

# Avoiding long Berge cycles

Zoltán Füredi\*

Alexandr Kostochka†

Ruth Luo‡

May 15, 2018

## Abstract

Let  $n \geq k \geq r + 3$  and  $\mathcal{H}$  be an  $n$ -vertex  $r$ -uniform hypergraph. We show that if

$$|\mathcal{H}| > \frac{n-1}{k-2} \binom{k-1}{r}$$

then  $\mathcal{H}$  contains a Berge cycle of length at least  $k$ . This bound is tight when  $k-2$  divides  $n-1$ . We also show that the bound is attained only for connected  $r$ -uniform hypergraphs in which every block is the complete hypergraph  $K_{k-1}^{(r)}$ .

We conjecture that our bound also holds in the case  $k = r + 2$ , but the case of short cycles,  $k \leq r + 1$ , is different.

**Mathematics Subject Classification:** 05D05, 05C65, 05C38, 05C35.

**Keywords:** Berge cycles, extremal hypergraph theory.

## 1 Definitions, Berge $F$ subhypergraphs

An  $r$ -uniform hypergraph, or simply  $r$ -graph, is a family of  $r$ -element subsets of a finite set. We associate an  $r$ -graph  $\mathcal{H}$  with its edge set and call its vertex set  $V(\mathcal{H})$ . Usually we take  $V(\mathcal{H}) = [n]$ , where  $[n]$  is the set of first  $n$  integers,  $[n] := \{1, 2, 3, \dots, n\}$ . We also use the notation  $\mathcal{H} \subseteq \binom{[n]}{r}$ .

**Definition 1.1** (Anstee and Salazar [1], Gerbner and Palmer [5]). *For a graph  $F$  with vertex set  $\{v_1, \dots, v_p\}$  and edge set  $\{e_1, \dots, e_q\}$ , a hypergraph  $\mathcal{H}$  contains a **Berge  $F$**  if there exist distinct vertices  $\{w_1, \dots, w_p\} \subseteq V(\mathcal{H})$  and edges  $\{f_1, \dots, f_q\} \subseteq E(\mathcal{H})$ , such that if  $e_i = v_{i_1}v_{i_2}$ , then  $\{w_{i_1}, w_{i_2}\} \subseteq f_i$ . The vertices  $\{w_1, \dots, w_p\}$  are called the **base vertices** of the Berge  $F$ .*

Of particular interest to us are Berge cycles.

**Definition 1.2.** *A **Berge cycle** of length  $\ell$  in a hypergraph is a set of  $\ell$  distinct vertices  $\{v_1, \dots, v_\ell\}$  and  $\ell$  distinct edges  $\{e_1, \dots, e_\ell\}$  such that  $\{v_i, v_{i+1}\} \subseteq e_i$  with indices taken modulo  $\ell$ .*

*A **Berge path** of length  $\ell$  in a hypergraph in a hypergraph is a set of  $\ell + 1$  vertices  $\{v_1, \dots, v_{\ell+1}\}$  and  $\ell$  hyperedges  $\{e_1, \dots, e_\ell\}$  such that  $\{v_i, v_{i+1}\} \subseteq e_i$  for all  $1 \leq i \leq \ell$ .*

---

\*Alfréd Rényi Institute of Mathematics, Hungary. E-mail: z-furedi@illinois.edu. Research supported in part by the Hungarian National Research, Development and Innovation Office NKFIH grant K116769, and by the Simons Foundation Collaboration Grant 317487.

†University of Illinois at Urbana–Champaign, Urbana, IL 61801 and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia. E-mail: kostochk@math.uiuc.edu. Research is supported in part by NSF grant DMS-1600592 and grants 18-01-00353A and 16-01-00499 of the Russian Foundation for Basic Research.

‡University of Illinois at Urbana–Champaign, Urbana, IL 61801, USA. E-mail: ruthluo2@illinois.edu.

Let  $\mathcal{H}$  be a hypergraph and  $p$  be an integer. The  $p$ -shadow,  $\partial_p \mathcal{H}$ , is the collection of the  $p$ -sets that lie in some edge of  $\mathcal{H}$ . In particular, we will often consider the 2-shadow  $\partial_2 \mathcal{H}$  of a  $r$ -uniform hypergraph  $\mathcal{H}$  in which each edge of  $\mathcal{H}$  yields a clique on  $r$  vertices.

