

On the Noether number of p -groups

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Abstract

A group of order p^n (p prime) has an indecomposable polynomial invariant of degree at least p^{n-1} if and only if the group has a cyclic subgroup of index at most p or it is isomorphic to the elementary abelian group of order 8 or the Heisenberg group of order 27.

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1 Introduction

Let G be a finite group and V a G -module over a field \mathbb{F} of characteristic not dividing the group order $|G|$. The Noether-number $\beta(G, V)$ is the maximal degree in a minimal generating set of the ring of polynomial invariants $\mathbb{F}[V]^G$. It is known that $\beta(G, V) \leq |G|$ (see [16], [9], [8]). Even more, it was observed that $\beta(G) := \sup_V \beta(G, V)$ (where V runs over all G -modules over the base field \mathbb{F}) is typically much less than $|G|$. For an algebraically closed base field of characteristic zero it was proved in [17] that $\beta(G) = |G|$ holds only if G is cyclic. Then it turned out that $\beta(G) \leq \frac{3}{4}|G|$ for any non-cyclic group G (see [7] and [19]). Moreover $\beta(G) \geq \frac{1}{2}|G|$ holds if and only if G has a cyclic subgroup of index at most two, with the exception of four particular groups of small order (see [2, Theorem 1.1]). Recently some asymptotic extensions of this result were given in [14]. Our goal in the present article is to establish the following strengthening of this kind of results for the class of p -groups:

Theorem 1. *If G is a finite p -group for a prime p and the characteristic of the base field \mathbb{F} is zero or greater than p then the inequality*

$$\beta(G) \geq \frac{1}{p}|G| \tag{1}$$

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holds if and only if G has a cyclic subgroup of index at most p or G is the elementary abelian group $C_2 \times C_2 \times C_2$ or the Heisenberg group of order 27.

The proof of Theorem 1 will be reduced to the study of a single critical case, the Heisenberg group H_p , which is the extraspecial group of order p^3 and exponent p for an odd prime p . We prove about this the following result:

Theorem 2. *For any prime $p \geq 5$ and base field \mathbb{F} of characteristic 0 or greater than p we have $\beta(H_p) < p^2$.*

The paper is organised as follows. Section 2 contains some technical results on zero-sum sequences over abelian groups that will be needed later. In Section 3 we reduce the proof of Theorem 1 to that of Theorem 2. Then in Section 4 we explain the main invariant theoretic idea behind the proof of Theorem 2 which is also applicable in a more general setting. The proof itself of Theorem 2 will then be carried out in full detail in Section 5. Finally, Section 6 completes our argument by showing that for the case $p = 3$ we have $\beta(H_3) = 9$ in any non-modular characteristic.

2 Some preliminaries on zero-sum sequences

We follow here in our notations and terminology the usage fixed in [5]. Let A be an abelian group noted additively. By a sequence S over a subset $A_0 \subseteq A$ we mean a multiset of elements of A_0 . They form a free commutative monoid with respect to concatenation, denoted by $S \cdot T$, and unit element the empty sequence \emptyset ; this has to be distinguished from 0, the zero element of A . The sequence $a \cdot a \cdots a$, obtained by the k -fold repetition of an element $a \in A$, is denoted by $a^{[k]}$; this has to be distinguished from the product $ka \in A$. The multiplicity of an element $a \in A$ in a sequence S is denoted by $\mathbf{v}_a(S)$. We also write $a \in S$ to indicate that $\mathbf{v}_a(S) > 0$. We say that T is a subsequence of S , and write $T \mid S$, if there is a sequence R such that $S = T \cdot R$. In this case we also write $R = S \cdot T^{[-1]}$. The *length* of a sequence, denoted by $|S|$, can be expressed as $\sum_{a \in A} \mathbf{v}_a(S)$, whereas the *sum* of a sequence $S = a_1 \cdots a_n$ is $\sigma(S) := a_1 + \dots + a_n \in A$ and by convention we set $\sigma(\emptyset) = 0$. We say that S is a *zero-sum sequence* if $\sigma(S) = 0$.

The relevance of zero-sum sequences for our topic is due to the fact that for an abelian group A the Noether number $\beta(A)$ coincides with the Davenport constant $\mathbf{D}(A)$, which is defined as the maximal length of a zero-sum sequence over A not containing any non-empty, proper zero-sum subsequence (see e.g. [5, Chapter 5]). Its value for p -groups is given by the following formula [12,

Theorem 5.5.9]:

$$D(C_{p^{n_1}} \times \cdots \times C_{p^{n_r}}) = \sum_{i=1}^r (p^{n_i} - 1) + 1. \quad (2)$$

A variant of this notion is the k th Davenport constant $D_k(A)$ defined for any $k \geq 1$ as the maximal length of a zero-sum sequence S that cannot be factored as the concatenation $S = S_1 \cdots S_{k+1}$ of non-empty zero-sum sequences S_i over A . Its numerical value is much less known (for some recent results see [10]); we shall only need the fact that according to [12, Theorem 6.1.5.2]:

$$D_k(C_p \times C_p) = kp + p - 1. \quad (3)$$

The following consequence of the definition of $D_k(A)$ will also be used:

Lemma 3 ([12], Lemma 6.1.2). *Any sequence S over an abelian group A of length at least $D_k(A)$ factors as $S = S_1 \cdots S_k \cdot R$ with some non-empty zero-sum sequences S_i .*

We define for any sequence S over A the set of all partial sums of S as $\Sigma(S) := \{\sigma(T) : \emptyset \neq T \mid S\}$. If $0 \notin \Sigma(S)$ then S is called *zero-sum free*. The next result could also be deduced from the Cauchy-Davenport theorem (see [12, Corollary 5.2.8.1]) but we provide here an elementary proof for the reader's convenience:

Lemma 4. *Let p be a prime. Then for any sequence S over $C_p \setminus \{0\}$ we have $|\Sigma(S)| \geq \min\{p, |S|\}$.*

