

RANDOM BASES FOR COPRIME LINEAR GROUPS

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ABSTRACT. The minimal base size $b(G)$ for a permutation group G , is a widely studied topic in the permutation group theory. Z. Halasi and K. Podoski [8] proved that $b(G) \leq 2$ for coprime linear groups. Motivated by this result and the probabilistic method used by T. Burness, M.W. Liebeck and A. Shalev, it was asked by L. Pyber [19] that for coprime linear groups $G \leq GL(V)$, whether there exists a constant c such that the probability of that a random c -tuple is a base for G tends to 1 as $|V| \rightarrow \infty$. While the answer to this question is negative in general, it is positive under the additional assumption that G is even primitive as a linear group. In this paper, we show that almost all 11-tuples are bases for coprime primitive linear groups.

1. INTRODUCTION

For a finite permutation group G , a subset B of Ω is called a base for G , if the pointwise stabilizer of B in G is trivial. The concept of base plays a fundamental role in the development of the algorithms for permutation groups and these algorithms are significantly faster if the size of the base is small (see the book of Á. Seress [22]). The minimal size of a base for G acting on Ω is denoted by $b(G)$. L. Pyber ([18]) showed that there exists a universal constant $c > 0$ such that almost all subgroups G of $\text{Sym}(n)$ satisfy that $b(G) > cn$.

On the other hand, there are several important group classes where the minimal base size $b(G)$ can be bounded by a fixed constant c . Á. Seress [21] showed that $b(G) \leq 4$ for a solvable primitive group G . For an almost simple primitive permutation group G , a famous conjecture of P. J. Cameron and W. M. Kantor [2] states that there exists an absolute constant such that $b(G) \leq c$ for all non-standard primitive permutation groups G . In [3], P. J. Cameron suggested that c can be chosen to 6 apart from the Mathieu group M_{24} with its natural action, where the minimal base size is 7. The Cameron–Kantor conjecture was proved by M. W. Liebeck and A. Shalev in [15]. However, this was an existence result for c , using probabilistic method without yielding any explicit value for this constant. Finally, T. C. Burness, M. W. Liebeck and A. Shalev [1] proved that if G is a finite almost simple group in a primitive faithful non-standard action, then $b(G) \leq 7$, with equality if and only if G is the Mathieu group M_{24} in its natural action of degree 24.

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Furthermore, they proved that if G is a finite almost simple group, and Ω is a primitive faithful non-standard G -set then the probability that a random 6-tuple in Ω is a base for G tends to 1 as $|\Omega| \rightarrow \infty$. For a finite vector space V , a linear group $G \leq GL(V)$ is called coprime, if $(|G|, |V|) = 1$. D. Gluck and K. Magaard [7] proved that for such a group the minimal base size of G is bounded by an absolute constant that is $b(G) \leq 94$. Z. Halasi and K. Podoski [8] improved this result by showing that $b(G) \leq 2$ and this estimation is sharp. Based on this result and the random base result of T. C. Burness, M. W. Liebeck and A. Shalev, L. Pyber [19] asked whether for a coprime linear group $G \leq GL(V)$ there exists an absolute constant c such that a random c -tuple in V is a base for G tends to 1 as $|V| \rightarrow \infty$. We answer this question affirmatively by showing that

Theorem 1.1. *Let V be a finite vector space and $G \leq GL(V)$ be a coprime primitive linear group, i.e. $(|G|, |V|) = 1$. Then the probability that a random 11-tuple in V is a base for G tends to 1 as $|V| \rightarrow \infty$.*

In fact, we also give lower bounds for this probability in terms of the base field and the dimension of V . Our bounds highly depends on the structure of G . As a part of our argument, we give a general structure theorem for maximal coprime primitive linear groups in Theorem 4.1.

For any positive integer c let us define the probability

$$Pb(c, G, V) := P(\text{random } v_1, \dots, v_c \in V \text{ is a base for } G).$$

The main goal of this paper is to prove the following

Theorem 1.2. *Let V be an n -dimensional vector space over the finite field \mathbb{F}_q and let $G \leq GL(V)$ be a coprime primitive linear group. Then for any $c \geq 11$, the probability $Pb(c, G, V)$ is close to zero if $|V|$ is large enough. More precisely, one of the following holds.*

$$(1) \quad Pb(c, G, V) \geq 1 - \frac{3}{q^{(\frac{c}{2}-5)\sqrt{n}}};$$

(2) *There is an \mathbb{F}_q^k vector space structure on V for some field extension $\mathbb{F}_q^k \geq \mathbb{F}_q$ (possibly $k = 1$) and a tensor product decomposition $V = V_1 \otimes_{\mathbb{F}_q^k} U$ over \mathbb{F}_q^k with $1 \leq \dim(U) < \dim(V_1) \leq n/k$ such that $G \leq \Gamma L(\mathbb{F}_q^k, n/k)$ and $H = G \cap GL((\mathbb{F}_q^k, n/k)$ preserves this tensor product decomposition. Furthermore, $H = H_1 \otimes H_2$ with $H_1 \leq GL(V_1)$, $H_2 \leq GL(U)$ are absolutely irreducible linear groups, and $S_1 = \text{Soc}(H_1/Z(H_1))$ is a non-Abelian simple group.*

(a) *If S_1 is not an alternating group, then*

$$Pb(c, G, V) \geq 1 - \left(\frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{2}{|V|^{(c-10)/80}} \right);$$

(b) *If $S_1 \simeq A_m$ for some m and V_1 is not an irreducible component of the natural permutation $\mathbb{F}_q^k A_m$ -module, then*

$$Pb(c, G, V) \geq 1 - \frac{3}{q^{\frac{c-10}{16}\sqrt{\dim(V)}}};$$

(c) *If $S_1 \simeq A_m$ for some m and V_1 is the non-trivial irreducible component of the natural permutation $\mathbb{F}_q^k A_m$ -module, then*

$$Pb(c, G, V) \geq 1 - \frac{3}{n^{c-2}}.$$

Remark 1.3.

- (1) Let $Z = Z(GL(V)) \simeq \mathbb{F}_q^\times$ denote the group of scalar transformations on V . If $G \leq GL(V)$ is a coprime linear group on V , then so is $GZ \geq G$ and we have $Pb(c, G, V) \geq Pb(c, GZ, V)$. Therefore, for the rest of this paper we will always assume that G contains Z .
- (2) The assumption ‘‘primitive’’ is necessary here. To see this, let $H \leq GL(n, q)$ be the group of all invertible diagonal matrices, so $H \simeq (\mathbb{F}_q^\times)^n$. Then $v_1, \dots, v_c \in \mathbb{F}_q^n$ is a base for H if and only if for each $1 \leq i \leq n$ the i -th component of some v_j is non-zero. For any fixed i , this has probability $(1 - 1/q^c)$, so we have

$$Pb(c, H, \mathbb{F}_q^n) = \left(1 - \frac{1}{q^c}\right)^n,$$

which is close to zero for any fixed c and big enough n . If $(q, n) = 1$, then one can add the regular permutation action of C_n on the components of \mathbb{F}_q^n to get the coprime irreducible linear group $G = H \rtimes C_n \leq GL(n, q)$ satisfying $\lim_{n \rightarrow \infty} Pb(c, G, \mathbb{F}_q^n) = 0$.

2. BOUNDS ON $Pb(c, G, V)$ IN TERMS OF SUPPORTS AND CHARACTER RATIOS

In order to prove Theorem 1.2, our primary tool will be the concept of support for elements of a linear group.

Definition 2.1. For a linear group $G \leq GL(V)$ and a $g \in G$ the *fixed-point space* and the *support* of g are defined as

$$\text{Fix}(g) := \{v \in V \mid g(v) = v\} \quad \text{and} \quad \text{Supp}(g) := \dim(V) - \dim(\text{Fix}_V(g)).$$

Furthermore, let the *minimum support* of G be defined as

$$\text{MinSupp}(G) := \min_{1 \neq g \in G} \text{Supp}(g).$$

We use the notation $\text{Fix}_V(g)$, $\text{Supp}_V(g)$ and $\text{MinSupp}_V(G)$ if we also want to highlight the vector space on which the group acts.