## 2 Background

Erdős and Gallai [3] proved the following result on the Turán number of paths.

**Theorem 2.1** (Erdős and Gallai [3]). *Let  $k \geq 2$  and let  $G$  be an  $n$ -vertex graph with no path on  $k$  vertices. Then  $e(G) \leq (k-2)n/2$ .*

This theorem is implied by a stronger result for graphs with no long cycles.

**Theorem 2.2** (Erdős and Gallai [3]). *Let  $k \geq 3$  and let  $G$  be an  $n$ -vertex graph with no cycle of length  $k$  or longer. Then  $e(G) \leq (k-1)(n-1)/2$ .*

Győri, Katona, and Lemons [6] extended Theorem 2.1 to Berge paths in  $r$ -graphs. The bounds depend on the relationship of  $r$  and  $k$ .

**Theorem 2.3** (Győri, Katona, and Lemons [6]). *Suppose that  $\mathcal{H}$  is an  $n$ -vertex  $r$ -graph with no Berge path of length  $k$ . If  $k \geq r+2 \geq 5$ , then  $e(\mathcal{H}) \leq \frac{n}{k} \binom{k}{r}$ , and if  $r \geq k \geq 3$ , then  $e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}$ .*

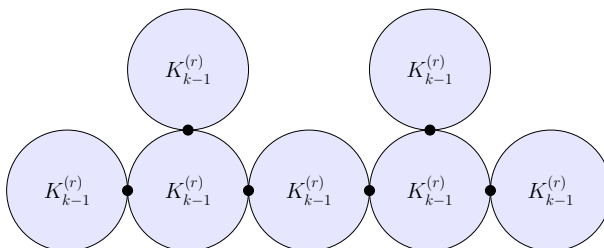
Both bounds in Theorem 2.3 are sharp for each  $k$  and  $r$  for infinitely many  $n$ . The remaining case of  $k = r+1$  was settled later by Davoodi, Győri, Methuku, and Tompkins [2]: *if  $\mathcal{H}$  is an  $n$ -vertex  $r$ -graph with  $|E(\mathcal{H})| > n$ , then it contains a Berge path of length at least  $r+1$ .* Furthermore, Győri, Methuku, Salia, Tompkins and Vizer [7] have found a better upper bound on the number of edges in  $n$ -vertex *connected*  $r$ -graphs with no Berge path of length  $k$ . Their bound is asymptotically exact when  $r$  is fixed and  $k$  and  $n$  are sufficiently large.

The goal of this paper is to present a similar result for cycles.

## 3 Main result: Hypergraphs without long Berge cycles

Our main result is an analogue of the Erdős–Gallai theorem on cycles for  $r$ -graphs.

**Theorem 3.1.** *Let  $r \geq 3$  and  $k \geq r+3$ , and suppose  $\mathcal{H}$  is an  $n$ -vertex  $r$ -graph with no Berge cycle of length  $k$  or longer. Then  $e(\mathcal{H}) \leq \frac{n-1}{k-2} \binom{k-1}{r}$ . Moreover, equality is achieved if and only if  $\partial_2 \mathcal{H}$  is connected and for every block  $D$  of  $\partial_2 \mathcal{H}$ ,  $D = K_{k-1}$  and  $\mathcal{H}[D] = K_{k-1}^{(r)}$ .*



Note that a Berge cycle can only be contained in the vertices of a single block of the 2-shadow. Hence the aforementioned sharpness examples cannot contain Berge cycles of length  $k$  or longer.

**Conjecture 3.2.** *The statement of Theorem 3.1 holds for  $k = r + 2$ , too.*

Similarly to the situation with paths, the case of short cycles,  $k \leq r + 1$ , is different. Exact bounds for  $k \leq r - 1$  and asymptotic bounds for  $k = r$  were found in [9]. The answer for  $k = r + 1$  is not known.

For convenience, below we will use notation

$$C_r(k) := \frac{1}{k-2} \binom{k-1}{r}. \quad (1)$$

(So  $C_2(k)(n-1) = (k-1)(n-1)/2$ .) Theorem 3.1 yields the following implication for paths.