Proof. We use induction on the length of S . For $|S| = 0$ the claim is trivial. Otherwise consider a sequence $S \cdot a$ where the claim holds for S . We have $\Sigma(S \cdot a) = \Sigma(S) \cup \{a\} \cup (a + \Sigma(S))$, where $a + \Sigma(S) := \{a + s : s \in \Sigma(S)\}$. Then either $|\Sigma(S \cdot a)| \geq |\Sigma(S)| + 1$, or else $a \in \Sigma(S)$ and $a + \Sigma(S) = \Sigma(S)$, that is when $\Sigma(S)$ is a subgroup of C_p containing a . But since C_p has only two subgroups and by assumption $\Sigma(S) \ni a \neq 0$, this means that $\Sigma(S) = C_p$. \square

Lemma 5 ([12], Theorem 5.1.10.1). *A sequence S over C_p (p prime) of length $|S| = p - 1$ is zero-sum free if and only if $S = a^{[p-1]}$ for some $a \in C_p \setminus \{0\}$.*

Lemma 6 ([12], Proposition 5.7.7.1). *Let p be a prime and S be a sequence over $C_p \times C_p$ of length $|S| \geq 3p - 2$. Then S has a zero-sum subsequence $X \mid S$ of length p or $2p$.*

We close this section with a technical result. Its motivation and relevance will become apparent through its application in the proof of Proposition 14. For any function π defined on A and any sequence S over A we will write $\pi(S)$ for the sequence obtained from S by applying π element-wise.

Lemma 7. *Let S be a sequence over C_p of length $|S| \geq p^2 - 1$. If we have $\nu_0(S) \geq p + 1$ then $S = S_1 \cdots S_\ell \cdot R$, where each S_i is a non-empty zero-sum sequence and $\ell \geq 2p - 1$.*

Proof. Let ℓ denote the maximal integer such that $S = S_1 \cdots S_\ell \cdot R$ for some non-empty zero-sum sequences S_i . Then each S_i is irreducible, hence $|S_i| \leq p$ and R is zero-sum free, hence $|R| \leq p - 1$. Assuming that $\ell \leq 2p - 2$ we get

$$p^2 - 1 \leq |S| \leq \nu_0(S) + (\ell - \nu_0(S))p + p - 1 \leq (p - 1)(2p + 1 - \nu_0(S))$$

whence $\nu_0(S) \leq p$ follows, in contradiction with our assumption. \square

Proposition 8. *Let $A = C_p \times C_p$ for some prime $p \geq 5$ and $\pi : A \rightarrow C_p$ the projection onto the first component. If S is a sequence over A with $|S| \geq p^2 - 1$ and $\nu_0(\pi(S)) \leq p$ then for any given subsequence $T \mid S$ of length $|T| \leq p - 1$ there is a factorisation $S = S_1 \cdots S_{p-1} \cdot R$, where each S_i is a non-empty zero-sum sequence over A , while $T \mid S \cdot (S_1 \cdot S_2)^{[-1]}$ and $\Sigma(\pi(S_1)) = C_p$.*

Proof. Let $S^* \mid S \cdot T^{[-1]}$ be the maximal subsequence such that $0 \notin \pi(S^*)$. Then by assumption $|S^*| \geq |S| - 2p + 1 \geq 3p$, as $p \geq 5$, so there is a zero-sum subsequence $X \mid S^*$ of length p or $2p$ by Lemma 6. We have two cases:

(i) If $|X| = 2p$ then $X = S_1 \cdot S_2$ for some non-empty zero-sum sequences S_1, S_2 such that $|S_1| \geq p$ and $|S_2| \leq p$, as $D(C_p \times C_p) = 2p - 1$ by (2).

(ii) If $|X| = p$ then we can take $S_1 := X$. Then we have $|S \cdot (S_1 \cdot T)^{[-1]}| \geq |S| - 2p + 1 \geq 3p$, so again by Lemma 6 we find a non-empty zero-sum sequence $S_2 \mid S \cdot (S_1 \cdot T)^{[-1]}$ of length $|S_2| \leq p$ as above.

In both cases $T \mid S \cdot (S_1 \cdot S_2)^{[-1]}$ and $|S_1 \cdot S_2| \leq 2p$ by construction. Consequently $|S \cdot (S_1 \cdot S_2)^{[-1]}| \geq |S| - 2p \geq p^2 - 2p - 1 = D_{p-3}(C_p \times C_p)$, hence by Lemma 3 we have a factorisation $S \cdot (S_1 \cdot S_2)^{[-1]} = S_3 \cdots S_{p-1} \cdot R$ with non-empty zero-sum sequences S_i for each $i \geq 3$. Finally, in both cases we had $|S_1| \geq p$ and $0 \notin \pi(S_1)$, hence $|\Sigma(\pi(S_1))| = p$ by Lemma 4. \square

3 Reduction of Theorem 1 to Theorem 2

Our main tool here will be the k th Noether number $\beta_k(G, V)$ which is defined for any $k \geq 1$ as the greatest integer d such that some invariant of degree d exists which is not contained in the ideal of $\mathbb{F}[V]^G$ generated by the products of at least $k + 1$ invariants of positive degree. This notion was introduced in [4, Section 2] with the goal of estimating the ordinary Noether number from information on its composition factors. This was made possible by [2, Lemma 1.4] according to which for any normal subgroup $N \triangleleft G$ we have:

$$\beta(G, V) \leq \beta_{\beta(G/N)}(N, V). \quad (4)$$

As observed in [5, Chapter 5], if A is an abelian group then $\beta_k(A)$ coincides with $D_k(A)$, so that we can use (3) in the applications of (4).

Proof of Theorem 1 (assuming Theorem 2). The “if” part follows from [17, Proposition 5.1] which states that $\beta(C) \leq \beta(G)$ for any subgroup $C \leq G$. So if C is cyclic of index at most p then $\beta(G) \geq \beta(C) = |C| = |G|/[G : C] \geq \frac{1}{p}|G|$. Moreover $\beta(C_2^3) = 4$ by (2) and $\beta(H_3) \geq 9$ by Proposition 17 below.

The “only if” part for $p = 2$ follows from [2, Theorem 1.1] so for the rest we may assume that $p \geq 3$. Let G be a group of order p^n for which (1) holds. If G is non-cyclic then it has a normal subgroup $N \cong C_p \times C_p$ by [1, Lemma 1.4]. We claim that G/N must be cyclic. For otherwise by applying [1, Lemma 1.4] to the factor group G/N we find a subgroup K such that $N \triangleleft K \triangleleft G$ and $K/N \cong C_p \times C_p$. But then we get using (4) and (3) that

$$\beta(K) \leq \beta_{\beta(C_p \times C_p)}(C_p \times C_p) = p(2p - 1) + p - 1 = 2p^2 - 1 < p^3 = \frac{1}{p}|K|.$$

As $\beta(G)/|G| \leq \beta(K)/|K|$ by [2, Lemma 1.2] we get a contradiction with (1).

