G}_x$ is the group of functional elements below $x; 1; x$ for $x \in J$. We have already shown that J is finite and $\mathfrak{G}_x$ is finite for all $x \in J$. Now, the atoms below $x; 1; y$ in $\mathfrak{C}$ are in one-to-one correspondence with $\mathfrak{G}_x/H_{xy}$ where H_{xy} is a normal subgroup of $\mathfrak{G}_x$, by the definition of an algebra determined by $\mathcal{G}$ (see [10, p.1171]). Since $\mathfrak{G}_x$ is finite, this means that there are finitely many atoms below $x; 1; y$, for any $x, y \in J$. Since J is finite, this means that $\mathfrak{C}$ is finite. Since $\mathfrak{A}$ is a subalgebra of $\mathfrak{C}$, we have that $\mathfrak{A}$ is finite and we have already shown that $\mathfrak{A}$ is measurable. □

There are many finite and infinite representable measurable relation algebras, these are described in [6]. Also, infinitely many finite and infinite non-representable measurable relation algebras are constructed in [2, sections 3, 4]. (We note that these are not weakly representable, either.) By using Theorem 3.1, the non-representable measurable algebras can be used to give answers to Problems 2,3,4 from [3] (see also Problems P5,P6,P7 in [19]). These problems were already solved in [5], but measurable algebras provide a different kind of examples for their solutions. Below, we elaborate on this.

A relation algebra is called *maximal* if it is finite, simple and has no proper simple relation algebra extension. It is known that the representable maximal relation algebras are exactly the finite full set relation algebras, that is to say, the concrete algebras of all subsets of $U \times U$ for some finite set U. Until 1997, these were the only known maximal relation algebras.

Problem 3 in [3] asks whether there are any non-representable simple absolute retracts in RA (or in the variety SA of semi-associative relation algebras). In semi-simple varieties V, being an absolute retract and being maximal among the simple algebras are equivalent [3, Lemma 3.3]. This problem was solved in [5], Frias and Maddux constructed infinitely many non-representable maximal relation algebras. These algebras are integral, i.e., the identity constant 1' is an atom in them. Theorem 3.1, together with [2, Theorem 4.2], provide different, non-integral maximal examples for Problem 3, as follows.

We have seen when proving (5) in the proof of Theorem 2.1 that a persistently finite algebra can have only finitely many simple extensions (up to isomorphism, in a discriminator variety of finite similarity type). Therefore, each simple persistently finite algebra can be extended to a maximal one. The maximal extension of a representable measurable relation algebra is a full set relation algebra, this provides many persistently finite and non-maximal simple relation algebras. On the other hand, the maximal extension of a non-representable measurable relation algebra is also non-representable, and thus provides new solutions for [3, Problem 3]. The simple non-representable measurable relation algebras are all non-integral, because in an integral simple measurable relation algebra the square $1';1;1' = 1$, and thus measurability implies that the unit is the supremum of functional elements and these are known to be all representable.

Problem 4 in [3] asks whether for an atom in a relation algebra satisfying the equality $p;1;p^{\smile} \leq 1'$ is necessary or not for being persistent. We note that any maximal non-representable relation algebra gives a negative answer for this problem, because a finite algebra is atomic, if it is maximal, then all its atoms are persistent, and it is known that if all atoms p in an algebra satisfy the given equality $p;1;p^{\smile} \leq 1'$, then the algebra must be representable. This problem was solved in the negative in [5], too, and the maximal non-representable measurable relation algebras provide further examples disproving an affirmative answer.

Problem 2 in [3] asks whether there exists a simple relation algebra that is not embeddable into a one-generated relation algebra. Any maximal relation

algebra that is not one-generated gives a negative answer to this problem. Indeed, [5] gave such integral examples. We believe that the smallest non-representable measurable relation algebra constructed from the two-element group in [1, section 5] and in [2, sections 3,4] is maximal, 2-generated but not one-generated. (However, we did not check the details of this claim.) That would provide a new example for a negative answer to [3, Problem 2].

We note that Problem 5 from [3] has been solved recently by Mohamed Khaled. He proved that the finitely generated free non-associative relation algebras are not atomic [18], and the finitely generated finite dimensional free non-commutative cylindric algebras are not atomic, either [17]. Problem 1 of [3] asking whether the free m-generated and free n-generated pairing algebras may be isomorphic for distinct m, n is still open, to our best knowledge. With this, we surveyed the present-day statuses of all the problems given in [3, section 11].

4 Completions of representable relation algebras

In this section we are going to apply Theorem 2.1 in the context of relation algebras. We begin with showing that the notion of a dense extension as defined below Definition 2.1 in this paper is equivalent with the one most widely used in the literature, for Boolean algebras with additional operators, and in particular, for relation algebras. (See, for example, [8, Definition 15.4(ii)], [13, Definition 2.6], [20, p.237].) Let $\leq$ denote the Boolean ordering, i.e., $x \leq y$ is defined by $x + y = y$.

