LIMIT THEOREMS FOR LOCAL AND OCCUPATION TIMES OF RANDOM WALKS AND BROWNIAN MOTION ON A SPIDER

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Abstract

A simple random walk and a Brownian motion are considered on a spider that is a collection of half lines (we call them legs) joined at the origin. We give a strong approximation of these two objects and their local times. For fixed number of legs we establish limit theorems for n step local and occupation times.

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1   Introduction

Consider the following collection of half lines on the complex plane

$$\mathbf{SP}(N) := \{v_N(x, j), \ x \geq 0, \ j = 1, 2, \ldots, N\},$$

where, with $i = \sqrt{-1}$,

$$v_N(x, j) = x \exp\left(\frac{2\pi i(j-1)}{N}\right).$$

We will call $\mathbf{SP}(N)$ a spider with N legs. Also,

$$v_N(0) := v_N(0, 1) = v_N(0, 2) = \ldots = v_N(0, N)$$

is called the body of the spider, and $L_j := \{v_N(x, j), \ x > 0\}$ is the j-th leg of the spider.

We define the distance on $\mathbf{SP}(N)$ by

$$|v_N(x, j) - v_N(y, j)| = |x - y|, \quad j = 1, \ldots, N$$

$$|v_N(x, j) - v_N(y, k)| = x + y, \quad j, k = 1, \ldots, N, \ j \neq k.$$

In this paper the number of legs N is fixed, so we will often suppress N in the notation and, instead of $v_N(x, j)$ or $v_N(0)$, we will simply write $v(x, j)$ or $v(0) = \mathbf{0}$, whenever convenient.

We consider a random walk $\mathbf{S}_n$, $n = 0, 1, 2 \ldots$, on $\mathbf{SP}(N)$ that starts from the body of the spider, i.e., $\mathbf{S}_0 = v_N(0) = \mathbf{0}$, with the following transition probabilities:

$$\mathbf{P}(\mathbf{S}_{n+1} = v_N(1, j)|\mathbf{S}_n = v_N(0)) = p_j, \quad j = 1, \ldots, N,$$

with

$$\sum_{j=1}^{N} p_j = 1,$$

and, for $r = 1, 2, \ldots, \quad j = 1, \ldots, N$,

$$\mathbf{P}(\mathbf{S}_{n+1} = v_N(r+1, j)|\mathbf{S}_n = v_N(r, j)) = \mathbf{P}(\mathbf{S}_{n+1} = v_N(r-1, j)|\mathbf{S}_n = v_N(r, j)) = \frac{1}{2}.$$

The thus defined random walk $\mathbf{S}_n$ on $\mathbf{SP}(N)$ will be called random walk on spider (RWS) or simply spider walk in this paper.

Observe that the particular case $N = 2$, $p_1 = p_2 = 1/2$ corresponds to the simple symmetric random walk $S(n)$, $n = 0, 1, 2, \ldots$, on the line. The spider walk $\mathbf{S}_n$ can be constructed from $S(n)$, $n = 0, 1, \ldots$, as follows. Consider the absolute value $|S(n)|$, $n = 1, 2, \ldots$, that consists of infinitely many excursions from zero, and let $G_1, G_2, \ldots$, denote the excursion intervals. Put these excursions, independently of each other, on leg j of the spider with probability p_j, $j = 1, 2, \ldots, N$. What we

obtain this way is the first n steps of the spider walk $\mathbf{S}_.$, created from the first n steps of the random walk $S(\cdot)$. Let γ_i, $i = 1, 2, \ldots$, be i.i.d. random variables with

$$P(\gamma_i = j) = p_j, \quad j = 1, 2, \ldots, N,$$

that are independent from the simple symmetric random walk $S(\cdot)$. Consequently, one can also define the just introduced spider walk as $\mathbf{S}_0 = \mathbf{0}$,

$$\mathbf{S}_n := \sum_{m=1}^{\infty} I\{n \in G_m\} v_N(|S(n)|, \gamma_m), \ \ n = 1, 2, \ldots, \tag{1.1}$$

if $S(n) \neq 0$, and $\mathbf{S}_n = v_N(0)$ if $S(n) = 0$.

In view of this definition, and the notations already used, in the sequel $\mathbf{S}_n$ will stand for a spider walk and $S(n)$ for a simple symmetric random walk on the line, with respective probabilities denoted by $\mathbf{P}$ and P.

The limit process is the so-called Brownian motion on spider (BMS), or simply Brownian spider, a version of Walsh's Brownian motion (cf Walsh [33], and Introduction of Csáki et al. [11]), that can be constructed from a standard Brownian motion $\{B(t), \, t \geq 0\}$ on the line as follows. The process $\{|B(t)| \, t \geq 0\}$ has a countable number of excursions from zero, and denote by $J_1, J_2, \ldots$, a fixed enumeration of its excursion intervals away from zero. Then, for any $t > 0$ for which $B(t) \neq 0$, we have that $t \in J_m$ for one of the values of $m = 1, 2, \ldots$. Let κ_m, $m = 1, 2, \ldots$, be i.i.d. random variables, independent of $B(\cdot)$ with

$$P(\kappa_m = j) = p_j, \quad j = 1, 2, \ldots, N.$$

We now construct the Brownian spider $\{\mathbf{B}(t), \, t \geq 0\}$ by putting the excursion whose interval is J_m to the κ_m-th leg of the spider $\mathbf{SP}(N)$. Hence we can define the Brownian spider $\mathbf{B}(\cdot)$ by

$$\mathbf{B}(t) := \sum_{m=1}^{\infty} I\{t \in J_m\} v_N(|B(t)|, \kappa_m), \quad \text{if} \quad B(t) \neq 0, \tag{1.2}$$

and

$$\mathbf{B}(t) := v_N(0) = \mathbf{0}, \quad \text{if} \quad B(t) = 0.$$

In [11] we investigated the heights of the spider walk $\mathbf{S}_n$ on the legs of the spider. We also established a strong approximation as follows.

Theorem 1.1 *On a rich enough probability space one can define a* BMS $\{\mathbf{B}(t), \, t \geq 0\}$ *and an* RWS $\{\mathbf{S}_n, \, n = 0, 1, 2, \ldots\}$, *both on* $\mathbf{SP}(N)$, *and both selecting their legs with the same probabilities* p_j, $j = 1, 2, \ldots, N$, *so that, as* $n \to \infty$, *we have*

$$|\mathbf{S}_n - \mathbf{B}(n)| = O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \quad a.s.$$

In this paper we keep N and $p_1, \ldots, p_N$ fixed and consider limit theorems concerning local and occupation times on the legs, as the number of steps n tends to infinity.

2 Local times

Throughout we use the notation $I\{A\}$ for the indicator of an event A in the brackets, i.e., $I\{A\} = 1$ if A occurs and $I\{A\} = 0$ otherwise.

The local time of an RWS $\mathbf{S}_n$ on $\mathbf{SP}(N)$ is defined as

$$\boldsymbol{\xi}((r,j),n) := \sum_{i=1}^{n} I\{\mathbf{S}_i = v(r,j)\}, \ \ r = 1,2,\ldots, j = 1,2,\ldots,N,$$

$$\boldsymbol{\xi}(0,n) := \sum_{i=1}^{n} I\{\mathbf{S}_i = v(0)\},$$

$n = 1,2,\ldots.$

The local time of a BMS is defined by

$$\boldsymbol{\eta}((x,j),t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I\{\mathbf{B}(s) \in (v(x-\varepsilon,j), v(x+\varepsilon,j))\}\,ds, \ \ x > 0, \qquad j = 1,2,\ldots N,$$

$$\boldsymbol{\eta}(0,t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \sum_{j=1}^{N} I\{\mathbf{B}(s) \in (0, v(\varepsilon,j))\}\,ds.$$

Note that by the constructions given in Section 1 we have

$$\boldsymbol{\xi}(0,n) = \xi(0,n) \quad \text{and} \quad \boldsymbol{\eta}(0,n) = \eta(0,n),$$

where $\xi(0,n)$ and $\eta(0,n)$, are the local times at zero of the simple symmetric random walk $S(\cdot)$ and standard Brownian motion $B(\cdot)$, respectively, that are used in the constructions.

