

# SMOOTHINGS OF SINGULARITIES AND SYMPLECTIC SURGERY

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**ABSTRACT.** Suppose that  $C = (C_1, \dots, C_m)$  is a configuration of 2-dimensional symplectic submanifolds in a symplectic 4-manifold  $(X, \omega)$  with connected, negative definite intersection graph  $\Gamma_C$ . We show that by replacing an appropriate neighborhood of  $\cup C_i$  with a smoothing  $W_S$  of a normal surface singularity  $(S, 0)$  with resolution graph  $\Gamma_C$ , the resulting 4-manifold admits a symplectic structure. This operation generalizes the rational blow-down operation of Fintushel-Stern for other configurations, and therefore extends Symington's result about symplectic rational blow-downs.

## 1. INTRODUCTION

Suppose that  $X$  is a closed, oriented 4-manifold. Recall that in the rational blow-down procedure (introduced by Fintushel and Stern [8] and extended by J. Park [19]) the tubular neighbourhood of a collection of embedded spheres  $S = (S_1, \dots, S_k)$  in  $X$  is replaced by a specific compact 4-manifold  $W_S$  with boundary, providing the closed 4-manifold  $X_S$ . The spheres intersect each other according to a linear graph  $\Gamma_S$ , and their self-intersections are determined by the continued fraction coefficients of the ratio  $-\frac{p^2}{pq-1}$  for some  $p > q > 0$  relatively prime integers. The 4-manifold  $W_S$  in the construction can be interpreted as a particular smoothing of the singularity which has the plumbing graph  $\Gamma_S$  as its resolution graph. (In fact, in the rational blow-down construction  $W_S$  admits the further property that it is a rational homology disk, that is,  $H_*(W_S; \mathbb{Q}) \cong H_*(D^4; \mathbb{Q})$ .)

The success of the rational blow-down construction stems from the fact that it produces 4-manifolds with interesting differential topology. The curiosity of the resulting smooth structure can be measured by the Seiberg-Witten invariants of  $X_S$ . In fact, for the rational blow-down construction there is a simple relation between the Seiberg-Witten invariants of the 4-manifold  $X$  and the resulting 4-manifold  $X_S$  [8]. In specific cases the nonvanishing of the Seiberg-Witten invariant of  $X_S$  can be explained using symplectic topology: according to a result of Symington [22, 23], if  $(X, \omega)$  is a symplectic 4-manifold and the spheres in the configuration  $S$  are symplectic submanifolds (intersecting  $\omega$ -orthogonally), then  $X_S$  admits a symplectic structure (hence by Taubes' theorem [24] it has nontrivial Seiberg-Witten invariants). This symplectic feature of the construction has been extended to further configurations of symplectic surfaces in symplectic 4-manifolds and further smoothings of singularities in [1, 9, 10, 11]. The general case, however, stayed open and was formulated as Conjecture 1.4 in [11]. The aim of the present paper is to prove this conjecture. Informally, the result says that if we have a configuration of symplectic surfaces in a symplectic 4-manifold which intersect each other according

to a negative definite matrix, we collapse them to a point, and deform the resulting complex singularity, then the deformation 'globalizes in the symplectic category'.

To formulate the theorem precisely, suppose that  $(X, \omega)$  is a closed symplectic 4-manifold and  $C = (C_1, \dots, C_m)$  is a collection of smooth, closed 2-dimensional submanifolds which satisfy the following properties:

- each  $C_i$  is a symplectic submanifold and  $C = \cup C_i$  is connected,
- $C_i$  intersects each other  $C_j$   $\omega$ -orthogonally in at most one point, and
- the intersection matrix  $I = (C_i \cdot C_j)$  (with the self-intersections in the diagonal) is negative definite.

Suppose that  $\Gamma_C$  is the (connected) plumbing graph corresponding to the curve configuration  $C$ . By our assumption it is a negative definite plumbing graph, where the vertex  $v$  corresponding to the surface  $C_v$  is decorated by the self-intersection  $e_v = C_v \cdot C_v < 0$  and by the genus  $g_v = g(C_v) \geq 0$ . According to a fundamental result of Grauert [13], for any such graph there is a normal surface singularity  $(S, 0)$  with resolution dual graph equal to  $\Gamma_C$ . (The analytic type of the singularity is not necessarily unique, although its topology is determined by the graph  $\Gamma_C$ .) Suppose that  $W_C$  is a smoothing (or Milnor fiber) of the chosen singularity  $(S, 0)$ .