**Corollary 3.3.** *Let  $r \geq 3$  and  $n \geq k + 1 \geq r + 4$ . If  $\mathcal{H}$  is a connected  $n$ -vertex  $r$ -graph with no Berge path of length  $k$ , then  $e(\mathcal{H}) \leq C_r(k)(n-1)$ .*

This gives a  $\frac{k-2}{k-r}$  times stronger bound than Theorem 2.3 for connected  $r$ -graphs for all  $r \geq 3$  and  $n \geq k + 1 \geq r + 4$  and not only for sufficiently large  $k$  and  $n$ . In particular, Corollary 3.3 implies the following slight sharpening of Theorem 2.3 for  $k \geq r + 3$ .

**Corollary 3.4.** *Let  $r \geq 3$  and  $n \geq k \geq r + 3$ . If  $\mathcal{H}$  is an  $n$ -vertex  $r$ -graph with no Berge path of length  $k$ , then  $e(\mathcal{H}) \leq \frac{n}{k} \binom{k}{r}$  with equality only if every component of  $\mathcal{H}$  is the complete  $r$ -graph  $K_k^{(r)}$ .*

In the next section, we introduce the notion of *representative pairs* and use it to derive useful properties of Berge  $F$ -free hypergraphs for rather general  $F$ . In Section 5, we cite Kopylov's Theorem and prove two useful inequalities. In Section 6 we prove our main result, Theorem 3.1, and in the final Section 7 we derive Corollaries 3.3 and 3.4.

## 4 Representative pairs, the structure of Berge $F$ -free hypergraphs

**Definition 4.1.** *For a hypergraph  $\mathcal{H}$ , a system of distinct representative pairs (SDRP) of  $\mathcal{H}$  is a set of distinct pairs  $A = \{\{x_1, y_1\}, \dots, \{x_s, y_s\}\}$  and a set of distinct hyperedges  $\mathcal{A} = \{f_1, \dots, f_s\}$  of  $\mathcal{H}$  such that for all  $1 \leq i \leq s$*

- $\{x_i, y_i\} \subseteq f_i$ , and
- $\{x_i, y_i\}$  is not contained in any  $f \in \mathcal{H} - \{f_1, \dots, f_s\}$ .

**Lemma 4.2.** *Let  $\mathcal{H}$  be a hypergraph, let  $(A, \mathcal{A})$  be an SDRP of  $\mathcal{H}$  of maximum size. Let  $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$  and let  $B = \partial_2 \mathcal{B}$  be the 2-shadow of  $\mathcal{B}$ . For a subset  $S \subseteq B$ , let  $\mathcal{B}_S$  denote the set of hyperedges that contain at least one edge of  $S$ . Then for all nonempty  $S \subseteq B$ ,  $|S| < |\mathcal{B}_S|$ .*

*Proof.* Suppose for contradiction there exists a nonempty set  $S \subseteq B$  such that  $|S| \geq |\mathcal{B}_S|$ . Choose a smallest such  $S$ .

We claim that  $|S| = |\mathcal{B}_S|$ . Indeed, if  $|S| > |\mathcal{B}_S|$  then  $|S| \geq 2$  because  $\mathcal{B}_S \neq \emptyset$  by definition. Take any edge  $e \in S$ . The set  $S \setminus e$  is nonempty and  $|S \setminus e| = |S| - 1 \geq |\mathcal{B}_S| \geq |\mathcal{B}_{S \setminus e}|$ , a contradiction to the minimality of  $S$ .

Consider the case  $|S| = |\mathcal{B}_S|$ . By the minimality of  $S$ , each subset  $S' \subset S$  satisfies  $|S'| < |\mathcal{B}_{S'}|$ . Therefore by Hall's theorem, one can find a bijective mapping of  $S$  to  $\mathcal{B}_S$ , where say the edge  $e_i \in S$  gets mapped to hyperedge  $f_i$  in  $\mathcal{B}_S$  for  $1 \leq i \leq |S|$ . Then  $(A \cup \{e_1, \dots, e_{|S|}\}, \mathcal{A} \cup \{f_1, \dots, f_{|S|}\})$  is a larger SDRP of  $\mathcal{H}$ , a contradiction.  $\square$