First we consider the case $N = 2$, i.e., the so called skew random walk (SRW) and skew Brownian motion (SBM). In this case we let $p_1 = p$, $p_2 = 1 - p = q$.

2.1 Skew random walk and skew Brownian motion

The skew random walk (SRW) $\{\mathbf{S}_n^*, \ n = 0,1,2,\ldots\}$ with parameter p is the particular case of an RWS with $N = 2$. It is a Markov chain on $\mathbb{Z}$ with transition probabilities

$$\mathbf{P}(0,1) = p, \ \mathbf{P}(0,-1) = q = 1-p, \ \mathbf{P}(x,x+1) = \mathbf{P}(x,x-1) = 1/2, \ x = \pm 1, \pm 2, \ldots$$

In this case the construction yielding (1.1) is equivalent to $\mathbf{S}_n^* := 0$, if $S(n) = 0$, and

$$\mathbf{S}_n^* := \sum_{m=1}^{\infty} I\{n \in G_m\} \delta_m |S(n)|, \qquad \text{if} \quad S(n) \neq 0, \ n = 1,2,\ldots, \tag{2.1}$$

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where δ_m, $m = 1, 2, \ldots$ are i.i.d. random variables with $P(\delta_1 = 1) = p = 1 - P(\delta_1 = -1)$, and $G_1, G_2, \ldots$ are excursion intervals of $S(\cdot)$.

The skew Brownian motion (SBM) $\mathbf{B}^*(\cdot)$ with parameter p can be defined as follows (cf. [1]). Let $B(\cdot)$ be a standard Brownian motion on the line, and let $J_1, J_2, \ldots$ be the excursion intervals from zero of $|B(\cdot)|$. Let δ_i, $i = 1, 2, \ldots$, be i.i.d. random variables, independent of $B(\cdot)$, with $P(\delta_1 = 1) = p = 1 - P(\delta_1 = -1)$. Then

$$\mathbf{B}^*(t) := \sum_{m=1}^{\infty} I\{t \in J_m\} \delta_m |B(t)|, \quad \text{if} \quad B(t) \neq 0, \tag{2.2}$$

$$\mathbf{B}^*(t) := 0, \qquad \text{if} \quad B(t) = 0. \tag{2.3}$$

The skew Brownian motion was introduced by Itô and McKean [22] (cf. also the Introduction of Harrison and Shepp [20]). Weak invariance principle between skew random walk and skew Brownian motion was established by Harrison and Shepp [20] and Cherny et al. [6]. We note that throughout this paper we use the above introduced respective abbreviations SRW and SBM. For further results concerning SRW and SBM, we refer to Revuz and Yor [31], Lejay [26], [27], Appuhamillage et al. [1], Hajri [19] and references given in these papers.

We note in passing that the above definition of SBM is a special case of BMS as in (1.2) with $N = 2$, $p_1 = p$, $p_2 = 1 - p = q$, so that, in this case, the first one of the two legs is the positive half-line, and the second one is the negative half-line. In particular, we thus have

$$v_2(|B(t)|, \kappa_m) = \delta_m |B(t)|.$$

The local time of SRW is defined by

$$\boldsymbol{\xi}^*(k, n) := \sum_{i=1}^{n} I\{\mathbf{S}_i^* = k\}, \quad k = 0, \pm 1, \pm 2, \ldots, \ n = 1, 2, \ldots,$$

and that of an SBM is defined by

$$\boldsymbol{\eta}^*(x, t) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I\{\mathbf{B}^*(s) \in (x - \varepsilon, x + \varepsilon)\} \, ds. \tag{2.4}$$

Various results for local times are given in Walsh [33], Burdzy and Chen [5], Lyulko [28], Gairat and Shcherbakov [18].

Here we first give a joint strong invariance principle for SBM and SRW and their local times.

Theorem 2.1 *A probability space with an SBM $\{\mathbf{B}^*(t), t \geq 0\}$ and an SRW $\{\mathbf{S}_n^*, n = 0, 1, 2, \ldots\}$ on it with the same parameter p, can be so constructed that, as $n \to \infty$, we have*

$$|\mathbf{S}_n^* - \mathbf{B}^*(n)| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad a.s. \tag{2.5}$$

and

$$\max_{1 \leq k \leq n} \sup_{x \in \mathbb{Z}} |\boldsymbol{\xi}^*(x, k) - \boldsymbol{\eta}^*(x, k)| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad a.s. \tag{2.6}$$

Proof. The statement of (2.5) is a special case of Theorem 1.1 when $N = 2$, with $p_1 = p$, $p_2 = 1 - p = q$.

The conclusion of (2.6) was established in Csörgő and Horváth [15] in the case of a simple symmetric random walk, i.e., when $p = 1/2$. We are to conclude (2.6) in the present context quite similarly to the proof of Theorem 1 in [15] with suitable modifications.

Let $\{\mathbf{B}^*(t), t \geq 0\}$ be an SBM and define $\tau_0 = 0$,

$$\tau_1 = \inf\{t : t > 0, |\mathbf{B}^*(t)| = 1\}, \quad \tau_n = \inf\{t : t > \tau_{n-1}, |\mathbf{B}^*(t) - \mathbf{B}^*(\tau_{n-1})| = 1\},$$

$n = 2, 3, \ldots$. Using these Skorokhod stopping times, in view of (1.2) and the proof of Theorem 1.1 in [8], it is easy to see that $\mathbf{S}_0^* = 0$,

$$\mathbf{S}_n^* = \sum_{i=1}^{n} X_i^*, \ n = 1, 2 \ldots, \ \text{ with } X_i^* = \mathbf{B}^*(\tau_i) - \mathbf{B}^*(\tau_{i-1}), \ i = 1, 2, \ldots,$$

is an SRW. We note in passing that, along these lines, we can again conclude the statement of (2.5).

Proving now (2.6), put

$$a_i(x) = \boldsymbol{\eta}^*(x, \tau_{\nu(i)+1}) - \boldsymbol{\eta}^*(x, \tau_{\nu(i)}),$$

where

$$\nu(1) = \min\{k : k \geq 0, \mathbf{B}^*(\tau_k) = \mathbf{S}_k^* = x\}, \ \nu(n) = \min\{k : k > \nu(n-1), \mathbf{B}^*(\tau_k) = \mathbf{S}_k^* = x\}, \ n \geq 2.$$

Observe that $a_i(x)$, $i = 1, 2 \ldots$ for any fixed $x \in \mathbb{Z}$, are i.i.d. random variables. Moreover, even though they are defined using an SBM, their common distribution is identical with those, which one gets using a standard Brownian motion. Consequently, we can use Borodin and Salminen [4], p. 173, (3.3.2), to conclude that the random variables $\{a_i(x), i \geq 1\}$, for all $x \in \mathbb{Z}$, are i.i.d. with exponential density function with parameter 1. Thus, for further use, for $i = 1, 2, \ldots$, and $x \in \mathbb{Z}$, we have

$$\mathbf{P}(a_i(x) \geq y) = e^{-y}, \quad y \geq 0. \tag{2.7}$$

Observe that, for all $n \geq 1$, we have

$$\mathbf{P}\left(\left|\sum_{i=1}^{\boldsymbol{\xi}^*(x,n)} a_i(x) - \boldsymbol{\eta}^*(x, \tau_n)\right| \leq \boldsymbol{\eta}^*(0, \tau_1) + a_{\boldsymbol{\xi}^*(x,n)}(x) \text{ for all } x \in \mathbb{Z}\right) = 1. \tag{2.8}$$

Based on (2.7), we have

$$\mathbf{P}\left(\max_{1 \leq i \leq n} \max_{-n \leq x \leq n} a_i(x) > 5 \log n\right) \leq 3n^2 \mathbf{P}(a_1(1) > 5 \log n) \leq \frac{1}{n^2}$$

and, therefore, as $n \to \infty$, by the Borel-Cantelli Lemma,

$$\max_{1 \leq i \leq n} \max_{-n \leq x \leq n} a_i(x) = O(\log n) \quad a.s. \tag{2.9}$$

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In view of (2.8) and (2.9), and observing that $\boldsymbol{\eta}^*(0, \tau_1)$ has the same distribution as $a_i(x)$, we conclude

$$\max_{-n \leq x \leq n} \left| \sum_{i=1}^{\xi^*(x,n)} a_i(x) - \boldsymbol{\eta}^*(x, \tau_n) \right| = O(\log n) \quad a.s., \tag{2.10}$$

as $n \to \infty$.