The main result of the present paper then reads as follows.

**Theorem 1.1.** *With the notations above, let  $\nu C$  denote a tubular neighbourhood of the union  $\cup C_i$ . Then, under the assumptions listed above, there is an orientation-reversing diffeomorphism  $\phi: \partial(X - \text{int}\nu C) \rightarrow \partial W_C$  such that the glued-up manifold*

$$X_C = (X - \text{int}\nu C) \cup_\phi W_C$$

*admits a symplectic structure which is equal to the restriction of  $\omega$  over  $X - \nu C$ .*

The main idea of the proof of the above result is the following: by [11] the configuration  $C = \cup_i C_i$  admits an  $\omega$ -convex neighbourhood  $U_C$ , with boundary  $\partial U_C$  supporting a compatible contact structure  $\xi_C$ . The smoothing  $W_C$ , on the other hand, admits the structure of a Stein domain, inducing the so-called *Milnor fillable* contact structure  $\xi_M$  on  $\partial W_C$ . By our assumption  $\partial U_C$  and  $\partial W_C$  are orientation preserving diffeomorphic 3-manifolds. The main tool in verifying Theorem 1.1 is the result showing that  $\xi_C$  and  $\xi_M$  are, in fact, contactomorphic. Therefore taking a contactomorphism  $\psi: (\partial U_C, \xi_C) \rightarrow (\partial W_C, \xi_M)$ , the gluing of symplectic 4-manifolds along contact hypersurfaces (as it is explained in [7], cf. also [18]) concludes the argument. In turn, the fact that the two contact structures  $\xi_C$  and  $\xi_M$  are contactomorphic will be proved by relating two compatible open book decompositions. The existence of this contactomorphism was verified in [1, 11] for specific families of configurations of  $C$ ; in this paper we extend the result of [11] by constructing an appropriate horizontal open book decomposition on  $\partial U_C$  compatible with  $\xi_C$  (cd. Theorem 3.2.)

The paper is organized as follows. In Section 2 we quickly recall some facts about horizontal open book decompositions, and in Section 3 we give the proof of the main result of the paper. In Section 4 we show an example where the resolution graph involves curves of high genus.

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## 2. HORIZONTAL OPEN BOOK DECOMPOSITIONS

By the Giroux correspondance [12], open book decompositions play a central role in contact topology. For completeness, in this section we recall some facts and constructions regarding specific open book decompositions on plumbed 3-manifolds. We start with a general definition.

**Definition 2.1.** *Suppose that  $Y$  is a given closed, oriented 3-manifold. The pair  $(B, \varphi)$  is an **open book decomposition** on  $Y$  if  $B \subset Y$  is an oriented 1-dimensional submanifold and  $\varphi: Y - B \rightarrow S^1$  is a locally trivial fibration with the property that a fiber  $\varphi^{-1}(t)$  is the interior of a Seifert surface of  $B$ . The submanifold  $B$  is called the **binding** of the open book, while the closure of a fiber  $\varphi^{-1}(t)$  is called a **page**. Two open book decompositions  $(B, \varphi)$  and  $(B', \varphi')$  of the diffeomorphic 3-manifolds  $Y$  and  $Y'$  are **equivalent** if there is an orientation preserving diffeomorphism  $g: Y \rightarrow Y'$  with the properties that  $g(B) = B'$  (as oriented 1-manifolds) and  $\varphi = \varphi' \circ g$ .*

According to the Giroux correspondence [12], an open book decomposition uniquely determines an isotopy class of compatible contact structures. Recall that the contact form  $\alpha$  is *compatible* with the open book decompositions  $(B, \varphi)$  if  $B$  is tangent to the Reeb flow  $R_\alpha$  defined by  $\alpha$ , while the interiors of the pages are transverse to the Reeb flow.

By a classical result of Stallings [20], in a rational homology 3-sphere an open book decomposition is determined by its binding. For manifolds with  $b_1 > 0$  this principle no longer holds. By considering specific classes of 3-manifolds and open book decompositions, however, a similar statement can be proved. For the statement we need a little preparation. (For related notions, see also [6].) Suppose that  $Y$  is a graph manifold, that is, it is given by the plumbing construction along a weighted graph  $\Gamma$ . This means that we consider circle bundles over the surfaces corresponding to the vertices (with Euler numbers specified by the framings of the graph) and plumb these pieces together.