**Lemma 4.3.** *Let  $\mathcal{H}$  be a hypergraph and let  $(A, \mathcal{A})$  be an SDRP of  $\mathcal{H}$  of maximum size. Let  $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$ ,  $B = \partial_2 \mathcal{B}$ , and let  $G$  be the graph on  $V(\mathcal{H})$  with edge set  $A \cup B$ . If  $G$  contains a copy of a graph  $F$ , then  $\mathcal{H}$  contains a Berge  $F$  on the same base vertex set.*

*Proof.* Let  $\{v_1, \dots, v_p\}$  and  $\{e_1, \dots, e_q\}$  be a set of vertices and a set of edges forming a copy of  $F$  in  $G$  such that the edges  $e_1, \dots, e_b$  belong to  $B$ . By Lemma 4.2, each subset  $S$  of  $\{e_1, \dots, e_b\}$  satisfies  $|S| < |\mathcal{B}_S|$ . So we may apply Hall's Theorem to match each of these  $e_i$ 's to a hyperedge  $f_i \in \mathcal{B}$ . The edges  $e_i \in A$  can be matched to distinct edges of  $\mathcal{A}$  given by the SDRP. Since  $\mathcal{A} \cap \mathcal{B} = \emptyset$  this yields a Berge  $F$  in  $\mathcal{H}$  on the same base vertex set.  $\square$

We have  $|\mathcal{H}| = |A| + |\mathcal{B}|$ . Note that the number of  $r$ -edges in  $\mathcal{B}$  is at most the number of copies of  $K_r$  in its 2-shadow. Therefore Lemma 4.3 gives a new proof for the following result of Gerbner and Palmer (cited in [4]): for any graph  $F$ ,

$$\text{ex}(n, K_r, F) \leq \text{ex}_r(n, \text{Berge } F) \leq \text{ex}(n, F) + \text{ex}(n, K_r, F).$$

Here  $\text{ex}_r(n, \{\mathcal{F}_1, \mathcal{F}_2, \dots\})$  denotes the *Turán number* of  $\{\mathcal{F}_1, \mathcal{F}_2, \dots\}$ , the maximum number of edges in an  $r$ -uniform hypergraph on  $n$  vertices that does not contain a copy of any  $\mathcal{F}_i$ .

The *generalized Turán function*  $\text{ex}(n, K_r, F)$  is the maximum number of copies of  $K_r$  in an  $F$ -free graph on  $n$  vertices.

## 5 Kopylov's Theorem and two inequalities

**Definition:** For a natural number  $\alpha$  and a graph  $G$ , the  $\alpha$ -*disintegration* of a graph  $G$  is the process of iteratively removing from  $G$  the vertices with degree at most  $\alpha$  until the resulting graph has minimum degree at least  $\alpha + 1$  or is empty. This resulting subgraph  $H(G, \alpha)$  will be called the  $(\alpha + 1)$ -*core* of  $G$ . It is well known (and easy) that  $H(G, \alpha)$  is unique and does not depend on the order of vertex deletion.

The following theorem is a consequence of Kopylov [8] about the structure of graphs without long cycles. We state it in the form that we need.<sup>1</sup>

**Theorem 5.1** (Kopylov [8]). *Let  $n \geq k \geq 5$  and let  $t = \lfloor \frac{k-1}{2} \rfloor$ . Suppose that  $G$  is a 2-connected  $n$ -vertex graph with no cycle of length at least  $k$ . Suppose that it is saturated, i.e., for every nonedge  $xy$  the graph  $G \cup \{xy\}$  has a cycle of length at least  $k$ . Then either*

<sup>1</sup>A proof and a recent application can be found in [10].

(5.1.1) the  $t$ -core  $H(G, t)$  is empty, the graph  $G$  is  $t$ -disintegrable; or

(5.1.2)  $|H(G, t)| = s$  for some  $t + 2 \leq s \leq k - 2$ , it is a complete graph on  $s$  vertices, and  $H(G, t) = H(G, k - s)$ , i.e., the rest of the vertices can be removed by a  $(k - s)$ -disintegration.

Note that in the second case  $2 \leq k - s \leq t$ .