Now let $g \in G$ be such that gN generates $G/N \cong C_{p^{n-2}}$. Then $g^{p^{n-2}} \in N$ has order p or 1. In the first case $\langle g \rangle$ has index p in G and we are done. In the other case $\langle g \rangle \cap N = \{1\}$ hence $G \cong N \rtimes \langle g \rangle$. If g acts trivially on N then G contains a subgroup $H \cong C_p \times C_p \times C_p$ for which we have $\beta(H) = 3p - 2$ by (2) hence $\beta(G)/|G| \leq \beta(H)/|H| < 3/p^2 \leq 1/p$, as $p \geq 3$, a contradiction. This shows that g must act non-trivially on $C_p \times C_p$. It is well known that $\text{Aut}(C_p \times C_p) = \text{GL}(2, p)$ has order $(p^2 - 1)(p^2 - p)$, so its Sylow p -subgroup must have order p and it is isomorphic to C_p . Therefore g^p must act trivially on N , so if $n \geq 4$ then $g^p \neq 1$ and the subgroup $\langle N, g^p \rangle$ is isomorphic to $C_p \times C_p \times C_p$, but this was excluded before. The only case which remains open is that $n = 3$ and $G \cong (C_p \times C_p) \rtimes C_p$, where the factor group C_p acts non-trivially on $C_p \times C_p$. This is the Heisenberg group denoted by H_p . By Theorem 2 we have $\beta(H_p) < p^2$ for all $p > 3$ under our assumption on the characteristic of the base field \mathbb{F} . So among the Heisenberg groups the inequality (1) can only hold for H_3 . \square

Remark 9. The precise value of the Noether number is already known for all the p -groups which satisfy (1) according to Theorem 1. As the Theorem states, equality holds in (1) for C_2^3 and H_3 . For the rest, the groups of order p^n which have a cyclic subgroup of index p were classified by Burnside (see e.g. [1, Theorem 1.2]) as follows:

(i) if G is abelian, then either G is cyclic with $\beta(G) = p^n$ or $G = C_{p^{n-1}} \times C_p$ in which case it has $\beta(G) = p^{n-1} + p - 1$ by (2)

(ii) if G is non-abelian and $p > 2$ then G is isomorphic to the modular group $M_{p^n} \cong C_{p^{n-1}} \rtimes C_p$. We have $\beta(M_{p^n}) = p^{n-1} + p - 1$ by [3, Remark 10.4].

(iii) if G is non-abelian and $p = 2$ then G is the dihedral group D_{2^n} or the semi-dihedral group SD_{2^n} or the generalised quaternion group Q_{2^n} . We have $\beta(Q_{2^n}) = 2^{n-1} + 2$ and $\beta(D_{2^n}) = \beta(SD_{2^n}) = 2^{n-1} + 1$ by [3, Theorem 10.3].

Altogether these results imply that for any non-cyclic p -group G we have

$$\beta(G) \leq \frac{1}{p}|G| + p \quad (5)$$

and this inequality is sharp only for the case $p = 2$.

Remark 10. The notion of the Davenport constant $D(G)$, originally defined only for abelian groups as in Section 2, was extended to any finite group G in [11, 13]. For the conjectural connection between the Noether number and this generalisation of the Davenport constant see [5, Section 5.1] and [6].

4 Invariant theoretic lemmas

Let us fix here some notations related to invariant rings. For any vector space V over a field \mathbb{F} we denote its coordinate ring by $\mathbb{F}[V]$. We say that a group G has a left action on V , or that V is a G -module, if a group homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$ is given and we abbreviate $\rho(g)(v)$ by writing $g \cdot v$ for any $g \in G$ and $v \in V$. By setting $f^g(v) := f(g \cdot v)$ for any $f \in \mathbb{F}[V]$ we obtain a right action of G on $\mathbb{F}[V]$. The ring of polynomial invariants is defined as $\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] : f^g = f \text{ for all } g \in G\}$. If the ring $\mathbb{F}[V]^N$ is already known for some normal subgroup $N \triangleleft G$ then $\mathbb{F}[V]^G$ as a vector space is spanned by its elements of the form $\tau_N^G(m)$, where m runs over the set of all monomials and $\tau_N^G : \mathbb{F}[V]^N \rightarrow \mathbb{F}[V]^G$ is the $\mathbb{F}[V]^G$ -module epimorphism defined as

$$\tau_N^G(m) = \frac{1}{|G/N|} \sum_{g \in G} m^g$$

(see e.g. [15, Chapter 2.2]). When N is trivial this definition amounts to the Reynolds operator $\tau := \tau_{\{1\}}^G$. Given any character $\chi \in \widehat{G} := \mathrm{Hom}(G, \mathbb{F}^\times)$ the set $\mathbb{F}[V]^{G,\chi} := \{f \in \mathbb{F}[V] : f^g = \chi(g)f\}$ constitutes the $\mathbb{F}[V]^G$ -module of G -semi-invariants of weight χ . If the restriction of χ to N is trivial, i.e. when $\chi \in \widehat{G/N}$, then these semi-invariants can be obtained by the projection map $\tau_\chi : \mathbb{F}[V]^N \rightarrow \mathbb{F}[V]^{G,\chi}$ defined with the analogous formula

$$\tau_\chi(u) = \frac{1}{|G/N|} \sum_{g \in G/N} \chi^{-1}(g)u^g.$$

$\mathbb{F}[V]$ and $\mathbb{F}[V]^G$ are graded rings: $\mathbb{F}[V]_d$ denotes for any $d \geq 0$ the vector space of degree d homogeneous polynomials and $\mathbb{F}[V]_d^G = \mathbb{F}[V]^G \cap \mathbb{F}[V]_d$. The

set $\mathbb{F}[V]_+^G := \bigoplus_{d \geq 1} \mathbb{F}[V]_d^G$ is a maximal ideal in $\mathbb{F}[V]^G$, while $\mathbb{F}[V]_+^G \mathbb{F}[V]$, the ideal of $\mathbb{F}[V]$ generated by all G -invariant polynomials of positive degree, is the so called *Hilbert-ideal*. This ideal will be our main object of interest since, as observed in [4, Section 3], the graded factor ring $\mathbb{F}[V]/\mathbb{F}[V]_+^G \mathbb{F}[V]$ is finite dimensional and its top degree, denoted by $b(G, V)$, yields an upper bound on the Noether number by an easy argument using the Reynolds operator:

$$\beta(G, V) \leq b(G, V) + 1. \quad (6)$$

It is well known that $\beta(G, V)$ is unchanged when we extend the base field so we will assume throughout this paper that \mathbb{F} is algebraically closed.

