

# A variation of a theorem by Pósa

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## Abstract

A graph  $G$  is  $\ell$ -hamiltonian if for any linear forest  $F$  of  $G$  with  $\ell$  edges,  $F$  can be extended to a hamiltonian cycle of  $G$ . We give a sharp upper bound for the maximum number of cliques of a fixed size in a non- $\ell$ -hamiltonian graph. Furthermore, we prove stability for the bound: if a non- $\ell$ -hamiltonian graph contains almost the maximum number of cliques, then it must be a subgraph of one of two examples.

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## 1 Background, $\ell$ -hamiltonian graphs

We use standard notation. In particular,  $V(G)$  denotes the vertex set of a graph  $G$ ,  $E(G)$  denotes the edge set of  $G$ , and  $e(G) = |E(G)|$ . Also, if  $v \in V(G)$ , then  $N(v)$  denotes the neighborhood of  $v$  and  $\deg(v) = |N(v)|$ . Call a graph  $\ell$ -hamiltonian if for every linear forest  $F$  with  $\ell$  edges contained in  $G$ ,  $G$  has a hamiltonian cycle containing all edges of  $F$ . In particular, ‘0-hamiltonian’ means ‘hamiltonian’. A well-known sufficient condition for a graph to be  $\ell$ -hamiltonian was proved by Pósa [12]:

**Theorem 1** (Pósa [12]). *Let  $n \geq 3$ ,  $1 \leq \ell < n$  and let  $G$  be an  $n$ -vertex graph such that  $\deg(u) + \deg(v) \geq n + \ell$  for every non-edge  $uv$  in  $G$ . Then  $G$  is  $\ell$ -hamiltonian.*

A family of extremal non- $\ell$ -hamiltonian graphs is as follows. For  $n, d, \ell \in \mathbb{N}$  with  $\ell < d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ , let the graph  $H_{n,d,\ell}$  be obtained from a copy of  $K_{n-d+\ell}$ , say with vertex set  $A$ , by adding  $d - \ell$  vertices of degree  $d$  each of which is adjacent to the same set  $B$  of  $d$  vertices in  $A$  (see the left side of Figure 1). Let

$$h(n, d, \ell) := |E(H_{n,d,\ell})| = \binom{n-d+\ell}{2} + (d-\ell)d. \quad (1)$$

For any  $\ell < d \leq \lfloor (n+\ell-1)/2 \rfloor$ , graph  $H_{n,d,\ell}$  is not  $\ell$ -hamiltonian: No linear forest  $F$  with  $\ell$  edges in  $G[B]$  can be completed to a hamiltonian cycle of  $H_{n,d,\ell}$ . Erdős [3] proved the following Turán-type result:

**Theorem 2** (Erdős [3]). *Let  $n$  and  $d$  be integers with  $0 < d \leq \lfloor \frac{n-1}{2} \rfloor$ . If  $G$  is a nonhamiltonian graph on  $n$  vertices with minimum degree  $\delta(G) \geq d$ , then*

$$e(G) \leq \max \left\{ h(n, d, 0), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor, 0) \right\}.$$

Graphs  $H_{n,d,0}$  and  $H_{n, \lfloor \frac{n-1}{2} \rfloor, 0}$  show the sharpness of Theorem 2.

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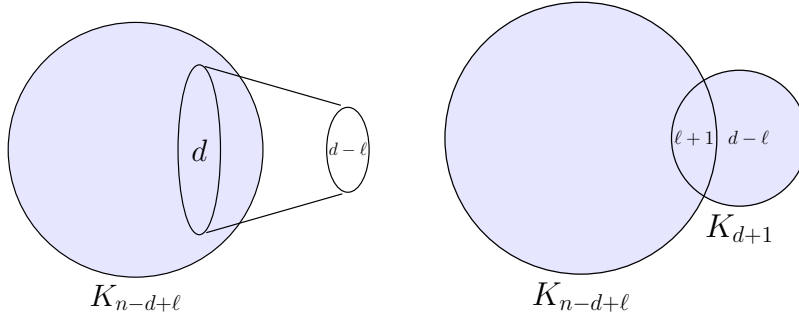


Figure 1: Graphs  $H_{n,d,\ell}$  and  $H'_{n,d,\ell}$ ; shaded ovals denote complete graphs.