**Definition 2.2.** *An open book decomposition  $(B, \varphi)$  on a graph manifold  $Y$  is **horizontal** if the binding  $B$  is the union of fibers of the individual fibrations, the pages are transverse to these fibrations and the orientation induced by the pages on the binding coincides with the orientation given by the fibration.*

Let  $\mathbf{n} = (n_v)$  denote the vector of nonnegative numbers specified by the binding components at each vertex  $v$  of the plumbing graph  $\Gamma$ . Now the version of Stallings' result (due to Caubel–Némethi–Popescu-Pampu) is the following:

**Theorem 2.3** ([3]). *Suppose that  $(B, \varphi)$  and  $(B', \varphi')$  are two horizontal open book decompositions of the plumbing 3-manifold  $Y = Y_\Gamma$ . If  $\mathbf{n} = \mathbf{n}'$  and for each vertex  $v$  we have  $n_v = n'_v > 0$  then the two open book decompositions are equivalent, and hence the compatible contact structures are contactomorphic.*  $\square$

By another result of Caubel–Némethi–Popescu-Pampu, horizontal open book decompositions compatible with the Milnor fillable contact structure  $\xi_M$  are easy to construct:

**Proposition 2.4** (Theorem 4.1 of [3]). *Let  $p : (\tilde{S}, E) \rightarrow (S, 0)$  be a good resolution of a normal surface singularity  $(S, 0)$ , where  $E$  is a normal crossing divisor in  $\tilde{S}$  having smooth components  $E_1, \dots, E_m$  with  $E = \sum_i E_i$ . Assume that the nonzero effective divisor  $D = \sum d_i E_i$  ( $d_i \in \mathbb{N}$ ) satisfies*

$$(D + E + K_\Sigma) \cdot E_i + 2 \leq 0 \text{ for any } i = 1, \dots, m.$$

*Then there exists a holomorphic function on  $(S, 0)$  with an isolated singularity at 0 such that  $\text{div}(f \circ p)$  is a normal crossing divisor on  $\tilde{S}$  and the exceptional part of  $\text{div}(f \circ p)$  is  $D$ . Moreover, for each  $i$ , the number of intersection points  $n_i = \text{div}(f \circ p)_s \cdot E_i$  is strictly positive, where  $\text{div}(f \circ p)_s$  is strict transform part of  $\text{div}(f \circ p)$ .  $\square$*

By [3, Remark 4.2], for any good resolution of the normal surface singularity  $(S, 0)$  there is a effective  $D$  which satisfies the condition of Proposition 2.4. Consider now the open book decomposition determined by a function  $f$  provided by Proposition 2.4: let  $B = f^{-1}(0) \cap \partial W_C$  and  $\varphi = \frac{f}{|f|}$ . As it was explained in [3], the resulting open book decomposition is horizontal, compatible with  $\xi_M$ , and with the notation  $n_v = -D \cdot E_v$  each  $n_v$  is strictly positive.

Therefore there are horizontal open book decompositions compatible with the Milnor fillable contact structure  $\xi_M$ , and indeed, we can find such open books with the extra condition that  $n_v > 0$  for all  $v$ . A useful simple observation shows that if  $\mathbf{n} = (n_v)$  appears as such a vector, then so does  $k \cdot \mathbf{n}$  for any positive integer  $k$ :

**Lemma 2.5.** *Let  $D$  be an effective divisor which satisfies the conditions of Proposition 2.4. Then any positive integer multiple  $k \cdot D$  of  $D$  also satisfies those conditions.*

*Proof.* Let  $D = \sum d_i E_i$  be an effective divisor satisfying  $(D + E + K)E_i + 2 \leq 0$  for all  $i = 1, \dots, m$ . Let  $k$  be a positive integer. Then  $k((D + E + K)E_i + 2) \leq 0$  for all  $i$ . Since  $0 \leq \sum_{j \neq i} E_j E_i + 2g(E_i) = (E - E_i) \cdot E_i + (E_i + K) \cdot E_i + 2 = (E + K)E_i + 2 \leq k((E + K)E_i + 2)$  for all  $i$ , we have

$$(k \cdot D + E + K)E_i + 2 \leq k((D + E + K)E_i + 2) \leq 0$$

for all  $i$ .  $\square$

### 3. HORIZONTAL OPEN BOOKS FOR $\xi_C$

Now we turn our attention to constructing horizontal open book decompositions compatible with the contact structure  $\xi_C$ . First of all, following [11, Section 4] we extend the notion of an open book decomposition for manifolds with boundary as follows: if  $M$  is a compact 3-manifold with nonempty boundary  $\partial M$ , then  $(B, \varphi)$  is an open book decomposition if  $B \subset M - \partial M$  is an oriented link and  $\varphi : M - B \rightarrow S^1$  is a map which behaves near  $B$  as a usual open book does and restricts to  $\partial M$  as a fibration  $\partial M \rightarrow S^1$ .