Lemma 11. *Let G be a finite group with a normal subgroup N such that G/N is abelian. Let W be a G -module over \mathbb{F} and assume that $|G| \in \mathbb{F}^\times$. Then $(\mathbb{F}[W]_+^N)^k \subseteq \mathbb{F}[W]_+^G \mathbb{F}[W]$ for any $k \geq \mathbf{D}(G/N)$.*

Proof. $\mathbb{F}[W]^N$ regarded as a G/N -module has the direct sum decomposition $\bigoplus_{\chi \in \widehat{G/N}} \mathbb{F}[W]^{G, \chi}$. (Here we used both our assumptions on \mathbb{F} .) This means that any element $u \in \mathbb{F}[W]_+^N$ can be written as a sum $u = \sum_{\chi \in \widehat{G/N}} \tau_\chi(u)$. Now for any $k \geq 1$ and $u_1, \dots, u_k \in \mathbb{F}[W]_+^N$ we have

$$\prod_{i=1}^k u_i = \prod_{i=1}^k \left(\sum_{\chi \in \widehat{G/N}} \tau_\chi(u_i) \right) = \sum_{\chi_1, \dots, \chi_k \in \widehat{G/N}} \tau_{\chi_1}(u_1) \cdots \tau_{\chi_k}(u_k). \quad (7)$$

The term $\tau_{\chi_1}(u_1) \cdots \tau_{\chi_k}(u_k)$ belongs to the ideal $\mathbb{F}[W]_+^G \mathbb{F}[W]$ whenever the sequence (χ_1, \dots, χ_k) over $\widehat{G/N} \cong G/N$ contains a non-empty zero-sum subsequence. But this holds for every term on the right of (7) as $k \geq \mathbf{D}(G/N)$. \square

Lemma 12. *If in Lemma 11 the factor group $G/N \cong C_p$ is cyclic of prime order then for any $g \in G/N$ and any elements $u_1, \dots, u_{p-1} \in \mathbb{F}[W]_+^N$ we have the relation:*

$$u_1 \cdots u_{p-1} - u_1^g u_2^{-g} u_3 \cdots u_{p-1} \in \mathbb{F}[W]_+^G \mathbb{F}[W]. \quad (8)$$

Proof. Observe that in (7) with $k = p-1$ the weight sequence $(\chi_1, \dots, \chi_{p-1})$ over \widehat{C}_p is zero-sum free if and only if $\chi_1 = \dots = \chi_{p-1}$ and χ_1 is non-trivial (by Lemma 5). As a result we get:

$$u_1 \cdots u_{p-1} \in \sum_{\chi \in \widehat{C}_p \setminus \{1\}} \tau_\chi(u_1) \cdots \tau_\chi(u_{p-1}) + \mathbb{F}[W]_+^G \mathbb{F}[W].$$

Replacing here u_1 and u_2 with u_1^g and u_2^{-g} , respectively, and observing that by definition we have $\tau_\chi(u^g) = \chi(g)\tau(u)$ for any $u \in \mathbb{F}[W]^N$ we infer that $u_1^g u_2^{-g} u_3 \cdots u_{p-1}$ must belong to the same residue class modulo the ideal $\mathbb{F}[W]_+^G \mathbb{F}[W]$ to which $u_1 \cdots u_{p-1}$ does belong. This proves our claim. \square

5 The Heisenberg group H_p

The Heisenberg group $H_p = \langle a, b \rangle$ can be defined by the presentation:

$$a^p = b^p = c^p = 1 \quad [a, b] = c \quad [a, c] = [b, c] = 1 \quad (9)$$

where $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$. The subgroups $A := \langle a, c \rangle$ and $B := \langle b, c \rangle$ are normal and isomorphic to $C_p \times C_p$. The Frattini-subgroup, the center and the derived subgroup of H_p all coincide with $\langle c \rangle$, so that H_p is extraspecial. In particular $H_p/\langle c \rangle$ is also isomorphic to $C_p \times C_p$. Taking into account only the subgroup structure of H_p the best upper bound that we can give about its Noether number by means of (4) and (3) is the following:

$$\beta(H_p) \leq \beta_{\beta(C_p)}(C_p \times C_p) = p^2 + p - 1. \quad (10)$$

Our goal in this section will be to enhance this estimate by analysing more closely the invariant rings of H_p .

Let \mathbb{F} be an algebraically closed field with $\text{char}(\mathbb{F}) \neq p$, so that there is a primitive p -th root of unity $\omega \in \mathbb{F}$ that will be regarded as fixed throughout this paper. The irreducible H_p -modules over \mathbb{F} are then of two types:

(i) Composing any group homomorphism $\rho \in \text{Hom}(C_p \times C_p, \mathbb{F}^\times)$ with the canonic surjection $H_p \rightarrow H_p/\langle c \rangle \cong C_p \times C_p$ yields p^2 non-isomorphic 1-dimensional irreducible representations of H_p .

(ii) For each primitive p -th root of unity $\omega^i \in \mathbb{F}$, where $i = 1, \dots, p-1$, take the induced representation $V_{\omega^i} := \text{Ind}_A^{H_p} \langle v \rangle$, where $\langle v \rangle$ is a 1-dimensional left A -module such that $a \cdot v = v$ and $c \cdot v = \omega^i v$. In the basis $\{v, b \cdot v, \dots, b^{p-1} \cdot v\}$ this representation is then given in terms of matrices in the following form, with I_p the $p \times p$ identity matrix:

$$a \mapsto \begin{pmatrix} 1 & & & \\ & \omega^i & & \\ & & \ddots & \\ & & & \omega^{i(p-1)} \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 1 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix} \quad c \mapsto \omega^i I_p. \quad (11)$$

Each V_{ω^i} is irreducible by Mackey's criterion (see e.g. [18]) and for $\omega^i \neq \omega^{i'}$ it is easily seen (e.g. from the matrix corresponding to c) that V_{ω^i} and $V_{\omega^{i'}}$ are non-isomorphic as G -modules.