Remark 2.2. If G strictly contains Z , then $\text{MinSupp}(G)$ equals

$$\min_{g \in G \setminus Z} \left(\dim(V) - \max_{\lambda \in \mathbb{F}_q^\times} (\dim(\ker(g - \lambda \cdot \text{id}_V))) \right).$$

In order to get bounds for $\text{MinSupp}_V(G)$ in case of $G \leq GL(V)$ is a quasisimple coprime linear group, we will use results from character ratios of complex irreducible characters of such groups.

Definition 2.3. For a finite group G and $\chi \in \text{Irr}(G)$ with $\chi(1) \neq 1$ let us define the *maximal character ratios*

$$\text{mr}(G, \chi) := \max_{g \notin Z(\chi)} \frac{|\chi(g)|}{\chi(1)} \quad \text{and} \quad \text{mr}(G) := \max_{\chi \in \text{Irr}(G), \chi(1) \neq 1} \text{mr}(G, \chi).$$

Clearly, $\text{mr}(G) < 1$ for every finite group G .

The connection between minimal support and maximal character ratio is described in the following Lemma.

Lemma 2.4. *Let V be an n -dimensional vector space over the finite field \mathbb{F}_q and let $G \leq GL(V)$ be a non-Abelian coprime irreducible linear group. Then we have*

$$\text{MinSupp}_V(G) \geq \frac{\dim(V)}{2} (1 - \text{mr}(G)).$$

Moreover, if $\chi \in \text{Irr}(G)$ is any irreducible component of the Brauer character associated to V , then

$$\text{MinSupp}_V(G) \geq \frac{1}{2} \left(\chi(1) - \max_{g \notin Z(\chi)} |\chi(g)| \right).$$

Proof. Let $\overline{\mathbb{F}}_q$ be the algebraic closure of \mathbb{F}_q and let $\overline{V} = V \otimes \overline{\mathbb{F}}_q$ be the $\overline{\mathbb{F}}_q G$ -module arising from the $\mathbb{F}_q G$ -module V . Let $\overline{V} = V_1 \oplus \dots \oplus V_t$ be the decomposition of \overline{V} into irreducible $\overline{\mathbb{F}}_q G$ -modules. Then the corresponding representations $G \mapsto GL(V_i)$ form a single Galois conjugacy class by [10, Theorem 9.21], so $\text{Supp}_{V_i}(g) = \frac{1}{t} \text{Supp}_V(g)$ holds for every $g \in G$. Let $\chi_i : G \mapsto \mathbb{C}$ be the irreducible Brauer character associated to V_i for each $1 \leq i \leq t$. Since $(q, |G|) = 1$, we get $\chi_i \in \text{Irr}(G)$ by [10, Theorem 15.13]. Furthermore,

$$\chi_i(1) = \dim(V_i) \quad \text{and} \quad [\chi_i \langle g \rangle, 1_{\langle g \rangle}] = \dim(\text{Fix}_{V_i}(g)).$$

For any $g \in G$ we have $\chi_1(g) = (\chi_1(1) - k) \cdot 1 + \varepsilon_1 + \dots + \varepsilon_k$ where $k = \text{Supp}_{V_1}(g)$ and $\varepsilon_1, \dots, \varepsilon_k$ are $o(g)$ -th root of unity. Then $|\chi_1(g)| \geq \chi_1(1) - 2k = \chi_1(1) - 2 \text{Supp}_{V_1}(g)$ holds, so $2 \text{MinSupp}_{V_1}(G) \geq \chi_1(1) - \max_{g \notin Z(\chi_1)} |\chi_1(g)|$. (Note that the assumption that G is non-Abelian implies that the χ_i are non-linear characters. Furthermore, if $1 \neq g \in Z(\chi_1)$, then $\text{Supp}_V(g) = \dim(V)$, so $\text{MinSupp}_V(G) = \min_{g \notin Z(\chi_1)} \text{Supp}_V(g)$ must hold.)

It follows that

$$\begin{aligned} 2 \text{MinSupp}_V(G) &= 2t \text{MinSupp}_{V_1}(G) \geq t(\chi_1(1) - \max_{g \notin Z(\chi_1)} |\chi_1(g)|) \\ &= t\chi_1(1)(1 - \text{mr}(G, \chi_1)) \geq \dim(V)(1 - \text{mr}(G)). \end{aligned}$$

Now, the first inequality proves the second claim, while the second inequality proves the first claim. \square

Lemma 2.5.

$$Pb(c, G, V) \geq 1 - \sum_{1 \neq g \in G} \frac{1}{q^{c \cdot \text{Supp}(g)}} \geq 1 - \frac{|G|}{q^{c \cdot \text{MinSupp}(G)}} \geq 1 - \frac{1}{|V|^{c(1 - \text{mr}(G))/2 - 2}}.$$

In particular, $Pb(c, G, V) \geq 1 - \frac{1}{|V|^c}$ for $c \geq \frac{4+2\varepsilon}{1 - \text{mr}(G)}$.

Proof.

$$\begin{aligned} P(\{v_1, \dots, v_c\} \subseteq V \text{ is not a base for } G) &\leq \sum_{1 \neq g \in G} P(g(v_i) = v_i, \forall 1 \leq i \leq c) \\ &= \sum_{1 \neq g \in G} \left(\frac{|\text{Fix}(g)|}{|V|} \right)^c = \sum_{1 \neq g \in G} \frac{1}{q^{c \cdot \text{Supp}(g)}} \leq \frac{|G|}{q^{c \cdot \text{MinSupp}(G)}} \\ &\leq \frac{|V|^2}{(q^n)^{c(1 - \text{mr}(G))/2}} = \frac{1}{|V|^{c(1 - \text{mr}(G))/2 - 2}}, \end{aligned}$$

and the claim follows. \square

3. BOUNDS FOR CHARACTER RATIOS AND FOR MINIMAL SUPPORTS OF QUASISIMPLE LINEAR GROUPS

The goal of this section is to give lower bounds for minimal supports of coprime quasisimple groups $G \leq GL(V)$ in terms of $|G|$ and $\dim(V)$.

First we handle the case when G is a sporadic group or a finite quasisimple group of Lie type. For such groups, we use bounds for their maximal character ratios $\text{mr}(G)$.

Theorem 3.1. *Let G be a finite quasisimple group such that $G/Z(G)$ is not an alternating group.*

- (1) *If $G/Z(G)$ is a sporadic simple group, then $\text{mr}(G) < 0.54$.*
- (2) *If $G = G(r)$ is a finite quasisimple group of Lie type over the field \mathbb{F}_r , then*

$$\text{mr}(G) \leq \begin{cases} \max\left(\frac{1}{\sqrt{r-1}}, \frac{9}{r}\right) & \text{if } r > 9; \\ \frac{19}{20} & \text{if } r \leq 9. \end{cases}$$

Proof. We checked part (1) for the covering groups of the sporadic simple groups by using the GAP [4] Character table library and also the undeposited GAP package FUtil to turn cyclotomic complex numbers into floating ones in order to be able to compare the values of $|\chi(g)|$ for various g and χ .

Regarding part (2), it is a simplified version of a result of Gluck [6]. (For a summary of his results, see also [14, Theorem 2.4]). \square

Remark 3.2. For simple groups of alternating type there is no general upper bound for $\text{mr}(G)$ smaller than 1. Moreover it can be shown that for every $\varepsilon > 0$, the number of irreducible characters $\chi \in \text{Irr}(S_m)$ (or $\chi \in \text{Irr}(A_m)$) satisfying $\text{mr}(S_m, \chi) > 1 - \varepsilon$ is not bounded if m is large enough.

Corollary 3.3. *Let V be a vector space over the finite field \mathbb{F}_q and let $G = Z \cdot G_0 \leq GL(V)$ where G_0 is a coprime quasisimple irreducible linear group which is not of alternating type. Then $\text{MinSupp}_V(G) \geq \frac{1}{40} \dim(V)$.*

Proof. By Theorem 3.1, we have $\text{mr}(G) = \text{mr}(G_0) \leq \frac{19}{20}$, so the claim follows from Lemma 2.4. \square

Now, we handle the case when $\text{Soc}(G/Z(G))$ is an alternating group.