Lemma 4.1 *Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras with operators and $\mathfrak{A} \subseteq \mathfrak{B}$. Then* (i) *and* (ii) *below are equivalent.*

(i) *$\mathfrak{A}$ is a dense subalgebra of $\mathfrak{B}$.*

(ii) *For all nonzero $b \in B$ there is a nonzero $a \in A$ such that $a \leq b$.*

Proof. Assume (i) and let $0 < b \in B$. Since $\mathfrak{A} \subseteq \mathfrak{B}$, we have that $0 \in A$. Then, $0 \in A$, $0 < b \in B$ and (i) imply, by Definition 2.1, that there is $a \in A$ such that $0 < a \leq b$, and we are done.

Assume now (ii), we want to show that A is dense in $\mathfrak{B}$. Each element of B has an element of A below it, namely $0 \in A$. Assume that $a < b \in B$ and

$a \in A$. Then $0 < b - a \in B$, so by (ii) there is $x \in A$ such that $0 < x \leq b - a$. Then $a + x \in A$ and $a < a + x \leq b$, and we are done. □

A completion $\mathfrak{R}^c$ of a relation algebra $\mathfrak{R}$ is defined as a complete, dense extension of $\mathfrak{R}$; this exists and is unique up to an isomorphism that leaves R fixed. (See, for example, [8, Definition 15.17], [13, Definition 2.25, Lemma 2.26], [20, p.323].) For a class K of relation algebras, let K^c denote the class of completions of elements of K. The following lemma implies that $\mathsf{Var}(\mathsf{K}^d) = \mathsf{Var}(\mathsf{K}^c)$ for any $\mathsf{K} \subseteq \mathsf{RA}$.

Lemma 4.2 $\mathbf{SK}^d = \mathbf{SK}^c \subseteq \mathsf{RA}$ *for any* $\mathsf{K} \subseteq \mathsf{RA}$.

Proof. To show $\mathsf{K}^d \subseteq \mathbf{SK}^c$, let $\mathfrak{A} \in \mathsf{K}$ and let $\mathfrak{B}$ be a dense extension of $\mathfrak{A}$. We want to show that $\mathfrak{B} \in \mathbf{SK}^c$. Let $\mathfrak{B}^c$ be a completion of $\mathfrak{B}$, this exists by $\mathsf{K} \subseteq \mathsf{RA}$. Then $\mathfrak{B}^c$ is a dense extension of $\mathfrak{A}$ and it is complete, thus $\mathfrak{B}^c$ is a completion of $\mathfrak{A}$, and therefore $\mathfrak{B} \subseteq \mathfrak{B}^c \in \mathsf{K}^c$. The other direction follows from the definition of a completion, this definition immediately implies that $\mathsf{K}^c \subseteq \mathsf{K}^d$. Since RA is closed under completions, we immediately get $\mathsf{K}^c \subseteq \mathsf{RA}^c = \mathsf{RA}$. □

It is well known that the variety RA of relation algebras is a discriminator variety (see, for example, [3, Corollary 5.7], or [13, Theorem 3.19], [20, p.386], [14]). Since RA has a Boolean reduct, it is an ordered discriminator variety with letting $\leq$ be the Boolean order. Thus, Theorem 2.1 can be applied with taking V to be the variety of relation algebras. We are ready to state the theorem that gives an answer to [21, problem 1.1(1)].

Theorem 4.1 $\mathsf{Var}(\mathsf{RRA}^c) \neq \mathsf{RA}$. *Moreover, the following* (i)-(iv) *hold.*

(i) *There are infinitely many finite simple integral relation algebras which are not in* $\mathsf{Var}(\mathsf{RRA}^c)$.

(ii) *There are infinitely many finite simple non-integral relation algebras which are not in* $\mathsf{Var}(\mathsf{RRA}^c)$.

(iii) *There are continuum many varieties* W *such that* $\mathsf{Var}(\mathsf{RRA}^c) \subset \mathsf{W} \subset \mathsf{RA}$.

(iv) *A finite measurable relation algebra is in* $\mathsf{Var}(\mathsf{RRA}^c)$ *exactly when it is representable.*