**Lemma 5.2.** *Let  $k, r, t, s, a$  nonnegative integers, and suppose  $k \geq r + 3 \geq 6$ ,  $t = \lfloor (k - 1)/2 \rfloor$ , and  $0 \leq a \leq s \leq t$ . Then*

$$a + \binom{s - a}{r - 1} \leq \frac{1}{k - 2} \binom{k - 1}{r} := C_r(k).$$

This is the part of the proof where we use  $k \geq r + 3$  because this inequality does not hold for  $k = r + 2$  (then the right hand side is  $(r + 1)/r$  while the left hand side could be as large as  $\lfloor (r + 1)/2 \rfloor$ ).

*Proof.* Keeping  $k, r, t, s$  fixed the left hand side is a convex function of  $a$  (defined on the integers  $0 \leq a \leq s$ ). It takes its maximum either at  $a = s$  or  $a = 0$ . So the left hand side is at most  $\max\{s, \binom{s}{r-1}\}$ . This is at most  $\max\{t, \binom{t}{r-1}\}$ . We have eliminated the variables  $a$  and  $s$ .

We claim that  $t \leq \frac{1}{k-2} \binom{k-1}{r}$ . Indeed, keeping  $k, t$  fixed, the right hand side is minimized when  $r = k - 3$ , and then it equals to  $(k - 1)/2$ . This is at least  $\lfloor (k - 1)/2 \rfloor = t$ .

Finally, we claim that  $\binom{t}{r-1} \leq \frac{1}{k-2} \binom{k-1}{r}$ . If  $t < r - 1$ , then there is nothing to prove. For  $t \geq r - 1$  rearranging the inequality we get

$$r \leq \frac{k - 1}{t} \times \frac{k - 3}{t - 1} \times \cdots \times \frac{k - r}{t - r + 2}.$$

Each fraction on the right hand side is at least 2. Since  $r < 2^{r-1}$ , we are done.  $\square$

**Lemma 5.3.** *Let  $w, r \geq 2$  and let  $\mathcal{H}$  be a  $w$ -vertex  $r$ -graph. Let  $\overline{\partial_2 \mathcal{H}}$  denote the family of pairs of  $V(\mathcal{H})$  not contained in any member of  $\mathcal{H}$  (i.e., the complement of the 2-shadow). Then*

$$|\mathcal{H}| + |\overline{\partial_2 \mathcal{H}}| \leq a_r(w) := \begin{cases} \binom{w}{2} & \text{for } 2 \leq w \leq r + 2, \\ \binom{w}{r} & \text{for } r + 2 \leq w. \end{cases}$$

Moreover, for  $2 \leq w \leq k - 1$  one has  $a_r(w) \leq (w - 1) \binom{k-1}{r} / (k - 2)$  with equality if and only if  $w = k - 1$  and

- $w > r + 2$  and  $\mathcal{H}$  is complete, or
- $w = r + 2$  and either  $\mathcal{H}$  or  $\overline{\partial_2 \mathcal{H}}$  is complete.

*Proof.* The case of  $w \geq r + 2$  is a corollary of the classical Kruskal-Katona theorem, but one can give a direct proof by a double counting. If  $\overline{\partial_2 \mathcal{H}}$  is empty, then  $|\mathcal{H}| = \binom{w}{r}$  if and only if  $\mathcal{H} = \binom{V(\mathcal{H})}{r}$ . Otherwise, let  $\overline{\mathcal{H}}$  denote the  $r$ -subsets of  $V(\mathcal{H})$  that are not members of  $\mathcal{H}$ ,  $\overline{\mathcal{H}} = \binom{V(\mathcal{H})}{r} \setminus \mathcal{H}$ . Each pair of  $\overline{\partial_2 \mathcal{H}}$  is contained in  $\binom{w-2}{r-2}$  members of  $\overline{\mathcal{H}}$  and each  $e \in \overline{\mathcal{H}}$  contains at most  $\binom{r}{2}$  edges of  $\overline{\partial_2 \mathcal{H}}$ . We obtain

$$|\overline{\partial_2 \mathcal{H}}| \binom{w - 2}{r - 2} \leq |\overline{\mathcal{H}}| \binom{r}{2}.$$