Theorem 3.4. *Let $G = S_m$ and $\chi = \chi^{(\lambda)} \in \text{Irr}(G)$ corresponding to the partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ of $[m]$. Then $\chi^\lambda(1) - \chi^\lambda((123)) \geq \frac{1}{m-1} \chi^\lambda(1)$ unless*

$$\lambda \in \{(m); (1, \dots, 1)\}.$$

Proof. First, we introduce some notation. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ be a partition of m , different from the two exceptional ones given in the theorem. For any natural numbers i_1, \dots, i_k let $\chi^{\lambda - \{i_1, \dots, i_k\}}$ be the character of S_{m-k} corresponding to the Young diagram obtained from the diagram of λ by deleting the last cells of the i_1 -th, \dots , i_k -th row in that order with the assumption that $\lambda - \{i_1, \dots, i_s\}$ is a valid Young diagram for each $1 \leq s \leq k$. Otherwise, we define $\chi^{\lambda - \{i_1, \dots, i_k\}}$ as the constant zero function on S_{m-k} .

By the Murnaghan-Nakayama rule (see [11, 21.1]),

$$\begin{aligned}\chi^\lambda((123)) &\leq \sum_{\nu \in \{\lambda - rh(3)\}} \chi^\nu(1) + \sum_{\nu \in \{\lambda - rh(1,1,1)\}} \chi^\nu(1) \\ &= \sum_{\nu \in \{\lambda - rh(3)\}} \chi^\nu(1) + \sum_{\nu \in \{\bar{\lambda} - rh(3)\}} \chi^\nu(1)\end{aligned}$$

where $\{\lambda - rh(*)\}$ denotes the set of partitions of $m - 3$ which we can get from the Young-diagram of λ by removing a rim 3-hook of type $(*)$ such that the remaining cells form a valid Young diagram.

On the other hand, by using the branching rule (three times) one gets

$$\chi^\lambda(1) = \sum_{i,j,k} \chi^{\lambda - \{i,j,k\}}(1).$$

Let $\nu \in \{\lambda - rh(3)\}$. Then $\nu = \lambda - \{i, i, i\}$ for some (unique) i . Now, there is a $j \neq i$ such that $\tau = \lambda - \{i, i, j\}$ is a valid Young diagram. Then both induced characters $(\chi^\tau)^{S_{m-2}}$ and $(\chi^\nu)^{S_{m-2}}$ contain $\chi^{\lambda - \{i, i, i\}}$ as a component which results $\chi^\nu(1) \leq \chi^{\lambda - \{i, i, i\}}(1) \leq (m - 2)\chi^\tau(1)$. The same argument can be applied to any $\nu \in \{\bar{\lambda} - rh(3)\}$. It follows that

$$\begin{aligned}\chi^\lambda(1) &\geq \sum_i \chi^{\lambda - \{i, i, i\}}(1) \left(1 + \frac{1}{m-2}\right) + \sum_i \chi^{\bar{\lambda} - \{i, i, i\}}(1) \left(1 + \frac{1}{m-2}\right) \\ &\geq \frac{m-1}{m-2} \chi^\lambda((123)).\end{aligned}$$

Hence $\chi^\lambda(1) - \chi^\lambda((123)) \geq \frac{1}{m-1} \chi^\lambda(1)$ which proves the claim. \square

This result will be adequate for our purposes only if the degree of χ is large enough. In order to get an overall picture about the form of Young diagrams defining characters of small degree, we will use a result of Rasala [20]. In what follows, we use the terminology from Rasala's paper. For any partition λ of m , let $|\lambda| = m$ be the order of λ and let λ^* be the partition dual to λ . The partition λ is called primary, if $\lambda \geq \lambda^*$, where \geq denotes the standard ordering on partitions. If $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ is a partition of k and $m \geq \lambda_1 + k$, then let m/λ denote the partition of m defined as $m/\lambda = (m - k \geq \lambda_1 \geq \dots \geq \lambda_k)$ and let $\varphi_\lambda(m) := \chi^{m/\lambda}(1)$ be the degree of the character of S_m associated to m/λ . (Note that $\varphi_\lambda(m)$ is a polynomial in m by [20, Theorem A].) For any set P of partitions of k and for m large enough, let $L(P, m) := \{\varphi_\lambda(m) \mid \lambda \in P\}$ and let $\delta(P, m)$ be the largest degree in $L(P, m)$. Then P is said to be m -minimal, if for every primary partition μ of m either $\chi^\mu(1) > \delta(L, P)$ or $\mu = m/\lambda$ for some $\lambda \in P$.

By [20, Main Theorem 1.] (for $k = 3$) we have

Theorem 3.5. *Let P_3 be the set of all partitions of order at most 3, that is, $P_3 = \{\emptyset; (1); (2); (1, 1); (3); (2, 1); (1, 1, 1)\}$. Then P_3 is m -minimal for every $m \geq 15$.*

Thus, by using the hook length formula and the Murnaghan-Nakayama rule we can calculate the exact values of $\chi^\lambda(1)$ and $\chi^\lambda((123))$ when $\chi^\lambda(1)$ is among the first seven smallest character degrees of S_m for $m \geq 15$. Otherwise, we get a reasonably large lower bound for $\chi^\lambda(1)$. (Note that λ or λ^* is primary and $\chi^\lambda(1) = \chi^{\lambda^*}(1)$, $\chi^\lambda((123)) = \chi^{\lambda^*}((123))$ holds for every partition λ of m .)

Corollary 3.6. *Let λ be a partition of m for $m \geq 15$ and let $\chi^\lambda \in \text{Irr}(S_m)$ be the character of S_m associated to λ . Then $\chi^\lambda(1)$ and $\chi^\lambda((123))$ are as given in Table 1 or $\chi^\lambda(1) > \frac{1}{3}m(m-2)(m-4)$.*

| λ or λ^* | $\chi^\lambda(1) = \chi^{\lambda^*}(1)$ | $\chi^\lambda((123)) = \chi^{\lambda^*}((123))$ |
|--------------------------|---|---|
| (m) | 1 | 1 |
| $(m-1, 1)$ | $m-1$ | $m-4$ |
| $(m-2, 2)$ | $\frac{1}{2}m(m-3)$ | $\frac{1}{2}(m-3)(m-6)$ |
| $(m-2, 1, 1)$ | $\frac{1}{2}(m-1)(m-2)$ | $\frac{1}{2}(m-4)(m-5)$ |
| $(m-3, 3)$ | $\frac{1}{6}m(m-1)(m-5)$ | $\frac{1}{6}(m-3)(m-4)(m-8) + 1$ |
| $(m-3, 2, 1)$ | $\frac{1}{3}m(m-2)(m-4)$ | $\frac{1}{3}(m-3)(m-5)(m-7) - 1$ |
| $(m-3, 1, 1, 1)$ | $\frac{1}{6}(m-1)(m-2)(m-3)$ | $\frac{1}{6}(m-4)(m-5)(m-6) + 1$ |

TABLE 1. Character values of S_m when the degree is small.

Now, we give an analogue of Corollary 3.3 for alternating-type groups.

Corollary 3.7. *Let V be a vector space over the finite field \mathbb{F}_q and let $G = Z \cdot G_0 \leq GL(V)$ where G_0 is a coprime irreducible linear group and $G_0/Z(G_0) \simeq A_m$ for some $m \geq 5$. Let us assume that V is not a component of the natural permutation $\mathbb{F}_q A_m$ -module. Then $\text{MinSupp}_V(G) \geq \frac{1}{16}\sqrt{\dim(V)}$.*