Proof. First we prove (iv). Assume that $\mathfrak{M}$ is a finite measurable relation algebra. If $\mathfrak{M}$ is representable, then it is its own completion since it is complete, and so it is in RRA^c. In the reverse direction, assume that $\mathfrak{M} \in \mathsf{Var}(\mathsf{RRA}^c)$, we will show that $\mathfrak{M}$ is representable. Since RA is a discriminator variety, $\mathfrak{M}$ is a subdirect product of some simple algebras $\mathfrak{C}_i$. We will show that each $\mathfrak{C}_i \in \mathsf{RRA}$, this will imply that $\mathfrak{M} \in \mathsf{RRA}$. Now, $\mathfrak{C}_i$ is in $\mathsf{Var}(\mathsf{RRA}^c)$ since it is a homomorphic image of $\mathfrak{M} \in \mathsf{Var}(\mathsf{RRA}^c)$. Each $\mathfrak{C}_i$ remains measurable, this is easy to check directly by using the definition of a measurable relation algebra, but this fact also follows from [11, Theorem 3.1] and [1, Theorems 6.1, 6.2]. Thus, each $\mathfrak{C}_i$ is persistently finite in RA by Theorem 3.1. Thus, we can use Theorem 2.1 with substituting RA, RRA and $\mathfrak{C}_i$ in place of V, K and $\mathfrak{A}$ and using that RA is an ordered discriminator variety and $\mathsf{Var}(\mathsf{RRA}^c) = \mathsf{Var}(\mathsf{RRA}^d)$. We get that $\mathfrak{C}_i \in \mathsf{VarRRA} = \mathsf{RRA}$, as we wanted, and so $\mathfrak{M} \in \mathsf{RRA}$ since it is a subdirect product of the $\mathfrak{C}_i$s.

(ii) follows from (iv) by using the main result of [2] that there are infinitely many simple finite non-representable measurable relation algebras. All these algebras are non-integral. This can be seen by looking at the construction, or by noticing that simple integral measurable relation algebras are functionally dense and hence representable.

The proof of (i) is similar to that of (ii): in place of the non-representable persistently finite measurable relation algebras of [2] we use the infinitely many non-representable integral simple maximal relation algebras constructed in [5].

To prove (iii), we can use any infinite sequence $\langle \mathfrak{A}_i : i \in I \rangle$ of simple, persistently finite non-representable relation algebras. We have seen in the proofs of (ii) and (i) that there is such a sequence. We have seen that each persistently finite relation algebra has only finitely many simple extensions (see the proof of (5) in the proof of Theorem 2.1). Therefore, there is an infinite sub-sequence $\langle \mathfrak{A}_j : j \in J \rangle$ of the original one such that no $\mathfrak{A}_j$ can be embedded into $\mathfrak{A}_k$ for distinct $j, k \in J$. Having this sequence, we will repeat the proof of [2, Theorem 5.1] with the necessary modifications. For $S \subseteq J$ let

$$\mathsf{V}(S) = \{\mathfrak{B} \in \mathsf{RA} : \mathfrak{A}_n \text{ cannot be embedded into } \mathfrak{B}, \text{ for all } n \in S\}.$$

First we show that $\mathsf{V}(S)$ is a variety. By [15, Theorem 7.1], each finite simple relation algebra is splitting in the class RA of all relation algebras, thus the biggest variety of RA not containing $\mathfrak{A}_n$ is the class of all relation algebras

into which $\mathfrak{A}_n$ cannot be embedded. This is called the conjugate variety of $\mathfrak{A}_n$, let us denote it by $\mathsf{V}^-(\mathfrak{A}_n)$, then

$$\mathsf{V}(S) = \bigcap \{\mathsf{V}^-(\mathfrak{A}_n) : n \in S\}.$$

This shows that $\mathsf{V}(S)$ is a variety since it is an intersection of varieties.

All the $\mathfrak{A}_n$ are simple, persistently finite and non-representable, so none of them can be embedded into an element of $\mathsf{Var}(\mathsf{RRA}^c)$, by Theorem 2.1. Thus

$$\mathsf{Var}(\mathsf{RRA}^c) \subseteq \mathsf{V}(S) \subseteq \mathsf{RA}.$$

We are going to show that $\mathsf{V}(S)$ is distinct from $\mathsf{V}(Z)$ for distinct subsets S, Z of J. This will suffice, because J is countably infinite, and so it has continuum many subsets. Indeed, let S, Z be distinct subsets of J. Then there is an $n \in J$ such that, say, $n \in S$ and $n \notin Z$. Then $\mathfrak{A}_n \notin \mathsf{V}(S)$ since it can be embedded into itself and $n \in S$. On the other hand, $\mathfrak{A}_n \in \mathsf{V}(Z)$ because $n \notin Z$, so no $\mathfrak{A}_m$ with $m \in Z$ can be embedded into $\mathfrak{A}_n$. This shows that $\mathsf{V}(S) \neq \mathsf{V}(Z)$ and we are done with proving (iii). □

The splittable relation algebras, see [4, Definition 4], all fail to be persistently finite, because once an algebra has a splittable atom, this atom can be split to arbitrarily many parts (see [4, Theorem 3]). On the other hand, the non-representable relation algebras that are shown to be in $\mathsf{Var}(\mathsf{RRA}^c)$ are gotten by splitting finite simple relation algebras [21]. Perhaps, the next question to ask is the following.

Problem 4.1 *Is every finite simple splittable relation algebra in* $\mathsf{Var}(\mathsf{RRA}^c)$*?*

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Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
Budapest, Reáltanoda st. 13-15, H-1053 Hungary
andreka.hajnal@renyi.mta.hu, nemeti.istvan@renyi.mta.hu