Adding the squares of the dimensions of the above irreducible H_p -modules we get $p^2 \cdot 1 + (p-1)p^2 = p^3 = |H_p|$, so that no other irreducible H_p -modules exist. As a result an arbitrary H_p -module W over \mathbb{F} has the canonic direct sum decomposition

$$W = U \oplus V_1 \oplus \dots \oplus V_{p-1} \quad (12)$$

where U consists only of 1-dimensional irreducible representations of H_p with $\langle c \rangle$ in their kernel, while each V_i is an isotypic H_p -module consisting of the direct sum of $n_i \geq 0$ isomorphic copies of the irreducible representation V_{ω^i} :

$$V_i = \underbrace{V_{\omega^i} \oplus \dots \oplus V_{\omega^i}}_{n_i \text{ times}}. \quad (13)$$

Next we recall how does the action of G on W extend to the coordinate ring $\mathbb{F}[W]$. When speaking of a coordinate ring $\mathbb{F}[V_{\omega^i}] = \mathbb{F}[x_{i,0}, \dots, x_{i,p-1}]$ we always tacitly assume that the variables $x_{i,k}$ form a dual basis of the basis used at (11). By our convention from Section 4, H_p acts from the right on the variables, i.e. $x^g(v) = x(g \cdot v)$ for all $g \in H_p$, so we can rewrite (11) as:

$$x_{i,k}^b = x_{i,(k-1) \bmod p} \quad x_{i,k}^a = \omega^{ik} x_{i,k} \quad x_{i,k}^c = \omega^i x_{i,k}. \quad (14)$$

(Here, by some abuse of notation, we identified the integers $k = 0, 1, \dots, p-1$ occurring as indexes with the modulo p residue classes they represent.) This shows that the action of the subgroup A on a variable $x_{i,k}$ is completely determined by the modulo p residue classes of the exponents ik and i of ω in (14); we will call $\phi(x_{i,k}) := (ik, i) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ the *weight* of the variable $x_{i,k}$. We shall also refer to the projections $\phi_a(x_{i,k}) = ik$ and $\phi_c(x_{i,k}) = i$. With this notation it is immediate from (14) that for any $n \in \mathbb{Z}$ and $x = x_{i,k}$

$$\phi_a(x^{b^n}) = \phi_a(x) - n \phi_c(x) \quad \text{and} \quad \phi_c(x^{b^n}) = \phi_c(x) \quad (15)$$

where the subtraction and multiplication with n is understood in $\mathbb{Z}/p\mathbb{Z}$. This implies the observation, which will be used frequently later on, that for any variable x with $\phi_c(x) \neq 0$ and any arbitrarily given $w \in \mathbb{Z}/p\mathbb{Z}$ there is always an element $g \in \langle b \rangle$ such that $\phi_a(x^g) = w$. Our discussion also shows that for a variable $y \in \mathbb{F}[W]$ we have $\phi_c(y) = 0$ if and only if $y \in \mathbb{F}[U]$, and otherwise the value $\phi_c(y) = i$ determines the isotypic H_p -module V_i such that $y \in \mathbb{F}[V_i]$.

Any monomial $u \in \mathbb{F}[W]$ is an A -eigenvector, too, hence we can associate a weight $\phi(u) := (j, i) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ to it so that $u^a = \omega^j u$ and $u^c = \omega^i u$. Obviously then $\phi(uv) = \phi(u) + \phi(v)$ for any monomials u, v . If $u = y_1 \cdots y_n$ for some variables $y_i \in \mathbb{F}[W]$, with repetitions allowed, then we can form the sequence $\Phi(u) := \phi(y_1) \cdots \phi(y_n)$ over A , which will be called the *weight sequence* of u . Obviously $\phi(u) = \sigma(\Phi(u)) = \phi(y_1) + \cdots + \phi(y_n)$ with the notations of Section 2. Observe that a monomial u is A -invariant if and only if $\phi(u) = 0$, that is if $\Phi(u)$ is a zero-sum sequence over A . Finally, we set $\Phi_a(u) := (\phi_a(y_1), \dots, \phi_a(y_n))$ and $\Phi_c(u) := (\phi_c(y_1), \dots, \phi_c(y_n))$.

Definition 13. We call two monomials $u, v \in \mathbb{F}[W]$ *homologous*, denoted by $u \sim v$, if $\deg(u) = \deg(v) = d$ and $u = \prod_{n=1}^d y_n$ while $v = \prod_{n=1}^d y_n^{g_n}$ for some variables $y_n \in \mathbb{F}[W]$ (with repetitions allowed) and group elements $g_n \in \langle b \rangle$.

Observe that a monomial v obtained from a monomial u by repeated applications of (8) will be homologous to it in the above sense.

Proposition 14. *Let $p \geq 5$. If $u \in \mathbb{F}[W]$ is a monomial with $\deg(u) \geq p^2 - 1$, $\mathbf{v}_0(\Phi_c(u)) \leq p$ and $v \mid u$ is a monomial such that $\deg(v) \leq p$ and $0 \notin \Phi_c(v)$ then for any homologous monomial $v' \sim v$ there is a homologous monomial $u' \sim u$ such that $v' \mid u'$ and $u' - u \in \mathbb{F}[W]_+^G \mathbb{F}[W]$.*

Proof. We use induction on the degree $d := \deg(v) = \deg(v')$. If $d = 0$ then $v = v' = 1$, so we are done by taking $u' = u$. Suppose now that the claim holds for some $d \leq p - 1$. It suffices to prove that for any given divisor $xv \mid u$, where x is a variable, $\deg(v) = d$, $0 \notin \Phi_c(xv)$, and for any $v' \sim v$ and $g \in \langle b \rangle$ a monomial $u'' \sim u$ exists such that $x^g v' \mid u''$ and $u'' - u \in \mathbb{F}[W]_+^G \mathbb{F}[W]$.