Proof. As in the proof of Lemma 2.4, $\text{MinSupp}_V(G) = t \cdot \text{MinSupp}_{V_1}(G)$ and $\dim(V) = t \cdot \dim(V_1)$ where V_1 is an (absolutely) irreducible component of $\overline{\mathbb{F}_q}G$ -module $V \otimes \overline{\mathbb{F}_q}$. Then the claim clearly follows if we prove that $\text{MinSupp}_{V_1}(G) \geq \frac{1}{16}\sqrt{\dim(V_1)}$. In other words, we can assume that V is absolutely irreducible. First let us assume that $G_0 \simeq A_m$ for some $m \geq 9$. Let $\varphi \in \text{Irr}(A_m)$ be the Brauer character associated to V and $\chi \in \text{Irr}(S_m)$ above φ , i.e. $[\chi_{A_m}, \varphi] \neq 0$. Then either $\chi_{A_m} = \varphi$ (if χ is not self-dual) or $\chi_{A_m} = \varphi + \varphi^{(12)}$ (if χ is self-dual). In the latter case $\varphi((123)) = \chi((123))/2$, since the conjugacy class $(123)^{S_m}$ does not split in A_m . Let ϵ be 1 or $1/2$ according to these cases, so $\varphi(1) = \epsilon\chi(1)$ and $\varphi((123)) = \epsilon\chi((123))$. By [20, Result 2.], we have $\dim(V) = \epsilon\chi(1) \geq \frac{1}{2}m(m-3)$. If $\varphi((123)) < 0$, then $\text{Supp}_V((123)) \geq \frac{1}{2}\dim(V) \geq \frac{1}{4}\sqrt{\dim(V)}$ holds trivially. Otherwise, by using Lemma 2.4 and Theorem 3.4 we get that

$$\begin{aligned} \text{Supp}_V((123)) &\geq \frac{1}{2}(\varphi(1) - |\varphi((123))|) = \frac{\epsilon}{2}(\chi(1) - \chi((123))) = \frac{\epsilon\chi(1)}{2(m-1)} \\ &= \frac{\dim(V)}{2(m-1)} \geq \frac{\sqrt{m(m-3)}/2\sqrt{\dim(V)}}{2(m-1)} \geq \frac{1}{4}\sqrt{\dim(V)}. \end{aligned}$$

For any element $1 \neq g \in A_m$ there are $x, y \in A_m$ such that $[g, x, y]$ is a three-cycle. Applying Lemma 4.2 twice, we get that $\text{Supp}_V(g) \geq \frac{1}{4}\text{Supp}_V((123)) \geq \frac{1}{16}\sqrt{\dim(V)}$.

Now, let us assume that $m > 7$ and G_0 is the universal covering group of A_m , so $G_0 \simeq 2.A_m$. Let $z \in G_0$ be the generator of $Z(G_0) \simeq C_2$ and let $\bar{g} \in A_m$ denote

the image of any $g \in G_0$ under the natural surjection by $G_0 \mapsto A_m$. Then z acts on V as a scalar transformation $z(v) = -v$ for all $v \in V$, so $\text{Supp}_V(z) = \dim(V)$. Let $t \in G_0$ such that $\bar{t} = (12)(34)$. By Theorem [9, Theorem 3.9], t and tz are conjugate, so $z = [h, t]$ for some $h \in G_0$. It follows that $\text{Supp}_V(t) \geq \frac{1}{2} \text{Supp}_V(z) = \frac{\dim(V)}{2}$ by Lemma 4.2. (In fact, by using this argument to tz instead of t one can prove equality here.) Now, for any $g \in G_0 \setminus Z$ one can choose $x, y \in G$ such that $[\bar{g}, \bar{x}, \bar{y}]$ is conjugate to \bar{t} . Using again Lemma 4.2 twice, we get that $\text{Supp}_V(g) \geq \frac{1}{4} \text{Supp}_V(t) = \frac{1}{8} \dim(V) \geq \frac{1}{16} \sqrt{\dim(V)}$.

For the remaining cases, $\dim(V) \leq \sqrt{|G_0|} < 16^2$, so $\frac{1}{16} \sqrt{\dim(V)} < 1 \leq \text{MinSupp}_V(G)$ follows. \square

The next result gives a bound to the order of most coprime quasisimple linear groups similar to that of $|G| \leq |V|^2 = q^{2 \dim(V)}$ but using the minimal support $\text{MinSupp}_V(G)$ instead of $\dim(V)$.

Theorem 3.8. *Let V be a vector space over the finite field \mathbb{F}_q and let $G = Z \cdot G_0 \leq GL(V)$ where G_0 is a coprime quasisimple irreducible linear group.*

Then one of the following holds:

- (1) $\log_q |G| \leq d \cdot \text{MinSupp}_V(G)$ with $d = 5$.
- (2) $G_0 \simeq A_m$ and V is the non-trivial irreducible component of the natural permutation module of A_m over \mathbb{F}_q .
- (3) $G_0 = G_0(r)$ is a finite quasisimple group of Lie type over the finite field \mathbb{F}_r with $r \leq 43$, and $|V|$ is bounded by an absolute constant.

Proof. For any sporadic group S , let \widehat{S} be its universal covering group and let $q(S)$ be the smallest prime not dividing the order of S . By using GAP [4], we checked that for every $\chi \in \text{Irr}(\widehat{S})$, the inequality

$$\log_{q(S)} |\widehat{S}| < d \cdot (\chi(1) - \max_{g \in \widehat{S} - Z(\chi)} |\chi(g)|) / 2$$

holds with $d > 4.22$. (The largest value is attained for $2.J_2$.) Now, if $G \leq GL(V)$ is any finite quasisimple group with sporadic simple quotient $S = G/Z(G)$, then G is a homomorphic image of \widehat{S} , and we can view V as an irreducible $\mathbb{F}_q \widehat{S}$ -module (where $q \geq q(S)$). Now, if $\chi \in \text{Irr}(\widehat{S})$ is any irreducible component of the Brauer character corresponding to $V \otimes \overline{\mathbb{F}_q}$, then

$$\begin{aligned} \log_q |G| &\leq \log_{q(S)} |\widehat{S}| < d \cdot (\chi(1) - \max_{g \in \widehat{S} - Z(\chi)} |\chi(g)|) / 2 \\ &\leq d \cdot \text{MinSupp}_V(\widehat{S}) \leq d \cdot \text{MinSupp}_V(G) \end{aligned}$$

also holds with $d > 4.22$ by Lemma 2.4.

Next, let $G \simeq A_m$ for some $m \geq 15$. Then we have $m < q$ by the coprime assumption. Let us assume that V is not a component of the natural permutation $\mathbb{F}_q A_m$ -module. Let $\varphi \in \text{Irr}(A_m)$ be an irreducible component of the Brauer character associated to V and $\chi \in \text{Irr}(S_m)$ above φ . By the proof of Corollary 3.7, we have $\varphi(1) = \varepsilon \chi(1)$, and $\varphi((123)) = \varepsilon \chi((123))$, where ε is $1/2$ or 1 if χ is self-dual or not.

If χ is one of the characters given in Table 1, then χ is not self-dual. In that case we have

$$\text{Supp}_V((123)) \geq \frac{1}{2} (\chi(1) - |\chi(123)|) \geq \frac{3}{2} (m - 3)$$

by using Lemma 2.4 and the last five rows of Table 1. Otherwise, $\chi(1) > \frac{1}{3}m(m-2)(m-4)$, so we have

$$\begin{aligned} \text{Supp}_V((123)) &\geq \frac{1}{2}(\varphi(1) - |\varphi(123)|) \geq \frac{1}{4}(\chi(1) - |\chi(123)|) \\ &\geq \frac{\chi(1)}{4(m-1)} > \frac{m(m-2)(m-4)}{12(m-1)} \geq m-3 \end{aligned}$$

holds if $\varphi(123) \geq 0$. However, if $\varphi(123) < 0$, then $\text{Supp}_V((123)) \geq \frac{1}{2} \dim(V) \geq m-3$ holds trivially. Thus, $\text{Supp}_V((123)) \geq m-3$ holds in any case. Now, for any element $1 \neq g \in A_m$ there are $x, y \in A_m$ such that $[g, x, y]$ is a three-cycle. Applying Lemma 4.2 twice, we get that $\text{Supp}_V(g) \geq \frac{1}{4} \text{Supp}_V((123)) \geq \frac{m-3}{4}$ holds for any $1 \neq g \in A_m$. Thus, $d \cdot \text{MinSupp}_V(G) \geq \frac{d(m-3)}{4} \geq m \geq \log_m(m!) \geq \log_q |G|$ holds for $d \geq 5$.

Now, let $m \geq 12$ and let G_0 be the universal covering group of A_m , so $G_0 \simeq 2.A_m$. By the proof of Corollary 3.7, we have $\text{MinSupp}_V(G) \geq \frac{1}{8} \dim(V)$. Using [13, Main Theorem] we get that

$$\begin{aligned} d \cdot \text{MinSupp}_V(G) &\geq \frac{d}{8} \dim(V) \geq \frac{d}{8} \min\{\chi(1) \mid \chi \in \text{Irr}(G), \chi(z) \neq \chi(1)\} \\ &\geq d \cdot 2^{\lfloor m/2 \rfloor - 4} \geq m \geq \log_q |G| \end{aligned}$$

holds for $d \geq 3.25$. For the remaining members of Alternating groups and their covers (i.e for A_m , $12 \leq m \leq 14$), for $2.A_m$ ($m = 5$ or $8 \leq m \leq 11$) and for $6.A_6, 6.A_7$ we used the same algorithm as for sporadic groups.