Observe that by the law of the iterated logarithm (LIL) of Kesten [23] for the maximum in $x \in \mathbb{Z}$ of the local time of the simple symmetric random walk $S(\cdot)$, we have

$$\limsup_{n \to \infty} (2n \log \log n)^{-1/2} \max_{x \in \mathbb{Z}} \xi(x, n) = 1 \quad a.s. \tag{2.11}$$

Consequently, for the maximum local time of $\mathbf{S}_n^*$, we have

$$\limsup_{n \to \infty} (2n \log \log n)^{-1/2} \max_{x \in \mathbb{Z}} \boldsymbol{\xi}^*(x, n) \leq 2 \quad a.s. \tag{2.12}$$

Using (2.7) we get

$$\mathbf{E}(e^{t(a_i(x)-1)}) = \frac{e^{-t}}{1-t}, \qquad \mathbf{E}(e^{t(1-a_i(x))}) = \frac{e^t}{1+t},$$

so for $0 \leq t \leq 1/2$ we have

$$\mathbf{E}(e^{t(a_i(x)-1)}) \leq e^{t^2}, \qquad \mathbf{E}(e^{t(1-a_i(x))}) \leq e^{t^2}.$$

This implies that we can apply the inequality of Petrov [29] (Section III.5, Supplement 10, page 58) for the maximum of partial sums to get that for $0 \leq z \leq K$

$$\mathbf{P}\left(\max_{1 \leq k \leq K} \left| \sum_{i=1}^{k} (a_i(x) - 1) \right| \geq z \right)$$

$$\leq \mathbf{P}\left(\max_{1 \leq k \leq K} \sum_{i=1}^{k} (a_i(x) - 1) \geq z \right) + \mathbf{P}\left(\max_{1 \leq k \leq K} \sum_{i=1}^{k} (1 - a_i(x)) \geq z \right) \leq 2 \exp\left(-\frac{z^2}{4K} \right).$$

Putting $K = 4(n \log \log n)^{1/2}$, we obtain

$$\mathbf{P}\left(\max_{1 \leq k \leq 4(n \log \log n)^{1/2}} \left| \sum_{i=1}^{k} (a_i(x) - 1) \right| > 4\sqrt{3}(n \log \log n)^{1/4} (\log n)^{1/2} \right) \leq \frac{2}{n^3},$$

and hence

$$\mathbf{P}\left(\max_{|x| \leq n} \max_{1 \leq k \leq 4(n \log \log n)^{1/2}} \left| \sum_{i=1}^{k} (a_i(x) - 1) \right| > 4\sqrt{3}(n \log \log n)^{1/4} (\log n)^{1/2} \right) \leq \frac{C}{n^2}$$

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with some constant $C > 0$. Using (2.12), by the Borel-Cantelli Lemma we obtain

$$\max_{|x|\leq n}\left|\sum_{i=1}^{\boldsymbol{\xi}^*(x,n)}(a_i(x)-1)\right| \leq \max_{|x|\leq n}\max_{1\leq k\leq 4(n\log\log n)^{1/2}}\left|\sum_{i=1}^{k}(a_i(x)-1)\right| = O((n\log\log n)^{1/4}(\log n)^{1/2})$$

almost surely, as $n \to \infty$. So we arrived to the conclusion that

$$\limsup_{n\to\infty} n^{-1/4}(\log\log n)^{-1/4}(\log n)^{-1//2}\max_{-n\leq x\leq n}\left|\sum_{i=1}^{\boldsymbol{\xi}^*(x,n)}a_i(x)-\boldsymbol{\xi}^*(x,n)\right| \leq C_1 \quad a.s. \qquad (2.13)$$

with some positive constant C_1.

It is well known that, in terms of the Skorokhod stopping times $\{\tau_i,\ i\geq 0\}$, the random variables $\{\tau_i - \tau_{i-1},\ i \geq 1\}$ are i.i.d. random variables with mean 1 and variance 1. Thus, by the LIL for partial sums, we have

$$\limsup_{n\to\infty}(2n\log\log n)^{-1/2}|\tau_n - n| = 1 \quad a.s.$$

The latter LIL combined with Theorem 3 of Csáki et al. [8] with $g(n) = 4(n\log\log n)^{1/2}$ yields

$$\limsup_{n\to\infty} n^{-1/4}(\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{x}|\boldsymbol{\eta}^*(x, n\pm g(n)) - \boldsymbol{\eta}^*(x,n)| \leq C_2 \quad a.s. \qquad (2.14)$$

with some positive constant C_2.

Also,

$$\boldsymbol{\xi}^*(x,n) = 0 \quad \text{if } |x| > n, \quad \text{and} \quad \lim_{n\to\infty}\sup_{|x|>n}\boldsymbol{\eta}^*(x,n) = 0 \quad a.s. \qquad (2.15)$$

As a consequence of (2.10), (2.12)–(2.14), we now conclude

$$\limsup_{n\to\infty} n^{-1/4}(\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{x\in\mathbb{Z}}|\boldsymbol{\xi}^*(x,n) - \boldsymbol{\eta}^*(x,n)| \leq C_1 + C_2 \quad a.s. \qquad (2.16)$$

that, in turn, implies (2.6), and hence also concludes the proof of Theorem 2.1. $\qquad \square$

Remark. In [15] Csörgő and Horváth show that, using the Skorokhod embedding scheme, the rate of convergence in their Theorem 1 is best possible. The latter result is based on their conclusion that, using the Skorokhod embedding in case of the simple symmetric random walk, one has

$$\limsup_{n\to\infty} n^{-1/4}(\log\log n)^{-1/4}(\log n)^{-1/2}\max_{1\leq k\leq n}|\xi(0,k) - \eta(0,k)| = C \quad a.s.$$

with some positive constant C. Since the local time of an SRW $\mathbf{S}_n^*$ at zero and that of an SBM $\mathbf{B}^*(t)$ respectively coincide with those of a simple symmetric random walk and standard Brownian motion, the above conclusion also implies that the rate of approximation in (2.6) of our Theorem 2.1 is also best possible when using the Skorokhod embedding scheme.

This invariance principle is suitable for proving so-called first order limit theorems for local times. For example, in view of having the LIL for the maximal local time of an SRW as in (2.12), we can conclude the same LIL for the maximal local time of an SBM. Establishing the exact constant for any one of these two LIL's, the same constant would be inherited by the other one in hand.

For the so-called second order limit theorems we introduce the following iterated Brownian motion, or iterated Wiener process. Let $W(t)$, $t \geq 0$ be a standard Wiener process on the line, and let $\tilde{\eta}(t)$ a Wiener local time at zero, independent of $W(\cdot)$. Put

$$Z(t) = W(\tilde{\eta}(t)), \quad t \geq 0.$$

In the sequel we call $Z(t)$ iterated Brownian motion (IBM). For fixed real number $x \neq 0$ we define $c(\cdot)$, for throughout use from now on, as

$$c(x) := \begin{cases} 2p, & x > 0, \\ 2q, & x < 0. \end{cases}$$

The next strong invariance principles deal with second order limit theorems for the local times in hand, at fixed location for large times.