Another non- $\ell$ -hamiltonian graph with many edges is  $H'_{n,d,\ell}$  obtained from a copy of  $K_{n-d+\ell}$  and a copy of  $K_{d+1}$  by identifying  $\ell + 1$  vertices (see Fig. 2, on the right). Similarly to  $H_{n,d,\ell}$ , no path of  $\ell$  edges spanning the  $\ell + 1$  dominating vertices in  $H'_{n,d,\ell}$  can be extended to a hamiltonian cycle.

Li and Ning [6] and independently the present authors [4] proved the following refinement of Theorem 2.

**Theorem 3.** *Let  $n \geq 3$  and  $d \leq \lfloor \frac{n-1}{2} \rfloor$ . Suppose that  $G$  is an  $n$ -vertex nonhamiltonian graph with minimum degree  $\delta(G) \geq d$  such that*

$$e(G) > \max \left\{ h(n, d+1, 0), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor, 0) \right\}. \quad (2)$$

*Then  $G$  is a subgraph of either  $H_{n,d,0}$  or  $H'_{n,d,0}$ .*

Recently, Ma and Ning [8] extended Theorem 3 to graphs with bounded circumference. Also, Luo [7] and Ning and Peng [9] bounded the number cliques of given size in graphs with bounded circumference. The goal of this note is to refine and extend Theorem 3 in a different direction—for non- $\ell$ -hamiltonian graphs. One can also view them as an extension of Theorem 1. We state our results in the next section and prove them in the remaining two sections.

## 2 Our results

For a graph  $G$ , let  $N(G, K_r)$  denote the number of copies of  $K_r$  in  $G$ . In particular,  $N(G, K_2) = e(G)$ . Let

$$h_r(n, d, \ell) := N(H_{n,d,\ell}, K_r) = \binom{n-d+\ell}{r} + (d-\ell) \binom{d}{r-1}.$$

We show that classical results easily imply the following extension of Theorem 2 for non- $\ell$ -hamiltonian graphs.

**Theorem 4.** *Let  $n, d, r, \ell$  be integers with  $r \geq 2$  and  $0 \leq \ell < d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ . If  $G$  is an  $n$ -vertex graph with minimum degree  $\delta(G) \geq d$ , and  $G$  is not  $\ell$ -hamiltonian, then*

$$N(G, K_r) \leq \max \left\{ h_r(n, d, \ell), h_r(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell) \right\}.$$

In particular,

$$e(G) \leq \max \left\{ h(n, d, \ell), h(n, \left\lfloor \frac{n + \ell - 1}{2} \right\rfloor, \ell) \right\}.$$

The graphs  $H_{n,d,\ell}$  and  $H_{n, \lfloor (n+\ell-1)/2 \rfloor, \ell}$  show that this bound is sharp for all  $0 \leq \ell < d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ .

Note that a partial case of Theorem 4 (when  $\ell = 1$  and  $r = 2$ ) was proved by Ma and Ning [8].

We also obtain a generalization of Theorem 3 which can be viewed as a stability version of Theorem 4.

**Theorem 5.** *Let  $n \geq 3$ ,  $r \geq 2$  and  $\ell < d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ . Suppose that  $G$  is an  $n$ -vertex not  $\ell$ -hamiltonian graph with minimum degree  $\delta(G) \geq d$  such that*

$$N(G, K_r) > \max \left\{ h_r(n, d + 1, \ell), h_r(n, \left\lfloor \frac{n + \ell - 1}{2} \right\rfloor, \ell) \right\}. \quad (3)$$

Then  $G$  is a subgraph of either  $H_{n,d,\ell}$  or  $H'_{n,d,\ell}$ .

### 3 Bound on the number of $r$ -cliques: Proof of Theorem 4

Beside the Ore-type condition of Theorem 1, Pósa [12] and independently Kronk [5] proved the following degree sequence version.