Finally, let $G_0 = G_0(r)$ be a quasisimple group of Lie type over a finite field \mathbb{F}_r with $(r, q) = 1$. First, suppose that $r \geq 47$. By part (2) of Theorem 3.1, we have

$$\text{mr}(G) \leq \max\left(\frac{1}{\sqrt{r}-1}, \frac{9}{r}\right) < \frac{1}{5}.$$

By using [17, Theorem 1.] and Lemma 2.4,

$$\log_q |G| \leq 2n = 5 \cdot \frac{2n}{5} \leq 5 \cdot \text{MinSupp}_V(G)$$

For the rest of the proof, suppose that $r \leq 43$. Since $\chi(1) - 2 \text{Supp}(g) \leq |\chi(g)|$ for any $\chi \in \text{Irr}(G)$, we have that

$$\frac{1}{2} \chi(1) \left(1 - \frac{|\chi(g)|}{\chi(1)}\right) \leq \text{Supp}(g).$$

Using that $\text{mr}(G) \leq \frac{19}{20}$ also holds for all quasisimple groups of Lie-type by part (2) of Theorem 3.1, we obtain that

$$\frac{\dim(V)}{8} \leq 5 \text{MinSupp}_V(G)$$

by using Lemma 2.4 again. Since (see [12, Table 5.3.A]) $\dim(V) \geq r^{\mathcal{O}(m)}$ (where m denotes the rank of $G_0(r)$) and $\log_q |G| = \mathcal{O}(m^2 \log r)$, there exist only finitely many possible pairs (m, r) such that $\log_q |G| > 5 \text{MinSupp}_V(G)$. Furthermore, for any fixed (m, r) , the inequality $\log_q |G| \leq 5 \text{MinSupp}_V(G)$ still holds provided that $|V|$ is large enough. \square

We close this section by handling the case (2) in Theorem 3.8. In this case $\text{MinSupp}_V(G)$ is bounded. (It is 1 and 2 for $G_0 \simeq S_m$ and $G_0 \simeq A_m$, respectively.) Therefore, we give a direct proof for Theorem 1.2 in this case.

Theorem 3.9. *Let U be an m -dimensional vector space over \mathbb{F}_q , and let $G = S_m$ with its natural permutation action on U . Assuming that $(|G|, |U|) = 1$, we have*

$$P(\text{random } \underline{u} \in U^c \text{ is a base for } G) > 1 - \frac{1}{m^{c-2}} \text{ for any } c \geq 3.$$

Hence three random vectors form a base for G with high probability if m is large.

Proof. The $\mathbb{F}_q G$ -module V^c can be naturally identified with $M^{m \times c}(q)$, the space of $m \times c$ -matrices over \mathbb{F}_q . Under this identification, G acts on $M^{m \times c}(q)$ by permuting the rows of each element of $M^{m \times c}(q)$ in a natural way. Hence, a matrix $a \in M^{m \times c}(q)$ is a base for G if and only if the rows of a are pairwise different elements of $M^{1 \times c}(q)$, the space of c -dimensional row vectors over \mathbb{F}_q . Thus, the probability in question is equal to the probability that m random elements of $M^{1 \times c}(q)$ are pairwise different, which is

$$\prod_{i=0}^{m-1} \frac{q^c - i}{q^c} > \left(\frac{q^c - q}{q^c} \right)^m \geq \left(1 - \frac{1}{m^{c-1}} \right)^m \geq 1 - \frac{1}{m^{c-2}},$$

where the first and second inequalities follows since $m < q$ by the coprime assumption. The claim follows. \square

Corollary 3.10. *Let V be an n dimensional vector space over the finite field \mathbb{F}_q and let $G = Z \cdot G_0 \leq GL(V)$ be a coprime linear group, where $G_0 \simeq S_m$ or $G_0 \simeq A_m$ and V is the non-trivial irreducible component of the natural $\mathbb{F}_q G_0$ -module. Then we have*

$$P(\text{random } \underline{v} \in V^c \text{ is a base for } G) > 1 - \frac{1}{n^{c-2}} \text{ for any } c \geq 3.$$

Proof. First, note that $\text{Fix}(g) = 0$ for every $g \in G \setminus G_0$, so a $\underline{v} \in V^c$ is a base for G if and only if it is a base for G_0 . Second, let $U = V \oplus U_0$, where U_0 is the trivial module for G_0 . For any random vectors $u_1, \dots, u_c \in U$ let v_i be the projection of u_i to V along U_0 . Then u_1, \dots, u_c is a base for G_0 if and only if v_1, \dots, v_c is a base for G_0 , so the claim follows from Theorem 3.9. \square

4. PROOF OF THEOREM 1.2

Let V be an n -dimensional vector space over the finite field \mathbb{F}_q and let $G \leq GL(V) = GL(n, q)$ be a coprime primitive linear group, which is maximal, i.e. there is no coprime subgroup $L \leq GL(V)$ strictly containing G . In the following, we give a structure theorem of such groups very similar to a result about maximal solvable primitive linear group (see [21, Lemma 2.2] and [23, §§19–20]). Our proof uses ideas similar to those can be found in [5], [8], and [23]. For the convenience of the reader, we give a self-contained proof here.

In the following, we extend the vector space structure on V by defining multiplication on V with elements from a (possibly) larger field $\mathbb{F}_{q^k} \geq \mathbb{F}_q$ for some $k \mid n$. In that way, V will be both an \mathbb{F}_q -vector space and an \mathbb{F}_{q^k} -vector space at the same time.

We will use the notation $V = V_n(q)$, $V = V_d(q^k)$ or $V = V(q^k)$ if we would like to highlight the base field and/or the dimension of V .

Theorem 4.1. *Let $V = V_n(q)$ be an n -dimensional vector space over the finite field \mathbb{F}_q and let $G \leq GL(V)$ be a maximal coprime primitive linear group. Then the following statements hold.*

- (1) *There is a unique maximal Abelian subgroup $Z \leq GL(V)$, which is normalised by G . Moreover, Z is contained in G .*
- (2) *Z is cyclic and $Z \cup \{0\} \simeq \mathbb{F}_{q^k}$ for some $k \mid n$.*
- (3) *There is a (unique and maximal) \mathbb{F}_{q^k} vector space structure $V = V_d(q^k)$ on V for $d = n/k$ such that $G \leq \Gamma L(d, q^k)$.*
- (4) *Let $H := G \cap GL(d, q^k)$. Then $Z \leq H = C_G(Z) \triangleleft G$, furthermore $Z = Z(GL(d, q^k))$ is the group of scalar transformations on $V_d(q^k)$ and G/H is included into the Galois group $\text{Gal}(\mathbb{F}_{q^k}, \mathbb{F}_q)$.*
- (5) *Let $N = F^*(H)$ be the generalised Fitting subgroup of H . Then N/Z is the socle of H/Z . Furthermore, $V_d(q^k)$ is an absolutely irreducible $\mathbb{F}_{q^k}N$ -module.*
- (6) *Let N_1, \dots, N_t be the set of minimal normal subgroups of H above Z . Then there is an absolutely irreducible $\mathbb{F}_{q^k}N_i$ -module V_i for every i such that $V \simeq V_1 \otimes_{\mathbb{F}_{q^k}} \dots \otimes_{\mathbb{F}_{q^k}} V_t$. Furthermore, $N = N_1 \otimes N_2 \otimes \dots \otimes N_t$ and $H = H_1 \otimes H_2 \otimes \dots \otimes H_t$ where $N_i \triangleleft H_i \leq GL(V_i(q^k))$ for every i .*
- (7) *If N_i/Z is Abelian, then $N_i = ZR_i$ where $R_i \leq N_i$ is an extraspecial r_i -group for some prime r_i of order $r_i^{2l_i+1}$. Furthermore, $|N_i/Z| = r_i^{2l_i}$ and $\dim_{\mathbb{F}_{q^k}}(V_i) = r_i^{l_i}$.*
- (8) *If N_i/Z is a direct product of s many isomorphic non-Abelian simple groups, then there is a tensor product decomposition $V_i = W_1 \otimes \dots \otimes W_s$ preserved by N_i . Then $N_i = K_1 \otimes \dots \otimes K_s$ where $K_i = S_i Z$ for each i , and the $S_i \leq GL(W_i)$ are isomorphic quasisimple absolutely irreducible groups. Finally, H_i permutes the K_i -s and the W_i -s in a transitive way.*

Proof. Let $A \leq GL(V)$ be any Abelian subgroup normalised by G and P is the (unique) Sylow- p subgroup of A for $p = \text{char}(\mathbb{F}_q)$. Then P is normalised by G . Then $0 \neq \text{Fix}_V(P) = \bigcap_{p \in P} \text{Fix}_V(p) \leq V$ is G -invariant. Since V is an irreducible $\mathbb{F}_q G$ -module, we get that $P = 1$, so $|A|$ is coprime to $|V|$. Therefore, $GA \geq G$ is a coprime linear group, so $A \leq G$ by the maximality of G and part of (1) is proved.