Theorem 2.2 *For any fixed integer $x \neq 0$, a probability space with an SRW $\{\boldsymbol{S}_n^*, \ n = 0, 1, \ldots\}$, and an IBM $\{Z_1(t), \ t \geq 0\}$ on it can be so constructed that for the local times of SRW, as $n \to \infty$, we have*

$$\boldsymbol{\xi}^*(x, n) - c(x)\boldsymbol{\xi}^*(0, n) = (c(x)(4|x| - 1) - c^2(x))^{1/2}\, Z_1(n) + O(n^\mu) \quad a.s. \tag{2.17}$$

with some $7/16 < \mu < 1/4$.

Theorem 2.3 *For any fixed real number $x \neq 0$, a probability space with an SBM $\{\boldsymbol{B}^*(t), \ t \geq 0\}$, and an IBM $\{Z_2(t), \ t \geq 0\}$, can be so constructed that for the local times of SBM, as $t \to \infty$, we have*

$$\boldsymbol{\eta}^*(x, t) - c(x)\boldsymbol{\eta}^*(0, t) = 2(c(x)|x|)^{1/2}\, Z_2(t) + O(t^\mu) \quad a.s.$$

with some $7/16 < \mu < 1/4$.

Proof of Theorem 2.2. Define

$$\rho_0 = 0, \ \rho_k = \min\{i > \rho_{k-1} : \ \boldsymbol{S}_i^* = 0\}, \ k = 1, 2, \ldots,$$

the return times of SRW to the origin. In fact, ρ_k, $k = 1, 2, \ldots$, are also the return times of the simple random walk $S(n)$ that was used to construct SRW in (2.1). Then, for fixed integer $x \neq 0$ we have

$$\mathbf{P}(\boldsymbol{\xi}^*(x, \rho_1) = 0) = 1 - \frac{c(x)}{2|x|},$$

$$\mathbf{P}(\boldsymbol{\xi}^*(x, \rho_1) = m) = \frac{c(x)}{4x^2} \left(\frac{2|x| - 1}{2|x|}\right)^{m-1}, \quad m = 1, 2, \ldots$$

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The two formulas above are simple consequences of Theorem 9.7 of Révész ([30], page 102), where the above distributions are given for a simple symmetric random walk, i.e., for $p = 1/2$. Our conclusions are modified, as the skew random walk has unequal probabilities for stepping right or left from the origin, yielding $c(x)$ in our formulas. Accordingly, one can obtain

$$\mathbf{E}(\boldsymbol{\xi}^*(x, \rho_1)) = c(x), \quad Var\boldsymbol{\xi}^*(x, \rho_1) = c(x)(4|x| - 1) - c^2(x).$$

Put

$$V_i = \boldsymbol{\xi}^*(x, \rho_i) - \boldsymbol{\xi}^*(x, \rho_{i-1}), \quad i = 1, 2, \ldots$$

Then V_i are i.i.d. random variables with distributions given above with finite moment generating function. So by strong invariance results of Komlós et al. [24] we may conclude

$$\boldsymbol{\xi}^*(x, \rho_r) - c(x)r = (c(x)(4|x| - 1) - c^2(x))^{1/2} W(r) + O(\log r) \quad a.s. \tag{2.18}$$

as $r \to \infty$, where $W(\cdot)$ is a standard Wiener process. On noting that, by the definition of Spitzer [32] (cf. also Csáki et al. [9]), the simple random walk $S(n)$ is aperiodic, from here on we can follow the method and proof of [9] to prove Theorem 2.2. We can exactly proceed as in [9], since in our case of skew random walk, the local time $\boldsymbol{\xi}^*(x, \rho_n)$ can be treated as the sum of i.i.d. random variables V_i, where the excursions (ρ_{i-1}, ρ_i), $i = 1, 2, \ldots$, are in fact those of a simple random walk. By Lemma 2.2 in [9] we can construct a positive stable $1/2$ process (inverse Brownian local time) U_n such that

$$|\rho_n - U_n| = O(n^{5/3}) \quad a.s.$$

as $n \to \infty$. But constructing $W(\cdot)$ and $U_\cdot$ jointly, as above, they are not independent. Following the method and proof of Proposition 1 in [9] we can construct independent standard Wiener process $W^{(2)}(\cdot)$ and inverse Brownian local time process $U_\cdot^{(1)}$ so that

$$\boldsymbol{\xi}^*(x, \rho_n) - c(x)n = (c(x)(4|x| - 1) - c^2(x))^{1/2} W^{(2)}(n) + O(n^{2\psi}) \quad a.s.$$

and

$$|\rho_n - U_n^{(1)}| = O(n^{4\mu}) \quad a.s.,$$

as $n \to \infty$, where $\mu < \psi/2$, and

$$\frac{1}{8} < \psi < \frac{1}{4}, \qquad \frac{7}{16} < \mu < \frac{1}{2}.$$

Note that δ in (2.8) and (2.9) of [9] can be choosen arbitrary large, since $\boldsymbol{\xi}^*(x, \rho_1)$ have moment generating function. In particular, $\mu = \psi = 15/62$ can be choosen. Now, just as Theorem 1 follows from Proposition 1 in [9], we can similarly obtain (2.17) in Theorem 2.2. □

Proof of Theorem 2.3. To prove Theorem 2.3, consider the skew Brownian motion $\mathbf{B}^*(\cdot)$ as a diffusion process with scale function

$$s(x) = \frac{x}{c(x)}, \quad x \neq 0,$$

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and speed measure

$$m(dx) = 2c(x)dx, \quad x \neq 0.$$

We note that in Itô and McKean [22], and also in Borodin and Salminen [4], Appendix 1.12, the scale function is given as twice of the above scale function and the speed measure as half of the above speed measure, but this is equivalent to the quantities given above, in that they yield the usual scale and speed for Brownian motion, i.e., for $p = 1/2$.

In order to apply a result of Csáki and Salminen [14], we note that the local time L_t^x is defined there with respect to the speed measure, while $\boldsymbol{\eta}^*(x,t)$ is defined in (2.4) with respect to the Lebesgue measure. One can easily see that

$$L_t^x = \frac{1}{2c(x)}\boldsymbol{\eta}^*(x,t), \quad x \neq 0,$$

$$L_t^0 = \frac{1}{2}\boldsymbol{\eta}^*(0,t).$$

Let $\tau := \min\{s : \boldsymbol{\eta}^*(0,s) = 1\}$. Using the formula (2.18) in [14] for $t = 1/2$, we obtain for $\beta > 0$,

$$\mathbf{E}\left(\exp\left(-\beta\boldsymbol{\eta}^*(x,\tau)\right)\right) = \exp\left(-\frac{c(x)\beta}{1+2\beta|x|}\right), \quad x \neq 0,$$

from which

$$\mathbf{E}(\boldsymbol{\eta}^*(x,\tau)) = c(x), \quad Var(\boldsymbol{\eta}^*(x,\tau)) = 4c(x)|x|, \quad x \neq 0.$$

Theorem 2.3 can now be proved similarly to Theorem 2.2, referring to [9]. Let $B(\cdot)$ be the standard Brownian motion that is used in construction (2.2) in [9], and let $U(\cdot)$ be its inverse local time at zero. Put

$$Y_i = \boldsymbol{\eta}^*(x, U(i)) - \boldsymbol{\eta}^*(x, U(i-1)), \; i = 1, 2, \ldots$$

Then

$$\boldsymbol{\eta}^*(x, U(n)) = \sum_{i=1}^{n} Y_i,$$

which is a sum of i.i.d. random variables that can be approximated by a Wiener process as in (2.18). Now using the method and proof of Theorem 2 and Proposition 2 in [9], we obtain Theorem 2.3 as asserted above. $\square$

Next we state some consequences of the strong approximations given in Theorems 2.2 and 2.3. First, we coclude the following Dobrushin [16]-type theorems, where $\overset{d}{\longrightarrow}$ means convergence in distribution.