By the inductive hypothesis we already have a monomial $u' \sim u$ such that $v' \mid u'$ and $u' - u \in \mathbb{F}[W]_+^G \mathbb{F}[W]$. As $u'/v' \sim u/v$ and x divides u/v there is a $t \in \langle b \rangle$ such that x^t divides u'/v' . By applying Proposition 8 to the weight sequences $S := \Phi(u')$, $T := \Phi(v')$ we obtain a factorisation $u' = u_1 \cdots u_{p-1} u_p$ such that $u_i \in \mathbb{F}[W]_+^A$ for all $i = 1, \dots, p - 1$, $u_p \in \mathbb{F}[W]$, v' divides $u'/u_1 u_2$ and $\Sigma(\Phi_c(u_1)) = \mathbb{Z}/p\mathbb{Z}$. We have two cases:

i) If $x^t \mid u_1$ (or similarly if $x^t \mid u_2$) then take $u'' := u_1^{-t+g} u_2^{t-g} u_3 \cdots u_{p-1} u_p$. We have $x^g v' \mid u''$ and $u'' \sim u' \sim u$, while $u'' - u' \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ by Lemma 12.

ii) Otherwise $x^t \mid u_k$ for some $k > 2$. By our assumption on $\Sigma(\Phi_c(u_1))$ there is a divisor $w \mid u_1$ with $\phi_c(w) = -\phi_c(x^t)$. As $\phi_c(x^t) = \phi_c(x) \neq 0$ there is an $h \in \langle b \rangle$ for which $\phi_a(w^h) = -\phi_a(x^t)$. Then for $\hat{u} := u_1^h u_2^{-h} u_3 \cdots u_{p-1} u_p$ we have $\hat{u} \sim u$ and $\hat{u} - u \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ by Lemma 12. Take the factorisation $\hat{u} = \hat{u}_1 \cdots \hat{u}_p$ where $\hat{u}_1 = w^h x^t$, $\hat{u}_2 = u_2^{-h}$, $\hat{u}_k = (u_k/x^t)(u_1^h/w^h)$ and $\hat{u}_i = u_i$ for the rest. By construction $\hat{u}_i \in \mathbb{F}[W]_+^A$ for all $i \leq p - 1$, v' divides $\hat{u}/\hat{u}_1 \hat{u}_2$ and $x^t \mid \hat{u}_1$, so this factorisation of \hat{u} falls under case i) and we are done. \square

We need some further notations. The decomposition (12) induces an isomorphism $\mathbb{F}[W] \cong \mathbb{F}[U] \otimes \mathbb{F}[V_1] \otimes \cdots \otimes \mathbb{F}[V_{p-1}]$ which in turn yields for any monomial $m \in \mathbb{F}[W]$ a factorisation $m = m_0 m_1 \cdots m_{p-1}$ such that $m_0 \in \mathbb{F}[U]$ and $m_i \in \mathbb{F}[V_i]$ for all i . Then for each i the decomposition (13) gives the identifications $\mathbb{F}[V_i] = \bigotimes_{j=1}^{n_i} \mathbb{F}[V_{\omega^i}] = \mathbb{F}[x_{i,k}^{(j)} : k = 0, \dots, p - 1; j = 1, \dots, n_i]$, where we set $x_{i,k}^{(j)} := 1 \otimes \cdots \otimes x_{i,k} \otimes \cdots \otimes 1$, i.e. the variable $x_{i,k}$ introduced at (14) is placed in the j th tensor factor. So for any monomial $m_i \in \mathbb{F}[V_i]$ we have a factorisation $m_i = m_i^{(1)} \cdots m_i^{(n_i)}$ where each monomial $m_i^{(j)}$ depends only on the set of variables $\{x_{i,k}^{(j)} : k = 0, 1, \dots, p - 1\}$. Observe finally that two monomials $u, v \in \mathbb{F}[V_1 \oplus \cdots \oplus V_{p-1}]$ are homologous, $u \sim v$, if and only if $\deg(u_i^{(j)}) = \deg(v_i^{(j)})$ for all $i = 1, \dots, p - 1$ and $j = 1, \dots, n_i$.

We shall also need the polarisation operators defined for any polynomial $f \in \mathbb{F}[W]$ by the formula

$$\Delta_i^{s,t}(f) := \sum_{k=0}^{p-1} x_{i,k}^{(t)} \partial_{i,k}^{(s)} f \quad (16)$$

where $\partial_{i,k}^{(s)}$ denotes partial derivation with respect to the variable $x_{i,k}^{(s)}$. All polarisation operations $\Delta := \Delta_i^{s,t}$ are degree preserving, $\deg(\Delta(f)) = \deg(f)$, and G -equivariant, i.e. $\Delta(f^g) = \Delta(f)^g$. Therefore by the Leibniz rule

$$\begin{aligned} \Delta(\mathbb{F}[W]_+^G \mathbb{F}[W]) &\subseteq \mathbb{F}[W]_+^G \mathbb{F}[W] \quad \text{and} \\ \Delta(\mathbb{F}[W]_+^G \mathbb{F}[W]_+) &\subseteq \mathbb{F}[W]_+^G \mathbb{F}[W]_+. \end{aligned} \quad (17)$$

Proposition 15. *Let $p \geq 5$ and assume that $\text{char}(\mathbb{F})$ is 0 or greater than p . If a monomial $m \in \mathbb{F}[W]$ has $\deg(m) \geq p^2 - 1$ then $m \in \mathbb{F}[W]_+^G \mathbb{F}[W]$.*

Proof. Consider the factorisation $m = m_0 m_1 \cdots m_{p-1}$ derived from (12) as described above. Observe that for the weight sequence $S = \Phi(m)$ we have $\mathbf{v}_0(\Phi_c(m)) = \deg(m_0)$. So if $\deg(m_0) \geq p + 1$ then $m \in (\mathbb{F}[W]_+^{(c)})^{2p-1} \mathbb{F}[W]$ by Lemma 7 and we are done, as $\mathbf{D}(G/\langle c \rangle) = \mathbf{D}(C_p \times C_p) = 2p - 1$ by (2) hence $(\mathbb{F}[W]_+^{(c)})^{2p-1} \subseteq \mathbb{F}[W]_+^G \mathbb{F}[W]$ by Lemma 11.