Since  $\binom{w-2}{r-2} \geq \binom{r}{r-2} = \binom{r}{2}$ ,  $|\overline{\partial_2 \mathcal{H}}| \leq |\overline{\mathcal{H}}|$  with equality only when  $w = r + 2$ . Furthermore, if  $\overline{\partial_2 \mathcal{H}}$  and  $\mathcal{H}$  are both nonempty, then for any  $xy \in \overline{\partial_2 \mathcal{H}}$  and  $uv \in \partial_2 \mathcal{H}$  (with possibly  $x = u$ ), any  $r$ -tuple  $e$  containing  $\{x, y\} \cup \{u, v\}$  is in  $\overline{\mathcal{H}}$  but contributes strictly less than  $\binom{r}{2}$  edges to  $\overline{\partial_2 \mathcal{H}}$ , implying  $|\overline{\partial_2 \mathcal{H}}| < |\overline{\mathcal{H}}|$ . This completes the proof of the case.

The case  $w \leq r + 1$  is easy, and the calculation showing  $a_r(w) \leq C_r(k)(w - 1)$  with equality only if  $w = k - 1$  is standard.  $\square$

## 6 Proof of Theorem 3.1, the main upper bound

*Proof.* Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph on  $n$  vertices with no Berge cycle of length  $k$  or longer ( $k \geq r + 3 \geq 6$ ). Let  $(A, \mathcal{A})$  be an SDRP of  $\mathcal{H}$  of maximum size. Let  $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$ ,  $B = \partial_2 \mathcal{B}$ . By Lemma 4.3 the graph  $G$  with edge set  $A \cup B$  does not contain a cycle of length  $k$  or longer.

Let  $V_1, V_2, \dots, V_p$  be the vertex sets of the standard (and unique) decomposition of  $G$  into 2-connected blocks of sizes  $n_1, n_2, \dots, n_p$ . Then the graph  $A \cup B$  restricted to  $V_i$ , denoted by  $G_i$ , is either a 2-connected graph or a single edge (in the latter case  $n_i = 2$ ), each edge from  $A \cup B$  is contained in a single  $G_i$ , and  $\sum_{i=1}^p (n_i - 1) \leq (n - 1)$ .

This decomposition yields a decomposition of  $A = A_1 \cup A_2 \cup \dots \cup A_p$  and  $B = B_1 \cup B_2 \cup \dots \cup B_p$ ,  $A_i \cup B_i = E(G_i)$ . If an edge  $e \in B_i$  is contained in  $f \in \mathcal{B}$ , then  $f \subseteq V_i$  (because  $f$  induces a 2-connected graph  $K_r$  in  $B$ ), so the block-decomposition of  $G$  naturally extends to  $\mathcal{B}$ ,  $\mathcal{B}_i := \{f \in \mathcal{B} : f \subseteq V_i\}$  and we have  $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_p$ , and  $B_i = \partial_2 \mathcal{B}_i$ .

We claim that for each  $i$ ,

$$|A_i| + |\mathcal{B}_i| \leq C_r(k)(n_i - 1), \quad (2)$$

and hence

$$|\mathcal{H}| = |A| + |\mathcal{B}| = \sum_{i=1}^p |A_i| + |\mathcal{B}_i| \leq \sum_{i=1}^p C_r(k)(n_i - 1) \leq C_r(k)(n - 1),$$

completing the proof.

To prove (2) observe that the case  $n_i \leq k - 1$  immediately follows from Lemma 5.3. From now on, suppose that  $n_i \geq k$ .

Consider the graph  $G_i$  and, if necessary, add edges to it to make it a saturated graph with no cycle of length  $k$  or longer. Let the resulting graph be  $G'$ . Kopylov's Theorem (Theorem 5.1) can be applied to  $G'$ . If  $G$  is  $t$ -disintegrable, then make  $(n_i - k + 2)$  disintegration steps and let  $W$  be the remaining vertices of  $V_i$  ( $|W| = k - 2$ ). For the edges of  $A_i$  and  $\mathcal{B}_i$  contained in  $W$  we use Lemma 5.3 to see that

$$|A_i[W]| + |\mathcal{B}_i[W]| < C_r(k)(|W| - 1).$$

In the  $t$ -disintegration steps, we iteratively remove vertices with degree at most  $t$  until we arrive to  $W$ . When we remove a vertex  $v$  with degree  $s \leq t$  from  $G'$ ,  $a$  of its incident edges are from  $A$ , and the remaining  $s - a$  incident edges eliminate at most  $\binom{s-a}{r-1}$  hyperedges from  $\mathcal{B}_i$  containing  $v$ . Therefore  $v$  contributes at most  $a + \binom{s-a}{r-1} \leq C_r(k)$  (by Lemma 5.2) to  $|\mathcal{B}_i| + |A_i|$ .