**Theorem 6** (Pósa [12], Kronk [5]). *Let  $G$  be a graph on  $n$  vertices and let  $0 \leq \ell \leq n - 2$ . The following two conditions (together) are sufficient for  $G$  to be  $\ell$ -hamiltonian:*

- (6.1) *for all integers  $k$  with  $\ell < k < \frac{n+\ell-1}{2}$ , the number of points of degree not exceeding  $k$  is less than  $k - \ell$ ,*
- (6.2) *the number of points of degree not exceeding  $\frac{n+\ell-1}{2}$  does not exceed  $\frac{n-\ell-1}{2}$ .*

We need the following easy claim.

**Claim 7.** *Let  $G$  be an  $n$  vertex graph with a set of  $s$  vertices with degree at most  $t$ . Then  $N(G, K_r) \leq \binom{n-s}{r} + s \binom{t}{r-1}$ .*

*Proof.* Let  $D$  be the set of  $s$  vertices with degree at most  $t$ . Then the number of  $K_r$ 's disjoint from  $D$  is at most  $\binom{n-s}{r}$ . Meanwhile, since each vertex  $v$  of  $D$  has degree at most  $t$ ,  $v$  is contained in at most  $\binom{t}{r-1}$  copies of  $K_r$ . Summing up over all  $v \in D$ , we obtain our result.  $\square$

The following lemma is a corollary of Theorem 6 using Claim 7.

**Lemma 8.** *Let  $G$  be an  $n$ -vertex, not  $\ell$ -hamiltonian graph with  $N(G, K_r) > h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$ . Then*

- (8.1)  *$V(G)$  contains a subset  $D$  of  $k - \ell$  vertices of degree at most  $k$  for some  $k$  with  $\ell < k \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ ;*
- (8.2)  *$k$  should be less than  $\lfloor \frac{n+\ell-1}{2} \rfloor$ ;*
- (8.3)  *$N(G, K_r) \leq h_r(n, k, \ell)$ .*

*Proof.* To estimate  $N(G, K_r)$  in the cases (6.2) and (6.1) in Theorem 6 apply Claim 7 with the values  $(s, t) = (\frac{n-\ell+1}{2}, \frac{n+\ell-1}{2})$  and with  $(s, t) = (\frac{n-\ell-2}{2}, \frac{n+\ell-2}{2})$ , respectively. In both cases the upper bound for  $N(G, K_r)$  is equal to  $h(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$ . Equation (8.3) is also implied by Claim 7.  $\square$

For any integer  $p \geq 1$  and real  $x$  define  $\binom{x}{p}$  as  $x(x-1)\dots(x-p+1)/p!$  for  $x \geq p-1$  and 0 for  $x < p-1$ . This function is non-negative and convex (investigate the second derivative with respect to  $x$  when the function is positive).

For fixed integers  $n$ ,  $\ell$ , and  $r$  with  $0 \leq \ell \leq n - 1$ , and  $r \geq 2$ , consider the function  $h_r(n, x, \ell)$  in the closed interval  $[\ell, \lfloor \frac{n+\ell-1}{2} \rfloor]$ . One can show that this function is also convex in  $x$  (since both terms are convex), and it is strictly convex where it is positive. We obtained the following.

**Claim 9.** *Let  $J \subseteq [\ell, \lfloor \frac{n+\ell-1}{2} \rfloor]$  be a closed interval. Then  $h_r(n, x, \ell)$  is maximized on  $J$  at either of its endpoints.*  $\square$

**Proof of Theorem 4.** Suppose that  $G$  is an  $n$ -vertex, not  $\ell$ -hamiltonian graph with minimum degree  $\delta(G) \geq d$ , for some  $1 \leq \ell \leq d \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ .

If  $N(G, K_r) \leq h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$ , then we are done. Otherwise, (8.3) in Lemma 8 implies that  $N(G, K_r) \leq h_r(n, k, \ell)$  for some  $\ell < k < \lfloor \frac{n+\ell-1}{2} \rfloor$ . We have that  $k \geq \delta(G) \geq d$ . So Claim 9 gives

$$N(G, K_r) \leq \max_{k \in [d, \lfloor \frac{n+\ell-1}{2} \rfloor]} h_r(n, k, \ell) = \max \left\{ h_r(n, d, \ell), h_r(n, \left\lfloor \frac{n+\ell-1}{2} \right\rfloor, \ell) \right\}. \quad \square$$

## 4 Stability: Proof of Theorem 5

Call an  $n$ -vertex graph  $G$   $\ell$ -saturated if  $G$  is not  $\ell$ -hamiltonian, but each  $n$ -vertex graph obtained from  $G$  by adding an edge is  $\ell$ -hamiltonian.