It remains that $\deg(m_0) \leq p$. Then we must have $\deg(m_i) \geq p$ for some $i \geq 1$, say $i = 1$, as otherwise $\deg(m) \leq \deg(m_0) + (p-1)^2 \leq p^2 - p + 1$ would follow. Take the factorisation $m_1 = m_1^{(1)} \cdots m_1^{(n_1)}$ corresponding to the direct decomposition (13). We proceed by induction on $\mu(m) := \max_{j=1}^{n_1} \deg(m_1^{(j)})$.

Assume first that $\mu(m) \geq p$. This means that $\deg(m_1^{(j)}) \geq p$ for some j , say $j = 1$. Now let v be an arbitrary divisor of $m_1^{(1)}$ with degree $\deg(v) = p$ and let $v' = \prod_{g \in \langle b \rangle} x^g$ for some variable $x \in \mathbb{F}[V_{\omega^1}^{(1)}]$. Then v' is b -invariant by construction. Moreover by (15) we have $\phi_c(v') = p\phi_c(x) = 0$ and $\phi_a(v') = p\phi_a(x) - (1 + 2 + \cdots + p - 1)\phi_c(x) = 0$, and consequently v' is G -invariant. Now as $v' \sim v$, we can find by Proposition 14 a monomial $m' \sim m$ such that $m - m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ and $v' \mid m'$. But then $m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ and we are done for this case.

Now let $\mu(m) < p$. As $\deg(m_1) \geq p$, we can take a divisor $v \mid m_1$ such that $v = v^{(i)} v^{(j)}$ for some indices $i \neq j \leq n_1$ where we have $\deg(v^{(i)}) = \mu(m)$ and $\deg(v^{(j)}) = 1$. Then the monomial $v' := (x_{1,1}^{(i)})^{\mu(m)} x_{1,1}^{(j)}$ is homologous with this v and consequently, by Proposition 14, a monomial $m' \in \mathbb{F}[W]$ exists such that $m - m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ and $v' \mid m'$. Our claim will now follow by proving that $m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]$.

To this end observe that for the monomial $\tilde{m} := x_{1,1}^{(i)}m'/x_{1,1}^{(j)}$ we have $\mu(\tilde{m}) = \mu(m) + 1$, hence by the induction hypothesis $\tilde{m} \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ already holds. Moreover $\Delta_1^{i,j}(\tilde{m}) = (\mu(m) + 1)m'$ by construction, hence $(\mu(m) + 1)m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]$ by (17) and we are finished because by our assumption on \mathbb{F} we are allowed to divide by $\mu(m) + 1 \leq p < \text{char}(\mathbb{F})$. \square

Proof of Theorem 2. From Proposition 15 we see that $\mathbb{F}[W]$ as a module over $\mathbb{F}[W]^G$ is generated by elements of degree at most $p^2 - 2$. Equivalently, for the top degree in the factor ring $\mathbb{F}[W]/\mathbb{F}[W]_+^G \mathbb{F}[W]$ we have the estimate $b(G, W) \leq p^2 - 2$, whence by (6) we conclude that $\beta(G, W) \leq p^2 - 1$. \square

6 The case $p=3$

Proposition 16. *Consider $V = V_\omega$ for a primitive third root of unity $\omega \in \mathbb{F}$ as given by (11). Then $\beta(H_3, V) \geq 9$.*

Proof. Let $\mathbb{F}[V] = \mathbb{F}[x, y, z]$ with the variables conforming our conventions. $\mathbb{F}[V]^{H_3}$ is spanned by the elements $\tau(m) := \tau_A^{H_3}(m) = \frac{1}{3}(m + m^b + m^{b^2})$ where m is any A -invariant monomial. An easy argument shows that xyz, x^3, y^3, z^3 are the only irreducible A -invariant monomials. Then by enumerating all A -invariant monomials of degree at most 8 we see that they have degree 3 or 6 so that for $d \leq 8$ we have $\mathbb{F}[V]_d^{H_3} = R_d$, where $R := \mathbb{F}[xyz, \tau(x^3), \tau(x^3y^3)]$. Now if we assume that $\beta(H_3, V) \leq 8$ then $\mathbb{F}[V]^{H_3} = R$ follows. Observe however that all the generators of R are symmetric polynomials, so that $R \subseteq \mathbb{F}[V]^{S_3}$. On the other hand $\tau(x^6y^3) \in \mathbb{F}[V]^{H_3}$ is not a symmetric polynomial, whence $\tau(x^6y^3) \notin R$. This is a contradiction which proves that $\beta(H_3, V) \geq 9$. \square

The upper bound on $\beta(H_3)$ will be obtained by an argument very similar to Propositions 8, 14 and 15, but since there are many different details, too, we preferred to give a self-contained treatment of this case here:

Proposition 17. *If $\text{char}(\mathbb{F}) \neq 3$ then $\beta(H_3) \leq 9$.*

Proof. Suppose that $\beta(H_3, W) \geq 10$ holds for a H_3 -module W . Then there is a monomial $m \in \mathbb{F}[W]^A$ with $\deg(m) \geq 10$ such that $m \notin \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ (as otherwise for any $d \geq 10$ the space $\mathbb{F}[W]_d^G$ spanned by the elements $\tau(m)$ would be contained in $(\mathbb{F}[W]_+^G)^2$). Let $S = \Phi_c(m)$, identify $\langle c \rangle$ with $\mathbb{Z}/3\mathbb{Z}$ and let $d_i = \mathbf{v}_i(S)$ for $i \in \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. Recall that we have the factorisation $m = m_0m_1m_2$ corresponding to the direct decomposition $W = U \oplus V_1 \oplus V_2$, so that $\deg(m_i) = d_i$ for $i = 0, 1, 2$. We may assume by symmetry that $d_1 \geq d_2$.