Suppose that  $Y$  is a plumbing 3-manifold along the plumbing graph  $\Gamma$ . Let  $v$  be a fixed vertex of the graph  $\Gamma$  and suppose that  $\{v_1, \dots, v_k\}$  are the further vertices connected to  $v$  in the graph. Recall that  $e_v$  denotes the framing fixed for  $v$ . Let  $N_v, N_{v_j}$  (for  $j = 1, \dots, k$ ) and  $n_v$  be positive integers satisfying

$$(1) \quad N_v e_v + \sum N_{v_j} = -n_v.$$

In short, if  $I$  denotes the intersection matrix of  $\Gamma$  (with the  $e_v$ 's in the diagonal), then  $\mathbf{N} \cdot I = -\mathbf{n}$ , where  $\mathbf{N} = (N_v)$  and  $\mathbf{n} = (n_v)$ .

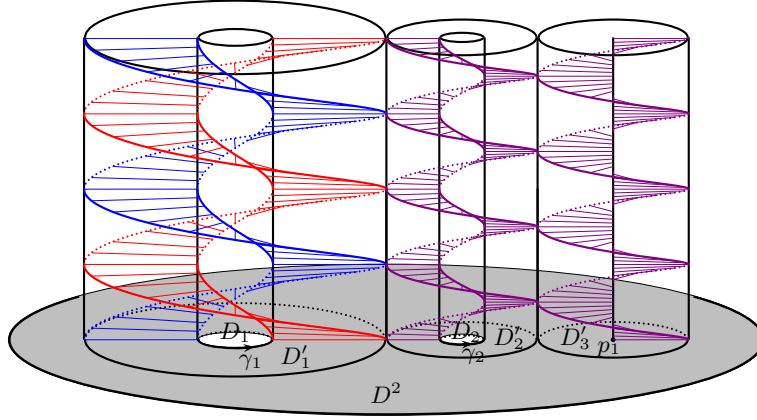


FIGURE 1. The diagram shows the special case when the vertex  $v$  has two neighbours  $v_1, v_2$ , furthermore  $N_v = 4, N_{v_1} = 2, N_{v_2} = 1$ , and the given value of  $n_v$  is equal to 1.

Let  $D^2$  be a 2-disk containing disjoint small disks  $D_1, \dots, D_k$  and let  $A = D^2 - \bigcup_{i=1}^k \text{int} D_i$ . Consider  $M = A \times S^1$  with coordinates  $\beta \in S^1$  and  $\gamma_j \in \partial D_j$  and  $\alpha \in \partial D^2$  ( $\alpha$  and  $\gamma_j$  with orientation as boundary of  $D^2$  and  $D_j$  respectively). Now an adaptation of [11, Lemma 4.1] gives the following result.

**Lemma 3.1.** *There exists an open book decomposition  $(B, \varphi)$  on  $M = A \times S^1$  such that the following conditions hold:*

- (1)  $\varphi|_{\partial D^2 \times S^1} = -e_v N_v \alpha + N_v \beta$ .
- (2)  $\varphi|_{\partial D_j \times S^1} = N_{v_j} \gamma_j + N_v \beta$ .
- (3) The pages  $\varphi^{-1}(\theta)$  are transverse to  $\partial_\beta$ .
- (4) The binding  $B$  is tangent to  $\partial_\beta$ .
- (5)  $B$  has  $n_v$  components  $B_1, \dots, B_{n_v}$ .
- (6) When the pages are oriented so that  $\partial_\beta$  is positively transverse, then  $B_1, \dots, B_{n_v}$  are oriented in the positive  $\partial_\beta$  direction.