Corollary 2.1 *For fixed integer $x \neq 0$ we have, as $n \to \infty$,*

$$\frac{\boldsymbol{\xi}^*(x,n) - c(x)\boldsymbol{\xi}^*(0,n)}{(c(x)(4|x|-1) - c^2(x))^{1/2}n^{1/4}} \overset{d}{\longrightarrow} U\sqrt{|V|},$$

where U and V are independent standard normal random variables.

Corollary 2.2 *For any fixed real number $x \neq 0$ we have, as $t \to \infty$,*

$$\frac{\boldsymbol{\eta}^*(x,t) - c(x)\boldsymbol{\eta}^*(0,t)}{(4c(x)|x|)^{1/2}t^{1/4}} \xrightarrow{d} U\sqrt{|V|},$$

where U and V are independent standard normal random variables.

Strassen-type theorems for the IBM, that also imply limsup results, are given in Csáki et al. [10], [13], and Hu et al. [21]. A Chung-type liminf result is shown in Hu et al. [21]. Applying these results, we also have the following corollaries to Theorems 2.2 and 2.3, respectively.

Corollary 2.3 *For any fixed integer $x \neq 0$, the set of functions*

$$g_n(s) = \frac{\boldsymbol{\xi}^*(x,[sn]) - c(x)\boldsymbol{\xi}^*(0,[sn])}{(c(x)(4|x|-1) - c^2(x))^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}}, \quad 0 \leq s \leq 1,$$

is relatively compact in $C[0,1]$ and the set of its limit points, as $n \to \infty$, is the set of functions $f(s)$, $0 \leq s \leq 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and

$$\int_0^1 |\dot{f}(s)|^{3/4}\,ds \leq 1.$$

Consequently,

$$\limsup_{n\to\infty} \frac{\boldsymbol{\xi}^*(x,n) - c(x)\boldsymbol{\xi}^*(0,n)}{(c(x)(4|x|-1) - c^2(x))^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}} = 1$$

almost surely. Also,

$$\liminf_{n\to\infty} \frac{(\log\log n)^{3/4}}{(c(x)(4|x|-1) - c^2(x))^{1/2}n^{1/4}} \max_{0\leq k\leq n} |\boldsymbol{\xi}^*(x,k) - c(x)\boldsymbol{\xi}^*(0,k)| = \left(\frac{3\pi^2}{8}\right)^{3/4}$$

almost surely.

Corollary 2.4 *For any fixed real number $x \neq 0$ the set of functions*

$$g_t(s) = \frac{\boldsymbol{\eta}^*(x,st) - c(x)\boldsymbol{\eta}^*(0,st)}{2(c(x)|x|)^{1/2}2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}}, \quad 0 \leq s \leq 1,$$

is relatively compact in $C[0,1]$ and the set of its limit points, as $t \to \infty$, is the set of functions $f(s)$, $0 \leq s \leq 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and

$$\int_0^1 |\dot{f}(s)|^{3/4}\,ds \leq 1.$$

Consequently,

$$\limsup_{t\to\infty} \frac{\boldsymbol{\eta}^*(x,t) - c(x)\boldsymbol{\eta}^*(0,t)}{2(c(x)|x|)^{1/2}2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}} = 1$$

almost surely. Also,

$$\liminf_{t\to\infty} \frac{(\log\log t)^{3/4}}{2(c(x)|x|)^{1/2}t^{1/4}} \max_{0\le s\le t} |\eta^*(x,s) - c(x)\boldsymbol{\eta}^*(0,s)| = \left(\frac{3\pi^2}{8}\right)^{3/4}$$

almost surely.

2.2 Local time on spider

First we give a joint invariance principle for the Brownian spider and the random walk on spider $\mathbf{SP}(N)$, and their local times, an analogue of Theorem 2.1.

Theorem 2.4 *On a rich enough probability space one can define a Brownian spider $\{\mathbf{B}(t),\, t \ge 0\}$ and random walk on the spider $\{\mathbf{S}_n,\, n = 0, 1, 2, \ldots\}$, both on $\mathbf{SP}(N)$, and both selecting their legs with the same probabilities p_j, $j = 1, 2, \ldots, N$, so that, as $n \to \infty$, we have*

$$|\mathbf{S}_n - \mathbf{B}(n)| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad a.s.$$

and

$$\sup_{(r,j)} |\boldsymbol{\xi}((r,j),n) - \boldsymbol{\eta}((r,j),n)| = O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}) \quad a.s.,$$

where sup *is taken for $r = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, N$.*

Proof. The proof can be reduced to the case $N = 2$. Namely, for a fixed j, consider the SRW and SBM with $p = p_j$, $q = 1 - p_j$. For this j, Theorem 2.4 follows from Theorem 2.1. Repeating the proof for all $j = 1, \ldots, N$, yields the proof of Theorem 2.4. $\square$

Concerning the so-called second order limit theorems for the local time, we have the following results, corresponding respectively to Theorems 2.2 and 2.3, on replacing $c(x)$ by $2p_j$.

Theorem 2.5 *For any fixed positive integer x and each $j = 1, 2, \ldots, N$, a probability space with an RWS $\mathbf{S}_n$ and an IBM $Z_3(t)$ can be so constructed that, as $n \to \infty$, we have*

$$\boldsymbol{\xi}((x,j),n) - 2p_j\boldsymbol{\xi}(0,n) = (2p_j(4x-1) - 4p_j^2)^{1/2}\, Z_3(n) + o(n^\mu) \quad a.s.,$$

where μ is as in Theorem 2.2.

Theorem 2.6 *For any fixed positive real number x and each $j = 1, 2, \ldots, N$, a probability space with an BMS $\mathbf{B}(t)$ and an IBM $Z_4(t)$ can be so constructed that, as $t \to \infty$, we have*

$$\boldsymbol{\eta}((x,j),t) - 2p_j\boldsymbol{\eta}(0,t) = (8p_jx)^{1/2}\, Z_4(t) + o(t^\mu) \quad a.s.,$$

where μ is as in Theorem 2.3.

Similarly to Corollaries 2.1-2.4, we have the following Dobrushin and Strassen type respective consequences of Theorems 2.5 and 2.6 for local times on a spider.

Corollary 2.5 *For any fixed integer $x > 0$ and each $j = 1, 2, \ldots, N$, we have, as $n \to \infty$,*

$$\frac{\boldsymbol{\xi}((x,j),n) - 2p_j\boldsymbol{\xi}(0,n)}{(2p_j(4x-1) - 4p_j^2)^{1/2}n^{1/4}} \xrightarrow{d} U\sqrt{|V|},$$

where U and V are independent standard normal random variables.

Corollary 2.6 *For any fixed real number $x > 0$ and each $j = 1, 2, \ldots, N$, we have, as $t \to \infty$,*

$$\frac{\boldsymbol{\eta}((x,j),t) - 2p_j\boldsymbol{\eta}(0,t)}{(8p_jx)^{1/2}t^{1/4}} \xrightarrow{d} U\sqrt{|V|},$$

where U and V are independent standard normal random variables.