Let $Z \leq GL(V)$ be a maximal Abelian subgroup normalised by G . By the previous paragraph, $Z \triangleleft G$. Since $G \leq GL(V)$ is primitive linear, V is a homogeneous $\mathbb{F}_q Z$ -module. If $V = V_1 \oplus \dots \oplus V_d$ is a decomposition of V into (isomorphic) irreducible $\mathbb{F}_q Z$ -modules, then $Z \simeq Z_{V_i} \leq \text{End}_Z(V_i) \simeq \mathbb{F}_{q^k}$ for some $k \geq 1$ by using Schur Lemma. Then $\langle Z \rangle_{\mathbb{F}_q}$ (the subalgebra of $\text{End}(V)$ generated by Z) is isomorphic to the field \mathbb{F}_q^k , and it is invariant under the conjugation by elements of G . It follows that $\langle Z \rangle_{\mathbb{F}_q} \setminus \{0\} \simeq \mathbb{F}_q^*$ is an Abelian subgroup of $GL(V)$ normalised by G . Therefore, (2) follows by the maximality of Z .

Identifying $Z \cup 0 \leq \text{End}(V)$ with \mathbb{F}_q^k , it defines an \mathbb{F}_{q^k} vector space structure on V . The conjugation action of G on $Z \cup \{0\} = \mathbb{F}_{q^k}$ defines a homomorphism $\sigma : G \mapsto \text{Gal}(\mathbb{F}_{q^k}, \mathbb{F}_q)$. Now, for any $g \in G$, $\alpha \in \mathbb{F}_q^k$ and $v \in V$ we have $g(\alpha v) = (g\alpha q^{-1})g(v) = \alpha^{\sigma(g)}(v)$, so G is included into the semilinear group $\Gamma L(V_d(q^k)) = \Gamma L(d, q^k)$. The subgroup H is just the kernel of σ , so (4) and part of (3) follows.

Let $B \triangleleft G$ be any Abelian normal subgroup, $\alpha \in Z$ a generator of Z and $b \in B$. Then $bab^{-1} = \alpha^{\sigma(b)} = \alpha^{q^s}$ for some $0 \leq s < k$, so $[b, \alpha] = \alpha^{q^s-1} \in B$ is centralised by b . Changing b to b^{-1} if necessary, we can assume that $0 \leq s \leq k/2$. This means $(\alpha^{q^s-1})^{q^s} = \alpha^{q^s-1}$, so $q^k - 1 \mid (q^s - 1)^2 < q^k - 1$. Therefore, $s = 0$. Thus, $B \leq C_G(Z)$, so $BZ \geq Z$ is an Abelian normal subgroup in G . By the maximality of Z , we get $B \leq Z$, which completes the proof of both (1) and (3).

Let $M = F(H)$ be the Fitting subgroup of H . Then $Z(M)$ is an Abelian normal subgroup of G , so $Z(M) = Z$ by the maximality of Z . Let n be the nilpotency class of M . If $n = 1$ then $M = Z$. Otherwise, we claim that $n = 2$. Assuming that $n \geq 3$, we have $1 \neq \gamma_n(M) \leq Z$, and $[\gamma_{n-1}(M), \gamma_{n-1}(M)] \leq [\gamma_2(M), \gamma_{n-1}(M)] \leq \gamma_{n+1}(M) = 1$, so $\gamma_{n-1}(M)$ is an Abelian normal subgroup of G , so it must be contained in Z . This forces $\gamma_n(M) = 1$, a contradiction. Therefore, $n \leq 2$, that is, M/Z is Abelian.

Let R be a Sylow- r -subgroup of M for some prime r dividing $|M/Z|$. The commutator map defines a symplectic bilinear function from R/Z into $Z(R) = R \cap Z$. Therefore, for any $x, y \in R$ we have $[x^r, y^r] = [x, y]^{r^2} = [x^{r^2}, y]$. If r^s is the exponent of $R/(R \cap Z)$ for some $s \geq 2$, then $R^{r^{s-1}}Z$ is an Abelian normal subgroup of G , so $R^{r^{s-1}} \leq Z$, a contradiction. Thus, we get $R/(R \cap Z)$ is an elementary Abelian r -group. Using this and the above commutator identity it also follows that $R' \leq Z$ is of exponent r . It follows that $R = (R \cap Z)R_0$ for some extraspecial r -group R_0 .

By the previous two paragraphs, $F(H)/Z$ is exactly the direct product of the minimal Abelian normal subgroups of H , so $F(H)/Z$ is contained in $\text{Soc}(H/Z)$. Since $N = F^*(H)$ is the central product of $F(H)$ and the layer $E(H)$, where $E(H)/Z$ is the direct product of the minimal non-Abelian normal subgroups of H/Z it follows that $N/Z = \text{Soc}(H/Z)$ as claimed. By [5, Lemma 12.1], $V_d(q^k)$ is an absolutely irreducible $\mathbb{F}_{q^k}H$ -module. If the irreducible $\mathbb{F}_{q^k}N$ -components of $V_d(q^k)$ were not be absolutely irreducible, then $Z(C_{GL(V_d(q^k))}(N))$ would be the multiplicative group of a proper field extension of \mathbb{F}_{q^k} normalised by G , which again contradicts with the maximality of Z . Now, let us assume that $V_d(q^k) = U \oplus \dots \oplus U$ is a direct sum of s many isomorphic absolutely irreducible $\mathbb{F}_{q^k}N$ -modules for some $s \geq 2$. By [12, Lemma 4.4.3(ii)], there is a tensor product decomposition $U \otimes_{\mathbb{F}_{q^k}} W$ of $V_d(q^k)$ such that $N \leq GL(U) \otimes 1_W \leq GL(U) \otimes GL(W)$ and $G \leq N_{GL(V)}(N) \leq N_{GL(V)}(GL(U) \otimes GL(W))$. Let $L = \{1_U \otimes h_W \mid \exists h_U \in GL(U) \text{ such that } h_U \otimes h_W \in H\}$. If $L = Z$, then $V_d(q^k)$ is not irreducible as an $\mathbb{F}_{q^k}H$ -module, a contradiction. We have $L \leq GL(V)$ is a coprime linear group normalised by G , so $LG \leq GL(V)$ is coprime. Using the maximality of G we get that $L < G$. But then $Z < L \leq H$ clearly centralises $N = F^*(H)$, a contradiction. So, $V_d(q^k)$ is an absolutely irreducible $\mathbb{F}_{q^k}N$ -module, and (5) is proved. Now, (6) follows by a combined use of [16, Corollary 18.2/(a)] and [12, Lemma 4.4.3(iii)].

If N_i/Z is Abelian, then it is a minimal Abelian normal subgroup of H/Z so it is elementary Abelian r_i -group for some prime r_i . Using the same argument as in paragraph 6 of this proof, one can find the extraspecial r_i -group R_i by taking the full inverse image of a maximal non-degenerate subspace of R/R' where R is the Sylow- r_i subgroup of N_i . For this subgroup, it clearly follows that $N_i = ZR_i$, and $|R_i| = r_i^{2l_i+1}$ for some integer. Furthermore, since V_i is an absolutely irreducible $\mathbb{F}_{q^k}N_i$ -module, it must be an absolutely irreducible $\mathbb{F}_{q^k}R_i$ -module. It is well-known that extraspecial r_i -group of order $r_i^{2l_i+1}$ has a unique faithful absolutely irreducible ordinary representation, and this representation has degree $r_i^{l_i}$, which finishes the proof of (7).