*Proof.* Let  $p_1, \dots, p_{n_v}$  be fixed points in  $A$ . We may assume that the centers of the disks  $D_1, \dots, D_k$  and the fixed points  $p_1, \dots, p_{n_v}$  lie on a line segment and that each  $D_j$  and  $p_i$  are contained in the interior of another disk  $D'_j$  for  $1 \leq j \leq k + n_v$  such that the disks  $D'_j$  and  $D'_{j+1}$  are tangent to each other at one point and the center of  $D'_j$  is equal either to the center of  $D_j$  or to  $p_{j-k}$ ; see Figure 1. The desired open book decomposition on  $M = A \times S^1$  will be built from pieces.

First we consider an index  $j$  between 1 and  $k$  and describe the open book decomposition on  $(D'_j - D_j) \times S^1$  over the annulus  $D'_j - D_j$ . Each such index  $j$  corresponds to an other vertex  $u$  of the plumbing graph  $\Gamma$  with the property that  $v$  and  $u$  are joined by an edge. Now consider the curve  $(N_v, -N_u)$  on  $\partial((D'_j - D_j) \times S^1)$ . Notice that the boundary  $\partial((D'_j - D_j) \times S^1)$  has two components, an inner and an outer one. In addition, the positive integers  $N_v$  and  $N_u$  are not necessarily relatively prime, therefore the resulting curve on one of the boundary components is not necessarily connected. Foliate the boundaries by these curves, and extend these foliations to a foliation of  $(D'_j - D_j) \times S^1$  by (possibly disjoint unions of) annuli. cf. the left hand portion of Figure 1.

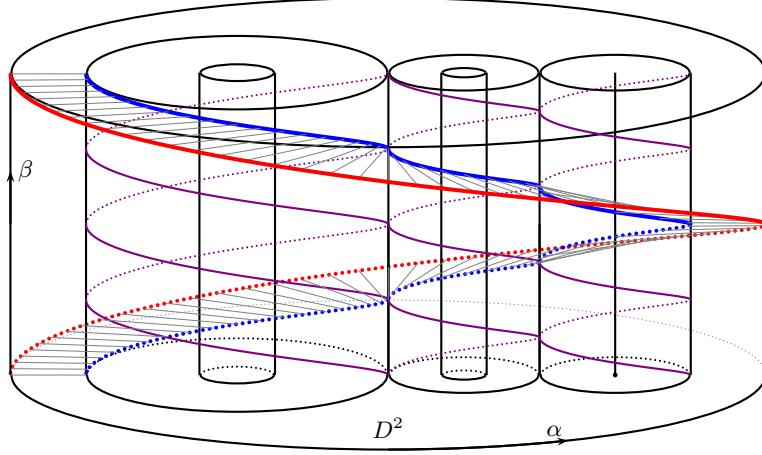


FIGURE 2. Computation of the slope of the curves on boundary of the union  $(\cup D'_j) \times S^1$ . The diagram depicts part of the page we get by gluing the pages of the individual open book decompositions together.

Consider now an index  $j$  between  $k+1$  and  $k+n_v$ . The disk  $D'_j$  then contains  $p_{j-k}$ . The open book decomposition on  $D'_j \times S^1$  will have binding equal to  $\{p_{j-k}\} \times S^1$  and the pages provide a foliation of  $\partial(D'_j \times S^1)$  by curves of type  $(N_v, -1)$ . (These conditions uniquely determine the open book decomposition.) For an illustrative example, see the right hand portion of Figure 1.

The union of the above pieces now provide an open book on  $(\cup D'_j) \times S^1$ . After trivial smoothings over the tangencies of the consecutive disks, this construction then extends to an open book decomposition on  $A \times S^1$ .

The proofs of the properties are routine exercises inside the individual pieces. There is one property which needs to be commented: we need to compute the slope of the fibration given by the pages on the outer boundary component  $\partial D^2 \times S^1$ . By construction, the curves we get by intersecting the boundary with the pages are of type  $(N_v, x)$ , where  $x$  is the sum of the corresponding coordinates on the individual pieces, cf. Figure 2. By our choices, we get that  $x = -n_v - \sum N_u$  where the summation goes for those vertices  $u$  which are connected in the graph  $\Gamma$  to  $v$ . By our choice of the vector  $\mathbf{N}$  (given in Equation (1)) we get that  $x = N_v e_v$ , verifying the first claimed property as well.  $\square$

According to a standard result (cf. [11, Corollary 3.4], for example), for a negative definite symmetric matrix  $I$  with nonnegative off-diagonals, and for any  $\mathbf{n} \in (\mathbb{R}^+)^m$  the vector  $-\mathbf{n} \cdot I^{-1}$  is in  $(\mathbb{R}^+)^m$ .