Corollary 2.7 *For any fixed integer $x > 0$ and each $j = 1, 2, \ldots, N$, the set of functions*

$$g_n(s) = \frac{\boldsymbol{\xi}((x,j),[sn]) - 2p_j\boldsymbol{\xi}(0,[sn])}{(2p_j(4x-1) - 4p_j^2)^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}}, \quad 0 \le s \le 1,$$

is relatively compact in $C[0,1]$ and the set of its limit points, as $n \to \infty$, is the set of functions $f(s)$, $0 \le s \le 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and

$$\int_0^1 |\dot{f}(s)|^{3/4}\, ds \le 1.$$

Consequently,

$$\limsup_{n \to \infty} \frac{\boldsymbol{\xi}((x,j),n) - 2p_j\boldsymbol{\xi}(0,n)}{(2p_j(4x-1) - 4p_j^2)^{1/2}2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}} = 1$$

almost surely. Also,

$$\liminf_{n \to \infty} \frac{(\log\log n)^{3/4}}{(2p_j(4x-1) - 4p_j^2)^{1/2}n^{1/4}} \max_{0 \le k \le n} |\boldsymbol{\xi}((x,j),k) - 2p_j\boldsymbol{\xi}(0,k)| = \left(\frac{3\pi^2}{8}\right)^{3/4}$$

almost surely.

Corollary 2.8 *For any fixed real number $x > 0$ and each $j = 1, 2, \ldots, N$, the set of functions*

$$g_t(s) = \frac{\boldsymbol{\eta}((x,j),st) - 2p_j\boldsymbol{\eta}(0,st)}{(8p_jx)^{1/2}2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}}, \quad 0 \le s \le 1,$$

is relatively compact in $C[0,1]$ and the set of its limit points, as $t \to \infty$, is the set of functions $f(s)$, $0 \le s \le 1$, with $f(0) = 0$, that are absolutely continuous with respect to Lebesgue measure and

$$\int_0^1 |\dot{f}(s)|^{3/4} \, ds \le 1.$$

Consequently,

$$\limsup_{t \to \infty} \frac{\boldsymbol{\eta}((x,j),t) - 2p_j \boldsymbol{\eta}(0,t)}{(8p_j x)^{1/2} 2^{5/4} 3^{-3/4} t^{1/4} (\log \log t)^{3/4}} = 1$$

almost surely. Also,

$$\liminf_{t \to \infty} \frac{(\log \log t)^{3/4}}{(8p_j x)^{1/2} t^{1/4}} \max_{0 \le s \le t} |\boldsymbol{\eta}((x,j),s) - 2p_j \boldsymbol{\eta}(0,s)| = \left(\frac{3\pi^2}{8} \right)^{3/4}$$

almost surely.

3 Occupation times.

In this section we consider the time the RWS, or the BMS, spends on particular legs. First we give the definitions of the occupation times. Recall that L_j denotes the leg j of the spider $\mathbf{SP}(N)$. Let $\mathbf{S}_j$, $j = 0, 1, \ldots$, be a random walk on the spider $\mathbf{SP}(N)$, with probabilities $p_1, \ldots, p_N$. The occupation time on leg j in n steps is defined by

$$T(j,n) := \sum_{k=1}^{n} I\{\mathbf{S}_k \in L_j\}.$$

The occupation time of the Brownian spider $\mathbf{B}(s)$, $s \ge 0$, up to time t is defined by

$$Z(j,t) := \int_0^t I\{\mathbf{B}(s) \in L_j\} \, ds.$$

Theorem 3.1 *On the probability space of* Theorem 2.4 *as $n \to \infty$, we have for any $\epsilon > 0$,*

$$\sup_{1 \le j \le N} |T(j,n) - Z(j,n)| = O(n^{3/4+\varepsilon}) \quad a.s.$$

Proof of Theorem 3.1. The occupation time of RWS on a particular leg of $\mathbf{SP}(N)$ can be considered as having only two legs, one to which the walker proceeds with probability p_j from the origin, and another one to which it goes with probability $1 - p_j$. Based on this idea, we consider the following setup.

Let $N = 2$, $p = p_1$, and $q = p_2 = 1-p$. This corresponds to a skew random walk $\{\mathbf{S}_i^*, \ i = 0, 1, \ldots\}$ with transition probabilities

$$\mathbf{P}(0,1) = p, \ \mathbf{P}(0,-1) = q = 1 - p, \ \mathbf{P}(x, x+1) = \mathbf{P}(x, x-1) = 1/2, \ x \neq 0.$$

Let $T^*(1,n)$ be the occupation time of the positive half-line of a skew random walk, i.e.,

$$T^*(1,n) = \sum_{i=1}^{n} I\{\mathbf{S}_i^* > 0\} = \sum_{i=1}^{\infty} \boldsymbol{\xi}^*(i,n).$$

Moreover, let $Z^*(1,t)$ be the occupation time of the positive half-line of a skew Brownian motion, i.e.,

$$Z^*(1,t) = \int_0^t I\{B^*(s) > 0\}ds = \int_0^\infty \boldsymbol{\eta}^*(x,t)\,dx.$$

The limiting density of $T^*(1,n)$ can be obtained from the density of $Z^*(1,t)$, for which we have

$$\mathbf{P}\left(\frac{Z^*(1,t)}{t} \in dx\right) = \frac{pq}{\pi\sqrt{x(1-x)}(p^2(1-x) + q^2x)}\,dx, \quad 0 < x < 1.$$

The latter formula is given in Lamperti [25] as a limiting distribution of occupation time of certain discrete time processes, including SRW. For this formula and related results, see also Appuhamillage et al. [1].

So we have to prove that for a suitable construction of SBM and SRW, as $n \to \infty$, we have for any $\varepsilon > 0$,

$$|T^*(1,n) - Z^*(1,n)| = O(n^{3/4+\varepsilon}) \quad a.s.,$$

It follows from the second statement of Theorem 2.1 and the law of the iterated logarithm that, as $n \to \infty$,

$$\sum_{i=1}^{\infty} |\boldsymbol{\xi}^*(i,n) - \boldsymbol{\eta}^*(i,n)| = O(n^{3/4+\varepsilon}) \quad a.s. \tag{3.1}$$

Also,

$$\left| Z^*(1,n) - \sum_{i=1}^{\infty} \boldsymbol{\eta}^*(i,n) \right| \leq \sum_{i=1}^{\infty} \sup_{i-1 < x \leq i} |\boldsymbol{\eta}^*(x,n) - \boldsymbol{\eta}^*(i,n)|.$$

Now, for Wiener local time Bass and Griffin [3], Lemma 5.3, proved that for every $\varepsilon > 0$

$$\sup_k \sup_{s \leq t} \sup_{z \in [k,k+1]} |\eta(z,s) - \eta(k,s)| = o(t^{1/4+\varepsilon/2})$$

almost surely, as $t \to \infty$. The skew Brownian motion can be obtained from standard Brownian motion by reflecting some excursions from negative to positive part or from positive to negative part. Hence for $i - 1 < x \leq i$,

$$|\boldsymbol{\eta}^*(x,n) - \boldsymbol{\eta}^*(i,n)| \leq |\eta(x,n) - \eta(i,n)| + |\eta(-x,n) - \eta(-i,n)|,$$

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and thus, by the law of the iterated logarithm

$$\left| Z^*(1,n) - \sum_{i=1}^{\infty} \boldsymbol{\eta}^*(i,n) \right| = O(n^{3/4+\varepsilon}) \quad a.s.,$$

as $n \to \infty$. This, combined with (3.1), proves Theorem 3.1. $\qquad \square$

Since the occupation time is usually of order n, using this strong invariance, it suffices to prove strong theorems either for RWS or BMS.

For the joint distribution of $\{Z(j,t)/t, \ j = 1,\ldots,N\}$, Barlow et al. [2] (see also Yano [35]) have shown the following equality in distribution: up to any time $t > 0$,

$$\left\{ \frac{Z(j,t)}{t}, \quad j = 1,\ldots,N \right\} \stackrel{d}{=} \left\{ \frac{p_j^2 U_j}{\sum_{k=1}^{N} p_k^2 U_k}, \quad j = 1,\ldots,N \right\}, \tag{3.2}$$

where $U_1,\ldots,U_N$ are independent one-sided stable $1/2$ random variables. This, via Theorem 3.1, also yields the joint limiting distribution of $\{T(j,n)/n, \ j = 1,\ldots,N\}$, as $n \to \infty$.