Finally, (8) can again be deduced from [16, Corollary 18.2/(a)] and from the fact that N_i/Z is a minimal normal subgroup in H_i/Z . \square

Lemma 4.2. *Let G be a group, K be a field and let V be an arbitrary finite dimensional KG -module.*

- (1) *For any $g, h \in G$ we have $\text{Supp}([g, h]) \leq 2 \text{Supp}(g)$.*
- (2) *If $N \triangleleft G$ such that V is absolutely irreducible as a KN -module, then $\text{MinSupp}(N) \leq 2 \text{MinSupp}(G)$.*

Proof. Let us consider the subspaces $U = \text{Fix}(g)$ and $W = \text{Fix}(h^{-1}gh)$ of V . Then we have

$$\dim(U) + \dim(W) - \dim(U \cap W) = \dim(U + W) \leq \dim(V).$$

Using that $\dim(U) = \dim(W) = \dim(V) - \text{Supp}(g)$ we get $\dim(U \cap W) \geq \dim(V) - 2 \text{Supp}(g)$. On the other hand $U \cap W \leq \text{Fix}([g, h])$ holds trivially, so

$$\begin{aligned} \text{Supp}([g, h]) &= \dim(V) - \dim(\text{Fix}([g, h])) \leq \dim(V) - \dim(U \cap W) \\ &\leq \dim(V) - (\dim(V) - 2 \text{Supp}(g)) = 2 \text{Supp}(g), \end{aligned}$$

and part (1) follows.

For part (2), let $1 \neq g \in G$ be any element. If $[g, N] = 1$, then g acts as a scalar transformation on V by [10, Theorem 9.2], so $\text{Supp}(g) = \dim(V) \geq \text{MinSupp}(N)$. Otherwise, there is an element $n \in N$ such that $[g, n] \neq 1$. Then we have $\text{MinSupp}(N) \leq \text{Supp}([g, n]) \leq 2 \text{Supp}(g)$. Thus, $\text{MinSupp}(N) \leq 2 \text{Supp}(g)$ for every $1 \neq g \in G$, which proves that $\text{MinSupp}(N) \leq 2 \text{MinSupp}(G)$. \square

Lemma 4.3. *Let V_1, \dots, V_k be finite dimensional vector spaces over the field \mathbb{F}_q and $Z < G_1 \leq GL(V_1), \dots, Z < G_k \leq GL(V_k)$ be coprime linear groups. Consider the group $G := G_1 \otimes \dots \otimes G_k$ acting on the tensor product $V := V_1 \otimes \dots \otimes V_k$ in a natural way.*

- (1) *Let $g = g_1 \otimes \dots \otimes g_k \in G$ with $g_j \in G_j$ for each j and let us assume that $g_i \notin Z$ for some i . Then*

$$\text{Supp}_V(g) \geq \text{MinSupp}_{V_i}(G_i) \cdot \frac{\dim(V)}{\dim(V_i)}$$

or $\text{Supp}_V(g) \geq \frac{1}{2} \dim(V)$.

- (2) *As a consequence*

$$\text{MinSupp}_V(G) = \min_i \left\{ \text{MinSupp}_{V_i}(G_i) \cdot \frac{\dim(V)}{\dim(V_i)} \right\},$$

or $\text{MinSupp}_V(G) \geq \frac{1}{2} \dim(V)$.

Proof. To prove part (1), first we consider the case $k = 2$. Let $n_1 = \dim(V_1)$, $n_2 = \dim(V_2)$, so $n = \dim(V) = n_1 n_2$. Furthermore, let $1 \neq g = g_1 \otimes g_2 \in G_1 \otimes G_2$ be an element of G with $g_1 \notin Z$. Since the action is coprime, g_1 and g_2 are diagonalisable over $\overline{\mathbb{F}}_q$. Let $\alpha_1, \dots, \alpha_s \in \overline{\mathbb{F}}_q$ be the different eigenvalues of g_1 with multiplicity k_1, k_2, \dots, k_s . We can assume that k_1 is the largest among the k_i . Let l_1, \dots, l_s be the multiplicities of $\alpha_1^{-1}, \dots, \alpha_s^{-1}$ in the characteristic polynomial of g_2 (Some of them can be zero). Then

$$\begin{aligned} \text{Supp}_V(g) &= \text{Supp}_V(g_1 \otimes g_2) = n - \dim(\text{Fix}_V(g_1 \otimes g_2)) \\ &= n - \sum_{i=1}^s k_i l_i \geq n - \sum_{i=1}^s k_1 l_i \geq (n_1 - k_1) n_2. \end{aligned}$$

If $\alpha_1 \in \mathbb{F}_q$, then we can substitute g_1 by $\alpha_1^{-1}g_1$ and g_2 by $\alpha_1 g_2$ (since both G_1 and G_2 contains all the scalar transformations), so we can assume that $\alpha_1 = 1$. Now, since $g_1 \neq 1$, we get $\text{Supp}_V(g) \geq (n_1 - k_1)n_2 = \text{Supp}_{V_1}(g_1)n_2 \geq \text{MinSupp}_{V_1}(G_1) \cdot \dim(V_2)$.

Now, let us assume that $\alpha_1 \notin \mathbb{F}_q$. Then there is an algebraic conjugate element of α_1 (different from α_1) under the action of $\text{Gal}(\overline{\mathbb{F}}_q, \mathbb{F}_q)$ which is also an eigenvalue of g_1 with the same multiplicity as α_1 . In particular, $k_1 \leq n_1/2$. Thus,

$$\text{Supp}_V(g) \geq (n_1 - k_1)n_2 \geq (n_1/2)n_2 = \frac{\dim(V)}{2}.$$

By changing the role of g_1 and g_2 in the proof and by using induction on k , we get the claim of part (1).

Finally, if $\text{Supp}_{V_i}(g_i) = \text{MinSupp}_{V_i}(G_i)$ for some $g_i \notin Z$, then

$$\text{Supp}_V(1 \otimes \dots \otimes 1 \otimes g_i \otimes 1 \otimes \dots \otimes 1) = \text{MinSupp}_{V_i}(G_i) \cdot \frac{\dim(V)}{\dim(V_i)},$$

so part (2) follows by using part (1). \square

Proof of Theorem 1.2. Let $G \leq GL(V)$ be a coprime primitive linear group. Without loss of generality we can assume that G is maximal among such subgroups of $GL(V)$. Let Z be the unique maximal Abelian subgroup in $GL(V)$ which is normalised by G and H be the intersection of G and $GL(d, q^k)$ as in Theorem 4.1. If $g \in G \setminus H$ then there is a $z \in Z$ such that $[g, z] \neq 1$. By Lemma 4.2, $\text{Supp}_V(g) \geq \frac{1}{2} \text{Supp}_V([g, z]) = \frac{1}{2} \dim(V)$. Therefore if $c > 4$, then

$$\sum_{g \in G \setminus H} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{|G \setminus H|}{q^{\frac{c}{2} \dim(V)}} \leq \frac{|V|^2}{|V|^{\frac{c}{2}}} \leq \frac{1}{|V|^{\frac{c}{2}-2}}.$$

Now let g be an element of $H = H_1 \otimes \dots \otimes H_t$. So $g = (g_1, \dots, g_t)$ where $g_i \in H_i$ for all $i \in [t]$ and g preserves the tensor product decomposition $V = V_1 \otimes \dots \otimes V_t$ over \mathbb{F}_q^k as in Theorem 4.1 (6) and $\dim_{\mathbb{F}_q^k}(V_i) = d_i$ for all i (therefore $d = \dim_{\mathbb{F}_q^k}(V) = \prod_{i=1}^t d_i$ and $\dim(V) = \dim_{\mathbb{F}_q}(V) = k \cdot \prod_{i=1}^t d_i$). We can assume that in this decomposition the dimensions of the vector spaces are decreasing, i.e. $d_1 \geq d_2 \geq \dots \geq d_t \geq 2$. Let $g_i \notin Z$ for some $i \neq 1$. Then by Lemma 4.3,

$$\text{Supp}_V(g) \geq \text{MinSupp}_{V_i}(H_i) \cdot \frac{\dim(V)}{\dim(V_i)} \geq k \prod_{j \neq i} d_j.$$