**Theorem 10** (Bondy and Chvátal (Theorem 9.11 in [1])). *Let  $G$  be an  $n$ -vertex  $\ell$ -saturated graph  $G$ . Then for each non-edge  $uv \notin E(G)$ ,  $\deg(u) + \deg(v) \leq n - 1 + \ell$ .*

They observed that the proof by Pósa [12] yields the following fact: If  $G$  is an  $n$ -vertex, not  $\ell$ -hamiltonian graph,  $uv \notin E(G)$ , and  $\deg(u) + \deg(v) \geq n + \ell$ , then  $G + uv$  is not  $\ell$ -hamiltonian either. Since this result implies both Theorems 1 and 6, to make this paper self-contained we include here a sketch of their proof.

Suppose on the contrary, that  $G$  has no hamiltonian cycle containing some linear forest on  $\ell$  edges  $F$  but  $G + uv$  has a hamiltonian cycle through  $F$ . Then we can order the vertices so that  $G$  has a hamiltonian path  $w_1 w_2 \dots w_n$  where  $w_1 = u$  and  $w_n = v$  containing  $F$ . Let  $N_G(u) = \{w_{i_1}, \dots, w_{i_k}\}$  where  $k = \deg_G(u)$ . If there exists a  $1 \leq j \leq k$  such that  $w_{i_j-1} \in N(v)$  and  $w_{i_j} w_{i_j-1} \notin E(F)$ , then

$$w_1 \dots w_{i_j-1} \cup w_{i_j-1} w_n \cup w_n \dots w_{i_j} \cup w_{i_j} w_1$$

is a hamiltonian cycle in  $G$  containing  $F$ . So either  $w_{i_j-1} w_{i_j} \in F$ , or  $w_{i_j-1} \notin N(v)$ . Since  $F$  contains  $\ell$  edges, we have  $k - \ell$  choices of  $w_{i_j-1} \in V(G) \setminus \{v\}$  satisfying  $vw_{i_j-1} \notin E(G)$ . This gives  $\deg_G(v) \leq (n - 1) - (k - \ell)$  yielding the contradiction  $\deg(u) + \deg(v) \leq n + \ell - 1$ .

To complete the proof of Theorem 10 suppose that  $G$  is  $\ell$ -saturated (note that  $G$  is not  $\ell$ -hamiltonian) and suppose that  $uv \notin E(G)$ . If  $\deg(u) + \deg(v) \geq n + \ell$ , then  $G + uv$  is not  $\ell$ -hamiltonian, a contradiction.  $\square$

We show a useful feature of the structure of saturated graphs with many edges.

**Lemma 11.** *Let  $G$  be an  $\ell$ -saturated  $n$ -vertex graph with  $N(G, K_r) > h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$ . Then for some  $\ell < k < \lfloor \frac{n+\ell-1}{2} \rfloor$ ,  $V(G)$  contains a subset  $D$  of  $k - \ell$  vertices of degree at most  $k$  such that  $G - D$  is a complete graph.*

**Proof.** Apply Lemma 8 (8.1) to  $G$ . It says that there is a subset of  $k - \ell$  vertices of degree at most  $k$  such that  $\ell < k \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ . Choose the maximum such  $k$ , and let  $D$  be the set of the vertices with degree at most  $k$ . Then Lemma 8 (8.2) implies that  $k < \lfloor \frac{n+\ell-1}{2} \rfloor$ . Then the maximality of  $k$  gives that  $|D| = k - \ell$ .