Now let

$$T_M(n) = \max_{1 \le j \le N} T(j,n), \qquad T_m(n) = \min_{1 \le j \le N} T(j,n).$$

and

$$Z_M(t) = \max_{1 \le j \le N} Z(j,t), \qquad Z_m(t) = \min_{1 \le j \le N} Z(j,t).$$

For the limsup of T_M, Z_M and liminf of T_m, Z_m, we show that the Chung-Erdős [7] result for simple symmetric walk remains valid.

Theorem 3.2 *Let $f(x)$ be a positive nondecreasing function for which $\lim_{x \to \infty} f(x) = \infty$, $x/f(x)$ is nondecreasing and $\lim_{x \to \infty} x/f(x) = \infty$.*

Let

$$I(f) := \int_{1}^{\infty} \frac{dx}{x(f(x))^{1/2}}.$$

Then

$$\mathbf{P}\left(T_M(n) > n\left(1 - \frac{1}{f(n)}\right) \ i.o. \ as \ n \to \infty \right) = 0 \ or \ 1,$$

$$\mathbf{P}\left(Z_M(t) > t\left(1 - \frac{1}{f(t)}\right) \ i.o. \ as \ t \to \infty \right) = 0 \ or \ 1,$$

$$\mathbf{P}\left(T_m(n) < \frac{n}{f(n)} \ i.o. \ as \ n \to \infty \right) = 0 \ or \ 1$$

and

$$\mathbf{P}\left(Z_m(t) < \frac{t}{f(t)} \ i.o. \ as \ t \to \infty\right) = 0 \ or \ 1$$

according as $I(f)$ converges or diverges.

Proof of Theorem 3.2. We will only prove the third statement, the proofs for the others go similarly. The limiting distribution of $T(i,n)$ for fixed i is the same as that of skew random walk, which is a general arcsin law (see Lamperti [25] or Watanabe, [34]). For small x it is of the same order as that of the arcsin law. Therefore,

$$c_1\sqrt{x} \le \mathbf{P}(T(i,n) \le nx) \le c_2\sqrt{x}$$

with some positive constants c_1 and c_2. As

$$\mathbf{P}(T(1,n) \le nx) \le \mathbf{P}(T_m(n) \le nx) = \mathbf{P}(\bigcup_{i=1}^{N}\{T(i,n) \le nx\})$$
$$\le N(\max_{i\le N}\mathbf{P}(T(i,n) \le nx),$$

it follows that for small x we have

$$c_3\sqrt{x} \le \mathbf{P}(T_m(n) \le nx) \le c_4\sqrt{x}$$

with positive constants c_3, c_4. Based on the proof of Chung-Erdős [7], we can prove the theorem as follows.

Convergent part: Let $n_k = 2^k$. Then we know that for a positive nondecreasing function $f(x)$, the integral $I(f) = \int_1^\infty \frac{dx}{x(f(x))^{1/2}}$ converges if and only if the sum $\sum_{k=1}^{\infty} \frac{1}{(f(2^k))^{1/2}}$ converges.

First we show that if $I(f) < \infty$, then

$$\sum_{k=1}^{\infty} \mathbf{P}\left(T_m(n_{k-1}) \le \frac{n_k}{f(n_k)}\right) < \infty. \tag{3.3}$$

We have

$$\mathbf{P}\left(T_m(n_{k-1}) \le \frac{2^k}{f(2^k)}\right) = \mathbf{P}\left(T_m(2^{k-1}) \le 2\frac{2^{k-1}}{f(2^{k-1})}\frac{f(2^{k-1})}{f(2^k)}\right)$$
$$\le \mathbf{P}\left(T_m(2^{k-1}) \le 2\frac{2^{k-1}}{f(2^{k-1})}\right) \le c_2\sqrt{2}\frac{1}{(f(2^{k-1}))^{1/2}},$$

where we used that $f(\cdot)$ is nondecreasing. By assumption, we now have that (3.3) is convergent. Thus, for k big enough,

$$T_m(n_{k-1}) \ge \frac{n_k}{f(n_k)} \quad a.s.$$

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Now for $n_{k-1} \leq n \leq n_k$, we have for n big enough that

$$T_m(n) \geq T_m(n_{k-1}) \geq \frac{n_k}{f(n_k)} \geq \frac{n}{f(n)}.$$

Divergent part: First we show that the theorem is valid for a skew random walk. To this end suppose that $N = 2$, and $p_1 = p \leq 1/2$ and $p_2 = q = 1 - p$. Clearly if $p = 1/2$ there is nothing to prove, in view of the original Chung-Erdős theorem. That is to say, we know for the time $T(1, n)$ spent on the positive side (spent on the first leg), we have $\mathbf{P}(T(1, n) < n/f(n) \ i.o.) = 1$, whenever $I(f) = \infty$. If $p < 1/2$, then we start with a simple symmetric walk again, and keep each excursion spent on the positive side with probability $2p$ and, with probability $1 - 2p$, we flip it to the negative side. Thus we get a skew randow walk with $p_1 = p < 1/2$. Denote the time spent on the positive side of this walk by $T^p(1, n)$. By construction, $T^p(1, n) \leq T(1, n)$, so $\mathbf{P}(T^p(1, n) < n/f(n) \ i.o.) = 1$, whenever $I(f) = \infty$. Similarly, if we have a skew random walk with $p_1 = p > 1/2$, we start again with a simple symmetric walk, but now apply the Chung-Erdős theorem for the time spent on the negative side (call it leg two). Then we have $\mathbf{P}(T(2, n) < n/f(n) \ i.o.) = 1$, whenever $I(f) = \infty$. Now construct the new skew random walk by keeping each excursion on leg two with probability $2q$, and flipping them to the first leg with probability $1 - 2q$. As before, $T^p(2, n) \leq T(2, n)$, hence $\mathbf{P}(T^p(2, n) < n/f(n) \ i.o.) = 1$, whenever $I(f) = \infty$. Consequently, for a skew random walk $T_m^p(n) = \min\{T^p(1, n), T^p(2, n)\}$, we have $\mathbf{P}(T_m^p(n) < n/f(n) \ i.o.) = 1$, whenever $I(f) = \infty$. Now for the spider random walk, it is enough to observe that selecting the leg (or one of the legs) with $p* := \min_{1 \leq i \leq N} p_i$ and putting all the excursions from the remaining legs to a second leg, we create a skew random walk for which $\mathbf{P}(T_m^p(1, n) < n/f(n) \ i.o.) = 1$, whenever $I(f) = \infty$. Clearly $T_m(n) \leq T^{p*}(1, n)$, so the theorem is now proved. $\square$

Recall the respective definitions of T_M, Z_M and T_m, Z_m, given right before Theorem 3.2. For the liminf of T_M, Z_M, and the limsup of T_m, Z_m, we have the following results.

Theorem 3.3

$$\liminf_{n \to \infty} \frac{T_M(n)}{n} = \limsup_{n \to \infty} \frac{T_m(n)}{n} = \frac{1}{N} \quad a.s. \tag{3.4}$$

and, similarly,

$$\liminf_{t \to \infty} \frac{Z_M(t)}{t} = \limsup_{t \to \infty} \frac{Z_m(t)}{t} = \frac{1}{N} \quad a.s. \tag{3.5}$$

Proof of Theorem 3.3. We show this result for Z_m and Z_M. The strong invariance result of Theorem 3.1 implies the conclusion for T_m and T_M. Recalling (3.2), we know that therein, $U_j, j = 1, 2, ..., N$, are independent stable $1/2$ random variables. Thus the events

$$\left\{ \frac{1 - \varepsilon}{N} \leq \min_{1 \leq j \leq N} \frac{p_j^2 U_j}{\sum_{k=1}^N p_k^2 U_k} \leq \frac{1}{N} \right\}$$

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and

$$\left\{ \frac{1}{N} \leq \max_{1 \leq j \leq N} \frac{p_j^2 U_j}{\sum_{k=1}^{N} p_k^2 U_k} \leq \frac{1+\varepsilon}{N} \right\}$$

have positive probability for $0 < \varepsilon < 1$.