It follows that

$$|A_i| + |\mathcal{B}_i| < \left( \sum_{v \in G' - W} C_r(k) \right) + C_r(k)(|W| - 1) = C_r(k)(n_i - 1).$$

This completes this case.

Next consider the case (5.1.2),  $W := V(H(G, t))$ ,  $|W| = s \leq k - 2$ . We proceed as in the previous case, making  $(n_i - s)$  disintegration steps. Apply Lemma 5.3 for  $|A_i[W]| + |\mathcal{B}_i[W]|$  and Lemma 5.2 for the  $(k - s)$ -disintegration steps (where  $k - s \leq t$ ) to get the desired upper bound (with strict inequality).

Furthermore, if  $e(\mathcal{H}) = |A| + |\mathcal{B}| = C_r(k)(n - 1)$ , then we have  $\sum_{i=1}^p (n_i - 1) = n - 1$  (so  $A \cup B$  is connected) and  $|A_i| + |\mathcal{B}_i| = C_r(k)(n_i - 1)$  for each  $1 \leq i \leq p$ . From the previous proof and Lemma 5.2, we see that this holds if and only if for each  $i$ ,  $n_i = k - 1$ , and either  $\mathcal{B}_i$  or  $A_i$  is complete. In particular, this implies that each block of  $A \cup B$  is a  $K_{k-1}$ . We will show that each  $G_i$  corresponds to a block in  $\mathcal{H}$  that is  $K_{k-1}^{(r)}$  with vertex set  $V_i$ .

In the case that  $\mathcal{B}_i$  is complete for all  $1 \leq i \leq p$ , we are done. Otherwise, if some  $A_i$  is complete (note  $r = k - 3$  by Lemma 5.2) then there are  $\binom{k-1}{2} = \binom{k-1}{k-3} = \binom{k-1}{r}$  hyperedges in  $\mathcal{A}$  containing  $V_i$ . If all such hyperedges are contained in  $V_i$ , again we get  $\mathcal{H}[V_i] = K_{k-1}^{(r)}$ . So suppose there exists a  $f \in \mathcal{A}$  which is paired with an edge  $xy \in A_i$  in the SDRP, but for some  $z \notin V_i$ ,  $\{x, y, z\} \subseteq f$ . Then  $z$  belongs to another block  $G_j$  of  $A \cup B$ . In  $A \cup B$ , there exists a path from  $x$  to  $z$  covering  $V_i \cup V_j$  which avoids the edge  $xy$ . Thus by Lemma 5.3, there is a Berge path from  $x$  to  $z$  with at least  $2(k - 1) - 1$  base vertices which avoids the hyperedge  $f$  (since edge  $xy$  was avoided). Adding  $f$  to this path yields a Berge cycle of length  $2(k - 1) - 1 > k$ , a contradiction.  $\square$

## 7 Corollaries for paths

In order to be self-contained, we present a short proof of a lemma by Györi, Katona, and Lemons [6].

**Lemma 7.1** (Györi, Katona, and Lemons [6]). *Let  $\mathcal{H}$  be a connected hypergraph with no Berge path of length  $k$ . If there is a Berge cycle of length  $k$  on the vertices  $v_1, \dots, v_k$  then these vertices constitute a component of  $\mathcal{H}$ .*