Suppose there exist  $x, y \in V(G) - D$  such that  $xy \notin E(G)$ . Among all such pairs, choose  $x$  and  $y$  with the maximum  $\deg(x)$ . Let  $D' := V(G) - N(x) - \{x\}$ . Here  $|D'| = n - \deg(x) - 1 > 0$ . By Theorem 10,

$$\deg(z) \leq n + \ell - 1 - \deg(x) = |D'| + \ell =: k' \quad \text{for all } z \in D'.$$

So  $D'$  is a set of  $k' - \ell$  vertices of degree at most  $k'$ . Since  $y \in D' \setminus D$ ,  $k' \geq \deg(y) > k$ . Thus by the maximality of  $k$ , we get  $k' = n + \ell - 1 - \deg(x) > \lfloor \frac{n+\ell-1}{2} \rfloor$ . Equivalently,  $\deg(x) < \lceil \frac{n+\ell-1}{2} \rceil$ , i.e.,  $\deg(x) \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ . We also get  $|D'| = n - 1 - \deg(x) > n - 1 - \lceil \frac{n+\ell-1}{2} \rceil = \lfloor \frac{n+\ell-1}{2} \rfloor - \ell$ .

For all  $z \in D'$ , either  $z \in D$  where  $\deg(z) \leq k \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ , or  $z \in V(G) - D$ , and so  $\deg(z) \leq \deg(x) \leq \lfloor \frac{n+\ell-1}{2} \rfloor$ . It follows that  $D'$  has a subset of  $\lfloor \frac{n+\ell-1}{2} \rfloor - \ell$  vertices of degree at most  $\lfloor \frac{n+\ell-1}{2} \rfloor$ . This contradicts Lemma 8 (8.2).

Thus,  $G - D$  is a complete graph. □

**Lemma 12.** *Under the conditions of Lemma 11, if  $k = \delta(G)$ , then  $G = H_{n,\delta(G),\ell}$  or  $G = H'_{n,\delta(G),\ell}$ .*

**Proof.** Set  $d := \delta(G)$ , and let  $D$  be a set of  $d - \ell$  vertices with degree at most  $d$ . Let  $u \in D$ . Since  $\delta(G) \geq |D| + \ell = d$ ,  $u$  has a neighbor  $w$  outside of  $D$ . Consider any  $v \in D - \{u\}$ . By Lemma 11,  $w$  is adjacent to all of  $V(G) - D - \{w\}$ . It also is adjacent to  $u$ , therefore its degree is at least  $(n - d + \ell - 1) + 1 = n - d + \ell$ . We obtain

$$\deg(w) + \deg(v) \geq (n - d + \ell) + d = n + \ell.$$

Then by (10),  $w$  is adjacent to  $v$ , and hence  $w$  is adjacent to all vertices of  $D$ .

Let  $W$  be the set of vertices in  $V(G) - D$  having a neighbor in  $D$ . We have obtained that  $|W| \geq \ell + 1$  and

$$N(u) \cap (V(G) - D) = W \quad \text{for all } u \in D. \tag{4}$$

Let  $G' = G[D \cup W]$ . Since  $|W| \leq \delta(G)$ ,  $|V(G')| \leq 2d - \ell$ . If  $|V(G')| = 2d - \ell$ , then by (4), each vertex  $u \in D$  has the same  $d$  neighbors in  $V(G) - D$ . Because  $\deg(u) = d$ ,  $D$  is an independent set. Thus  $G = H_{n,d,\ell}$ . Otherwise,  $(\ell + 1 + d - \ell) \leq |V(G')| \leq 2d - \ell - 1$ .

If  $|V(G')| = d + 1$ , then  $|W| = \ell + 1$ . Because  $\delta(G) \geq d$ , each vertex in  $D$  has at least  $d - 1$  neighbors in  $D$ . But this implies that  $D$  is a clique, and  $G = H'_{n,d,\ell}$ .

So we may assume  $d + 2 \leq |V(G')| \leq 2d - \ell - 1$ . That is,  $|W| \geq \ell + 2$ . We will show that in this case  $G$  is  $\ell$ -hamiltonian, a contradiction.

Let  $F$  be a linear forest in  $G$  with  $\ell$  edges, set  $F_1 := F \cap G'$ ,  $F_2 := F - F_1$ . Let  $ab$  be any edge within  $G[W]$  such that  $ab \notin E(F)$  and  $F \cup ab$  is a linear forest in  $G$ . Such an edge must exist because  $G[W]$  is a clique and either  $F_1$  is a path that occupies at most  $\ell + 1$  vertices in  $W$ , or  $F_1$  is a disjoint union of paths and we can choose  $ab$  to join endpoints of two different components of  $F_1$ .