**Theorem 3.2.** *Suppose that  $Y = Y_\Gamma$  is a given plumbing manifold, where  $\Gamma$  is a negative definite plumbing graph on  $m$  vertices. Suppose that  $\mathbf{n} = (n_1, \dots, n_m)$  is a given vector in  $\mathbb{N}^m$ . Then there is  $k \in \mathbb{N}$  and a horizontal open book decomposition  $(B, \varphi)$  on  $Y$  which is compatible with  $\xi_C$  and at each vertex  $v$  it has  $kn_v$  binding components.*

*Proof.* Let  $\Gamma$  be the given negative definite plumbing graph with vertices corresponding to the surfaces  $C_v$  ( $v = 1, \dots, m$ ) with self-intersection numbers  $C_v^2 = e_v$  and genera  $g(C_v) = g_v$ . The graph defines a 4-manifold (containing the surfaces  $C_v$ ) and  $Y$  is the boundary of this plumbing 4-manifold.

For the given positive integral vector  $\mathbf{n}$  consider  $\mathbf{N} \in \mathbb{Q}^m$  satisfying the relation given by Equation (1)  $\mathbf{N} \cdot I = -\mathbf{n}$ , where (as before)  $I$  is a intersection matrix of the plumbing graph. The observation preceding Theorem 3.2 implies that  $N_v$  is nonnegative for all  $v$ . Notice that *a priori* each  $N_v$  is a rational number, but after multiplying both sides of Equation (1) by an appropriate positive integer  $k$ , we may assume that the resulting vector (which we still denote by  $\mathbf{N}$ ) is integral.

For constructing a horizontal open book decomposition on  $Y$  with the required properties, we decompose the neighborhood of the union  $C = \cup C_v$  into fibered pieces: For the vertex  $v$  of the graph  $\Gamma$  we consider the  $S^1$ -bundle over  $C_v$  with Euler number  $e_v$ . Let  $\widehat{C}_v$  denote the punctured surface  $C_v - D^2$ . Suppose that  $\partial \widehat{C}_v \times S^1$  has coordinates such that  $\partial \widehat{C}_v \times \{1\} = m_v$ ,  $\{p\} \times S^1 = l_v$ , and similarly  $\partial D^2 \times S^1$  has coordinates  $\partial D^2 \times \{1\} = \alpha_v$  and  $\{p'\} \times S^1 = \beta_v$ . We orient  $\alpha_v$  by the boundary orientation of  $D^2$  and  $m_v$  with the orientation opposite to the boundary orientation of  $\widehat{C}_v$ . The  $S^1$ -bundle over  $C_v$  with Euler number  $e_v$  is given by gluing  $D^2 \times S^1$  and  $\widehat{C}_v \times S^1$  with the gluing map given by

$$(2) \quad \alpha_v + e_v \beta_v \rightarrow m_v \quad \text{and} \quad \beta_v \rightarrow l_v.$$

Assume that each  $C_v$  meets the union  $\cup_{u \neq v} C_u$  in  $k_v$  points. For each such intersection point (that is, for each index  $i$  between 1 and  $k_v$ ) choose a small disk  $D_i \subset D^2$ . Let  $A_v = D^2 - \text{int}(D_1 \cup \dots \cup D_{k_v}) \subset C_v$  be the complement of (the interiors) of the chosen disks. Near  $C_v$  therefore we can decompose the 3-manifold as the union of  $A_v \times S^1$  and  $\widehat{C}_v \times S^1$ .

On  $A_v \times S^1$  we take the horizontal open book decomposition provided by Lemma 3.1. On  $\widehat{C}_v \times S^1$  we consider the horizontal open book decomposition given by the map  $\pi : \widehat{C}_v \times S^1 \rightarrow S^1$  defined as  $\pi = N_i l_i$ .

By Property (1) of Lemma 3.1, when we glue  $A_v \times S^1$  and  $\widehat{C}_v \times S^1$  via the gluing map specified by Equation (2), the open book decompositions also glue together. Therefore we have a horizontal open book decomposition near the surface  $C_v$ .