A. *We claim that $d_1 \geq 5$.*

S is a zero-sum sequence over $\mathbb{Z}/3\mathbb{Z}$ and this is only possible if $d_1 - d_2 \equiv 0 \pmod{3}$. So let $d_1 - d_2 = 3k$ for some integer $k \geq 0$. Denoting by $\ell(S)$ the maximum number of non-empty zero-sum sequences into which S can be factored, we have $\ell(S) = d_0 + d_2 + k \leq 5$, as otherwise by Lemma 11 applied with $N = \langle c \rangle$ we get $m \in (\mathbb{F}[W]_+^{(c)})^6 \subseteq \mathbb{F}[W]_+^G \mathbb{F}[W]_+$, since $H_3/N \cong C_3 \times C_3$ and $D(C_3^2) = 5$ by (2). On the other hand $|S| = d_0 + d_1 + d_2 \geq 10$. Subtracting from this inequality the previous one yields $d_1 - k \geq 5$, whence the claim.

B. For any $w \mid m_1$ with $\deg(w) \leq 2$ there is a factorisation $m = u_1 u_2 u_3$ with $u_i \in \mathbb{F}[W]_+^A$ such that $w \mid u_3$ and $y \mid u_1$ for some variable $y \mid m_1$.

As $\deg(m/w) \geq 8 = D_2(C_3^2)$ there is a factorisation $m/w = u_1 u_2 r$ with $u_1, u_2 \in \mathbb{F}[W]_+^A$. Setting $u_3 = rw$ enforces $u_3 \in \mathbb{F}[W]_+^A$. Here $\deg(u_3) \leq D(A) = 5$ as otherwise $u_3 \in (\mathbb{F}[W]_+^A)^2$ and $m \in (\mathbb{F}[W]_+^A)^4 \subseteq \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ by Lemma 11, a contradiction. Therefore we cannot have $m_1 \mid u_3$, for then by **A.** we have $5 \leq \deg(m_1) \leq \deg(u_3) \leq 5$, so that $m_1 = u_3$ and $\Phi_c(m_1) = 1^{[5]}$, contradicting the assumption that $\Phi(u_3)$ is a zero-sum sequence over A . As a result there is a variable $y \mid m_1$ not dividing u_3 , whence the claim.

C. For any divisor $v \mid m_1$ with $\deg(v) \leq 3$ and any monomial $v' \sim v$ there is a monomial $m' \in \mathbb{F}[W]$ such that $v' \mid m'$ and $m - m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$.

Let $v = xw$ and $v' = x^g w'$ where $\deg(w) \leq 2$, $w' \sim w$ and $g \in \langle b \rangle$. By induction on $\deg(v)$ assume that we already have a monomial $m'' \sim m$ such that $m'' - m \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ and $x^t w' \mid m''$ for some $t \in \langle b \rangle$. According to **B.** there are factorisations $m'' = u_1 u_2 u_3$ with $u_i \in \mathbb{F}[W]_+^A$ such that $w' \mid u_3$ and $y \mid u_1$ for some variable $y \in \mathbb{F}[V_1]$. We have two cases: i) If we can take $y = x^t$ in one these factorisations then for $m' := u_1^{-t+g} u_2^{t-g} u_3 \sim m''$ we have $m' - m'' \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ by Lemma 12 so we are done as $v' \mid m'$. ii) Otherwise necessarily $x^t w' \mid u_3$ and $y \neq x^t$. Still however, there is an $h \in \langle b \rangle$ such that $\phi(y^h) = \phi(x^t)$ hence for $\tilde{m} := u_1^h u_2^{-h} u_3 \sim m''$ we have $m - \tilde{m} \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ by Lemma 12 and we obtain a factorisation $\tilde{m} = \tilde{u}_1 \tilde{u}_2 \tilde{u}_3$ falling under case i) by setting $\tilde{u}_1 = x^t u_1^h / y^h$, $\tilde{u}_2 = u_2^{-h}$, $\tilde{u}_3 = y^h u_3 / x^t$, so we are done again.

D. Now we proceed as in the proof of Proposition 15. For the sake of simplicity from now on we rename our variables so that $\mathbb{F}[V_1] = \bigotimes_{i=1}^{n_1} \mathbb{F}[V_\omega] = \mathbb{F}[x_i, y_i, z_i : i = 1, \dots, n_1]$. Moreover we abbreviate $\Delta_1^{s,t}$ as $\Delta^{s,t}$.

1) If we have $\deg(m_1^{(i)}) \geq 3$ for some $1 \leq i \leq n_1$ then we can apply **C.** with $v' := x_i y_i z_i \in \mathbb{F}[W]_+^G$, concluding that $m \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$, a contradiction.

2) Otherwise if $\deg(m_1^{(i)}) = 2$ for some i then still there is a $j \neq i$ such that $\deg(m_1^{(j)}) \geq 1$. After an application of **C.** we may assume that m is divisible by $x_i^2 x_j$. But then $m = \frac{1}{3} \Delta^{j,i}(\tilde{m})$ for the monomial $\tilde{m} := m x_i / x_j$ which falls under case 1) hence $m \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ by (17), a contradiction.

3) Finally, if $\deg(m_1^{(1)}) = \dots = \deg(m_1^{(n_1)}) = 1$ then after an application

of \mathbf{C} . we may assume that $x_1y_2z_3 \mid m$. Now consider the relation:

$$\Delta^{1,2}(x_1y_1z_3) + \Delta^{2,3}(x_1y_2z_2) + \Delta^{3,1}(x_3y_2z_3) = 3x_1y_2z_3 + \tau(x_3y_2z_1) \quad (18)$$

After multiplying (18) with $m' := m/x_1y_2z_3$ we get on the left hand side $\Delta^{1,2}(my_1/y_2) + \Delta^{2,3}(mz_2/z_3) + \Delta^{3,1}(mx_3/x_1) \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ by (17), as all the three monomials occurring here fall under case 2), and on the right hand side $\tau(x_3y_2z_1)m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$, whence $3m = 3x_1y_2z_3m' \in \mathbb{F}[W]_+^G \mathbb{F}[W]_+$ follows. This contradiction completes our proof. \square

Now comparing Proposition 16 and 17 immediately gives:

Corollary 18. *If $\text{char}(\mathbb{F}) \neq 3$ then $\beta(H_3) = 9$.*

Remark 19. It would be interesting to know if Theorem 2 also extends to the whole non-modular case, i.e. for any field \mathbb{F} whose characteristic does not divide $|G|$, just as it is the case for $p = 3$ by the above result.

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