Hence, via (3.2), for some $\alpha > 0$ and all $t > 0$, we have

$$\mathbf{P}\left(\frac{1}{N} \leq \frac{Z_M(t)}{t} \leq \frac{1+\varepsilon}{N} \right) \geq \alpha$$

and

$$\mathbf{P}\left(\frac{1-\varepsilon}{N} \leq \frac{Z_m(t)}{t} \leq \frac{1}{N} \right) \geq \alpha.$$

Consequently, the events

$$\frac{1}{N} \leq \liminf_{t \to \infty} \frac{Z_M(t)}{t} \leq \frac{1+\varepsilon}{N} \tag{3.6}$$

and

$$\frac{1-\varepsilon}{N} \leq \limsup_{t \to \infty} \frac{Z_m(t)}{t} \leq \frac{1}{N} \tag{3.7}$$

have positive probability, for all $0 < \varepsilon < 1$. It follows from the 0-1 law for Brownian spider (cf. Evans [17], Theorem 1.8) that the respective events as in (3.6) and (3.7) hold true almost surely. Since ε is arbitrary, (3.5) follows. $\quad \square$

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References

[1] APPUHAMILLAGE, T.A., BOKIL, V. THOMANN, E., WAYMIRE, E. AND WOOD, B. (2011). Occupation and local times for skew Brownian motion with applications to dispersion across an interface. *Ann. Appl. Probab.* **21** 183–214.

[2] BARLOW, M.T., PITMAN J.W., YOR M. (1989). Une extension multidimensionelle de la loi de l'arc sinus. *Sém. Prob. XXIII, Lect.Notes in Math.,* **1372**, Springer, Berlin, Heidelberg, New York 294-314.

[3] BASS, R.F. AND GRIFFIN, P.S. (1985). The most visited site of Brownian motion and simple random walk. *Z. Wahrsch. verw. Gebiete* **70** 417–436

[4] BORODIN, A.N. AND SALMINEN, P. (1996). *Handbook of Brownian Motion – Facts and Formulae*, Birkhäuser, Basel, Boston, Berlin.

[5] BURDZY, K. AND CHEN, Z-Q. (2001). Local time flow related to skew Brownian motion. *Ann. Probab.* **29** 1693–1715

[6] CHERNY, A., SHIRYAEV, A. AND YOR, M. (2004). Limit behavior of the "horizontal-vertical" random walk and some extension of the Donsker-Prokhorov invariance principle. *Theory Probab. Appl.* **47** 377–394

[7] CHUNG, K.L. AND ERDŐS, P. (1952). On the application of the Borel-Cantelli lemma. *Trans. Am. Math. Soc.* **64** 179–186.

[8] CSÁKI, E., CSÖRGŐ, M., FÖLDES, A. AND RÉVÉSZ, P. (1983). How big are the increments of the local time of a Wiener process? *Ann. Probab.* **11** 593–608

[9] CSÁKI, E., CSÖRGŐ, M., FÖLDES, A. AND RÉVÉSZ, P. (1992). Strong approximation of additive functionals. *J. Theoret. Probab.* **5** 679–706

[10] CSÁKI, E., CSÖRGŐ, M., FÖLDES, A. AND RÉVÉSZ, P. (1995). Global Strassen-type theorems for iterated Brownian motions. *Stochast. Process. Appl.* **59** 321–341.

[11] CSÁKI, E., CSÖRGŐ, M., FÖLDES, A. AND RÉVÉSZ, P. (2016). Some limit theorems for heights of random walks on a spider. *J. Theoret. Probab.* **29** 1685–1709.

[12] CSÁKI, E. AND FÖLDES, A. (1998). On asymptotic independence of partial sums. In: Szyszkowicz, B. (ed.) *Asymptotic Methods in Probability and Statistics.* A Volume in Honor of Miklós Csörgő, Elsevier, Amsterdam, pp. 373–381.

[13] CSÁKI, E., FÖLDES, A. AND RÉVÉSZ, P. (1997). Strassen theorems for a class of iterated processes. *Trans. Amer. Math. Soc.* **349** 1153–1167.

[14] CSÁKI, E. AND SALMINEN, P. (1996). On additive functionals of diffusion processes. *Studia Sci. Math. Hungar.* **31** 47–62

[15] CSÖRGŐ, M. AND HORVÁTH, L. (1989). On best possible approximations of local time. *Statist. Probab. Lett.* **8** 301–306

[16] DOBRUSHIN, R.L. (1955). Two limit theorems for the simplest random walk on a line (in Russian). *Uspehi Mat. Nauk* (N.S.) **10**, (3)(65), 139–146

[17] EVANS, S.N. (2000). Snakes and spiders: Brownian motion on $\mathbb{R}$-trees. *Probab. Theory Relat. Fields* **117** 361–386

[18] GAIRAT, A. AND SHCHERBAKOV, V. (2016). Density of skew Brownian motion and its functionals with application in finance. *Mathematical Finance* DOI: 10.1111/mafi.12120

[19] Hajri, H. (2012). Discrete approximations to solution flows of Tanaka's SDE related to Walsh Brownian motion. *Sém. Prob. XXIII, Lect. Notes in Math.,* **2046**, Springer, Berlin, Heidelberg, New York 167–190

[20] Harrison, J. and Shepp, L. (1981). On skew Brownian motion. *Ann. Probab.* **9** 309–313

[21] Hu, Y., Pierre-Loti-Viaud, D. and Shi, Z. (1995). Laws of the iterated logarithm for iterated Wiener processes. *J. Theor. Probab.* **8** 303–319

[22] Itô, K. and McKean, H. (1974). *Diffusion and Their Sample Paths*, 2nd ed., Springer, Berlin, Heidelberg, New York.

[23] Kesten, H. (1965). An iterated logarithm law for the local time. *Duke Math. J.* **32** 447–456

[24] Komlós, M., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent r.v.'s and sample df. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131

[25] Lamperti, J. (1958). An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.* **88** 380–387

[26] Lejay, A. (2006). On the constructions of the skew Brownian motion. *Probab. Surveys* **3** 413–466

[27] Lejay, A. (2016). Estimation of the biais parameter of the Skew Random Walk and application to the Skew Brownian Motion. <hal-01319319>

[28] Lyulko, Ya.A. (2012). On the distribution of time spent by a Markov chain at different levels until achieving a fixed state. *Theory Probab. Appl.* **56** 140–149

[29] Petrov, V.V. (1975). *Sums of Independent Random Variables.* Springer, Berlin.

[30] Révész, P. (2013). *Random Walk in Random and Non-Random Environment*, 3d ed., World Scientific, Singapore.

[31] Revuz, D. and Yor, M. (1991). *Continuous Martingales and Brownian Motion*, Springer, Berlin.

[32] Spitzer, F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.

[33] Walsh, J.B. (1978). A diffusion with discontinuous local time. *Astérisque* **52-53** 37–45

[34] Watanabe, S. (1995). Generalized arc-sine laws for one-dimensional diffusion processes and random walks. Proc. Sympos. Pure Math. 57, Stoch. Analysis, Cornell University (1993), Amer. Math. Soc., 157–172.

[35] Yano, Y. On the joint law of the occupation times for a diffusion process on multiray. *J. Theoret. Probab.* DOI 10.1007/s10959-015-0654-4