*Proof.* Let  $V = \{v_1, \dots, v_k\}$ ,  $E = \{e_1, \dots, e_k\}$  form the Berge cycle in  $\mathcal{H}$ . If some edge, say  $e_1$  contains a vertex  $v_0$  outside of  $V$ , then we have a path with vertex set  $\{v_0, v_1, \dots, v_\ell\}$  and edge set  $E$ . Therefore each  $e_i$  is contained in  $V$ . Suppose  $V \neq V(\mathcal{H})$ . Since  $\mathcal{H}$  is connected, there exists an edge  $e_0 \in \mathcal{H}$  and a vertex  $v_{k+1} \notin V$  such that for some  $v_i \in V$ , say  $i = k$ ,  $\{v_k, v_{k+1}\} \subseteq e_0$ . Then  $\{v_1, \dots, v_k, v_{k+1}\}, \{e_1, \dots, e_{k-1}, e_0\}$  is a Berge path of length  $k$ .  $\square$

*Proof of Corollary 3.3.* Suppose  $n \geq k + 1$  and  $\mathcal{H}$  is a connected  $n$ -vertex  $r$ -graph with  $e(\mathcal{H}) > C_r(k)(n - 1)$ . Then by Theorem 3.1,  $\mathcal{H}$  has a Berge cycle of length  $\ell \geq k$ . If  $\ell \geq k + 1$ , then removing any edge from the cycle yields a Berge path of length at least  $k$ . If  $\ell = k$ , then by Lemma 7.1,  $\mathcal{H}$  again has a Berge path of length  $k$ .  $\square$

Now Theorem 3.1 together with Corollary 3.3 directly imply Corollary 3.4.

**Proof of Corollary 3.4:** Suppose  $k \geq r + 3 \geq 6$  and  $\mathcal{H}$  is an  $r$ -graph. Let  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_s$  be the connected components of  $\mathcal{H}$  and  $|V(\mathcal{H}_i)| = n_i$  for  $i = 1, \dots, s$ .

If  $n_i \leq k - 1$ , then  $|\mathcal{H}_i| \leq \binom{n_i}{r} < \frac{n_i}{k} \binom{k}{r}$ . If  $n_i \geq k + 1$ , then by Corollary 3.3,  $|\mathcal{H}_i| \leq C_r(k)(n_i - 1) < \frac{n_i}{k} \binom{k}{r}$ . Finally, if  $n_i = k$ , then  $|\mathcal{H}_i| \leq \binom{k}{r} = \frac{n_i}{k} \binom{k}{r}$ , with equality only if  $\mathcal{H}_i = K_k^{(r)}$ . This proves the corollary.  $\square$

**Acknowledgment.** The authors would like to thank Jacques Verstraëte for suggesting this problem and for sharing his ideas and methods used in similar problems.

## References

- [1] R. Anstee and S. Salazar, Forbidden Berge hypergraphs, *Electron. J. Combin.* **24** (2017), Paper 1.59, 21 pp.
- [2] A. Davoodi, E. Gyóri, A. Methuku, and C. Tompkins, An Erdős-Gallai type theorem for hypergraphs, *European J. Combin.* **69** (2018), 159–162.
- [3] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–356.
- [4] D. Gerbner, A. Methuku, and M. Vizer, Asymptotics for the Turán number of Berge- $K_{2,t}$ , [arxiv:1705.04134](https://arxiv.org/abs/1705.04134), (2017), 24 pp.
- [5] D. Gerbner and C. Palmer, Extremal results for Berge-hypergraphs, *SIAM Journal on Discrete Mathematics*, **31** (2017), 2314–2337.
- [6] E. Gyóri, Gy. Y. Katona, and N. Lemons, Hypergraph extensions of the Erdős-Gallai theorem, *European Journal of Combinatorics*, **58** (2016), 238–246.
- [7] E. Gyóri, A. Methuku, N. Salia, C. Tompkins and M. Vizer, On the maximum size of connected hypergraphs without a path of given length, [arxiv:1710.08364](https://arxiv.org/abs/1710.08364), (2017), 6 pp.
- [8] G. N. Kopylov, Maximal paths and cycles in a graph, *Dokl. Akad. Nauk SSSR* **234** (1977), 19–21. (English translation: *Soviet Math. Dokl.* **18** (1977), no. 3, 593–596.)
- [9] A. Kostochka, and R. Luo, On  $r$ -uniform hypergraphs with circumference less than  $r$ , in preparation.
- [10] R. Luo, The maximum number of cliques in graphs without long cycles, *Journal of Combinatorial Theory, Series B*, **128** (2018), 219–226.