For any  $x, x' \in V(G')$ ,

$$\deg_{G'}(x) + \deg_{G'}(x') \geq d + d \geq |V(G')| + \ell + 1.$$

Therefore by Theorem 1,  $G'$  has a hamiltonian cycle  $C$  that passes through  $F_1 \cup ab$ . In particular, we obtain an  $(a, b)$ -hamiltonian path  $P_1$  in  $G'$  which passes through  $F_1$ . Since  $G'' := G - (V(G') - \{a, b\})$  is a complete graph, it contains an  $(a, b)$ -hamiltonian path  $P_2$  that passes through  $F_2$ . Then  $P_1 \cup P_2$  is a hamiltonian cycle of  $G$  containing  $F$ , a contradiction. □

**Proof of Theorem 5.** Let  $G'$  be obtained by adding edges to  $G$  until it is  $\ell$ -saturated. If  $N(G', K_r) \leq h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell)$ , then we are done. Otherwise, by (8.2) in Lemma 8  $G'$  contains a set of  $k - \ell$  vertices with degree at most  $k$  where  $\ell < k < \lfloor (n + \ell - 1)/2 \rfloor$ . If  $k = d$ , then  $G' \in \{H_{n,d,\ell}, H'_{n,d,\ell}\}$  by Lemma 12, and thus  $G$  is a subgraph of one of these two graphs. If  $k \geq d + 1$ , then  $N(G, K_r) \leq N(G', K_r) \leq h_r(n, k, \ell)$  for some

$d + 1 \leq k < \lfloor \frac{n+\ell-1}{2} \rfloor$  by (8.3) in Lemma 8.

So the convexity by Claim 9 gives that in both cases

$$N(G, K_r) \leq \max_{k \in [d+1, \lfloor \frac{n+\ell-1}{2} \rfloor]} h_r(n, k, \ell) = \max \left\{ h_r(n, d+1, \ell), h_r(n, \lfloor \frac{n+\ell-1}{2} \rfloor, \ell) \right\},$$

a contradiction. □

### Concluding remarks

Since in the proof of Theorem 4 for the upper bound for  $N(G, K_r)$  we only use the degree sequence of  $G$ , it seems to be likely that one can obtain similar results for graph classes whose degree sequences are well understood.

Let  $P$  be a property defined on all graphs of order  $n$  and let  $k$  be a nonnegative integer. Bondy and Chvátal [1] call  $P$  to be  $k$ -stable if whenever  $G + uv$  has property  $P$  and  $\deg_G(u) + \deg_G(v) \geq k$ , then  $G$  itself has property  $P$ . The  $k$ -closure  $\text{Cl}_k(G)$  of a graph  $G$  is the (unique) smallest graph  $H$  of order  $n$  such that  $E(G) \subseteq E(H)$  and  $\deg_H(u) + \deg_H(v) < k$  for all  $uv \notin E(H)$ . The  $k$ -closure can be obtained from  $G$  by a recursive procedure of joining nonadjacent vertices with degree-sum at least  $k$ . Thus, if  $P$  is  $k$ -stable and  $\text{Cl}_k(G)$  has property  $P$ , then  $G$  itself has property  $P$ . They prove  $k$ -stability (with appropriate values of  $k$ ) for a series of graph properties, e.g.,  $G$  contains  $C_s$  ( $k = 2n - s$ ),  $G$  contains a path  $P_s$  ( $k = n - 1$ ),  $G$  contains a matching  $sK_2$  ( $k = 2s - 1$ ),  $G$  contains a spanning  $s$ -regular subgraph ( $k = n + 2s - 4$ ),  $G$  is  $s$ -connected ( $k = n + s - 2$ ),  $G$  is  $s$ -wise hamiltonian, i.e., every  $n - s$  vertices span a  $C_{n-s}$  ( $k = n + s - 2$ ).

These graph classes are good candidates to find the maximum number of  $r$ -cliques. But the proof of stability (like in Theorem 5) might require more insight.

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