Since $2 \prod_{j \neq i} d_j \geq 2^{t-1} d_1 \geq \sum_{i=1}^t d_1$ and $k \prod_{j \neq i} d_j \geq k \sqrt{\prod_{j=1}^t d_j} \geq \sqrt{\dim(V)}$ we get that

$$c \cdot \text{Supp}_V(g) \geq 2k \sum_{i=1}^t d_i + (c-4) \sqrt{\dim(V)}.$$

Hence,

$$\begin{aligned} \sum_{g \in H \setminus H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} &\leq \frac{\prod_{i=1}^t |H_i|}{q^{2k \sum_{i=1}^t d_i + (c-4) \sqrt{\dim(V)}}} \\ &\leq \frac{q^{2k \sum_{i=1}^t d_i}}{q^{2k \sum_{i=1}^t d_i + (c-4) \sqrt{\dim(V)}}} = \frac{1}{q^{(c-4) \sqrt{\dim(V)}}}. \end{aligned}$$

Now assume that $g \in H_1$. In this case $\text{Supp}_V(g) = \text{Supp}_{V_1}(g) \cdot \frac{d}{d_1}$. By Theorem 4.1, $Z \leq N_1 \leq H_1 \leq GL(V_1(q^k))$ where N_1 is a minimal normal subgroup above Z and N_1/Z is characteristically simple. Therefore, it is either an elementary Abelian group, a direct product of non-Abelian simple groups, or a non-Abelian simple group.

First, if N_1/Z is elementary Abelian, then $N_1 = Z \cdot P$ where P is an extraspecial r -group for a prime r with $r \mid q^k - 1$. Then $V_1(q^k)$ is an absolutely irreducible $\mathbb{F}_{q^k}P$ -module. If $n \in P \setminus Z$ then n has exactly r different eigenvalues on V_1 (or on V) each with the same multiplicity. It follows that $\text{MinSupp}_V(N_1) \geq \frac{r-1}{r} \dim(V) \geq \frac{1}{2} \dim(V)$, so $\text{MinSupp}_V(H_1) \geq \frac{1}{4} \dim(V)$ by Lemma 4.2. In this case,

$$\sum_{g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{|H_1|}{q^{\frac{c}{4} \dim(V)}} \leq \frac{|V|^2}{q^{2 \dim(V) + (\frac{c}{4} - 2) \dim(V)}} \leq \frac{1}{|V|^{\frac{c}{4} - 2}}.$$

Next, let N_1/Z is a direct product of $s \geq 2$ many isomorphic non-Abelian simple groups. By Theorem 4.1 (8), the action of $N_1 = K_1 \otimes \dots \otimes K_s$ on V_1 preserves a tensor product decomposition $V_1 = W_1 \otimes \dots \otimes W_s$ over \mathbb{F}_q^k , where $\dim_{\mathbb{F}_q^k}(W_i) = \sqrt[s]{d_1} \geq 2$ for every i . Using [17, Theorem 1], we get that

$$|N_1| \leq \prod_{i=1}^s |K_s| \leq \prod_{i=1}^s |W_i|^2 = q^{2ks \sqrt[s]{d_1}}.$$

On the other hand, H_1/N_1 acts faithfully on $\{W_1, \dots, W_s\}$ and $|H_1/N_1|$ is coprime to q , so $|H_1/N_1| \leq q^s$ by [8, Corollary 2.4]. Therefore, $|H_1| \leq q^{2ks \sqrt[s]{d_1} + s}$. By Lemma 4.2 and by Lemma 4.3,

$$\text{Supp}_{V_1}(g) \geq \frac{1}{2} \text{MinSupp}_{V_1}(N_1) \geq \frac{k}{2} \cdot \frac{d_1}{\sqrt[s]{d_1}}.$$

Therefore,

$$c \text{Supp}_V(g) \geq 5kd_1^{(s-1)/s} + \left(\frac{c}{2} - 5\right)k\sqrt{d_1} \cdot \frac{d}{d_1} \geq 2ks \sqrt[s]{d_1} + s + \left(\frac{c}{2} - 5\right)\sqrt{\dim(V)}.$$

So,

$$\begin{aligned} \sum_{1 \neq g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} &\leq \frac{|H_1|}{q^{c \cdot \text{MinSupp}_V(H_1)}} \leq \frac{q^{2ks \sqrt[s]{d_1} + s}}{q^{2ks \sqrt[s]{d_1} + s + (\frac{c}{2} - 5)\sqrt{\dim(V)}}} \\ &\leq \frac{1}{q^{(\frac{c}{2} - 5)\sqrt{\dim(V)}}}. \end{aligned}$$

Finally, let N_1/Z be a non-Abelian simple group. If $d_1 \leq \sqrt{d}$, then we can use the same argument as in the previous paragraph to get that

$$\sum_{1 \neq g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{1}{q^{(c-2)\sqrt{\dim(V)}}}.$$

Summarizing the bounds given until this point, we get that

$$\begin{aligned} P_b(c, G, V) &\geq 1 - \sum_{1 \neq g \in G} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \geq 1 - \left(\frac{1}{|V|^{\frac{c}{4} - 2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} \right. \\ &\quad \left. + \frac{1}{q^{(\frac{c}{2} - 5)\sqrt{\dim(V)}}} \right) \geq 1 - \frac{3}{q^{(\frac{c}{2} - 5)\sqrt{\dim(V)}}}, \end{aligned}$$

which is case (1) of Theorem 1.2.

Now, let us assume that $d_1 \geq \sqrt{d}$. If $|V_1| = q^{kd_1}$ is bounded by the constant appearing in part (3) of Theorem 3.8, then $|V|$ is also bounded. Hence we can assume that either part (1) or part (2) of Theorem 3.8 holds. By Lemma 4.2, we also have $\text{MinSupp}_V(H_1) \geq \frac{1}{2} \text{MinSupp}_V(N_1)$.

If N_1/Z is not an alternating group, then $\text{MinSupp}_V(N_1) \geq \frac{1}{40} \dim(V)$ and $5 \cdot \text{MinSupp}_V(N_1) \geq \log_q |H_1|$ by using Corollary 3.3, Theorem 3.8/(1) and Lemma 4.3/(2). Thus, we have

$$\sum_{1 \neq g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{1}{|V|^{(c-10)/80}}.$$

So, in this case we get that

$$\begin{aligned} Pb(c, G, V) &\geq 1 - \left(\frac{1}{|V|^{\frac{c}{2}-2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{1}{|V|^{(c-10)/80}} \right) \\ &\geq 1 - \left(\frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{2}{|V|^{(c-10)/80}} \right), \end{aligned}$$

which is case (2)/a of Theorem 1.2.

Finally, let $N_1/Z \simeq A_m$ for some m . If V_1 is not an irreducible component of the natural $\mathbb{F}_q^k A_m$ permutation module, then we have $\text{MinSupp}_V(N_1) \geq \frac{1}{16} \sqrt{\dim(V)}$ and $5 \cdot \text{MinSupp}_V(N_1) \geq \log_q |H_1|$ by using Corollary 3.7, Theorem 3.8/(1) and Lemma 4.3/(2). Thus, we have

$$\sum_{1 \neq g \in H_1} \frac{1}{q^{c \cdot \text{Supp}_V(g)}} \leq \frac{1}{q^{\frac{c-10}{16} \sqrt{\dim(V)}}}$$

and

$$\begin{aligned} Pb(c, G, V) &\geq 1 - \left(\frac{1}{|V|^{\frac{c}{2}-2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + \frac{1}{q^{\frac{c-10}{16} \sqrt{\dim(V)}}} \right) \\ &\geq 1 - \frac{3}{q^{\frac{c-10}{16} \sqrt{\dim(V)}}}. \end{aligned}$$

Finally, if V_1 is the non-trivial irreducible component of the natural $\mathbb{F}_q^k A_m$ -module, then with the use of Corollary 3.10 we get that

$$Pb(c, G, V) \geq 1 - \left(\frac{1}{|V|^{\frac{c}{2}-2}} + \frac{1}{q^{(c-4)\sqrt{\dim(V)}}} + 1 - Pb(c, H_1, V) \right) \geq 1 - \frac{3}{n^{c-2}},$$

which completes the proof of Theorem 1.2. \square

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