Let  $q$  be an intersection point of  $C_v$  and  $C_u$  with  $q \in D_{i_1} \subset C_v$  and  $q \in D_{i_2} \subset C_u$ . When we plumb the two bundles at  $q$ , we glue the circle bundles with the map  $\gamma_{i_1} \rightarrow \beta_{i_2}$  and  $\beta_{i_1} \rightarrow \gamma_{i_2}$  (where the curves  $\gamma_j$  are as in Lemma 3.1). Thus  $N_{i_2} \gamma_{i_1} - N_{i_1} \beta_{i_1}$  maps to  $-(N_{i_1} \gamma_{i_2} - N_{i_2} \beta_{i_2})$ . By Property (2) of Lemma 3.1 the curve  $N_{i_2} \gamma_{i_1} - N_{i_1} \beta_{i_1}$  is part of the boundary of a page of the open book decomposition on  $A_v \times S^1$ . After plumbing, the pages of the open book decomposition are obtained by gluing pages of the open book decompositions on  $A_v \times S^1$  and  $A_u \times S^1$  along  $N_{i_2} \gamma_{i_1} - N_{i_1} \beta_{i_1} \subset \partial(A_v \times S^1)$  and  $N_{i_1} \gamma_{i_2} - N_{i_2} \beta_{i_2} \subset \partial(A_u \times S^1)$ . In conclusion, the pages of the individual open book decompositions glue together when performing the plumbing operation. This procedure therefore results a horizontal open book decomposition of  $Y = Y_\Gamma$  with binding number  $\mathbf{n} = (n_v)$ .

Finally we need to check that this open book decomposition is compatible with the contact structure  $\xi_C$ . More precisely, we have to check that the Reeb vector field for a contact form representing  $\xi_C$  is transverse to the pages and tangent to the binding components. Note that Reeb vector field can be chosen to be a positive multiple of  $\partial_\theta$  on  $f_v^{-1}(t)$  (cf. [11, Proposition 4.2]), and a positive multiple of

$b_1\partial_{q_1} + b_2\partial_{q_2}$  for some  $b_1, b_2 > 0$  on  $f_e^{-1}(t)$ . (Here we follow the notations in [11, Proposition 4.2] for  $f_v^{-1}(t)$ ,  $f_e^{-1}(t)$ ,  $q_1, q_2$ . In the above proof  $\{q_1, q_2\} = \{\gamma_j, \beta_j\}$  on the neighborhood  $D_j$ .) Therefore the open book decomposition is horizontal, concluding the proof.  $\square$

As a corollary of the arguments given above, now we can show that the two contact structures  $\xi_M$  and  $\xi_C$  are contactomorphic.

**Corollary 3.3.** *Suppose that  $C \subset (X, \omega)$  is a configuration of symplectic 2-manifolds as before, with  $\omega$ -convex neighbourhood  $U_C$  and induced contact structure  $\xi_C$  on  $\partial U_C$ . Let  $\xi_M$  be the Milnor fillable contact structure on the link of a singularity with resolution graph  $\Gamma_C$ . Then  $\xi_M$  and  $\xi_C$  are contactomorphic.*

*Proof.* Let  $\{E_v\}$  denote the irreducible components of the exceptional curve in the resolution. These curves correspond to the vertices of the resolution graph  $\Gamma_C$ . Let  $D = \sum d_i E_i$  be an effective divisor satisfying the assumptions of Proposition 2.4. (By [3, Remark 4.1] such  $D$  always exists.) As it is verified by Proposition 2.4, the existence of  $D$  shows that there is a horizontal open book decomposition on  $Y_{\Gamma_C}$  compatible with  $\xi_M$  which has  $n_v = -D \cdot E_v > 0$  binding component at the vertex  $v$ .

Define the vector  $\mathbf{N} = (N_v)$  of positive rational numbers by the identity  $\mathbf{N} \cdot I = -\mathbf{n}$ , where  $I$  is the intersection matrix of plumbing graph  $\Gamma_C$  and  $\mathbf{N} = (N_v)$ ,  $\mathbf{n} = (n_v)$ . Suppose that with the choice  $k \in \mathbb{N}$  the products  $k \cdot N_v$  are integers for all  $v$ . Consider the horizontal open book decomposition corresponding to the divisor  $k \cdot D$ . (As Lemma 2.5 shows, this divisor also satisfies the assumptions of Proposition 2.4.) This procedure provides a horizontal open book decomposition of  $Y_{\Gamma_C}$  which is compatible with  $\xi_M$  and has  $kn_v > 0$  binding components at each vertex  $v$  of  $\Gamma_C$ .

Now apply Theorem 3.2 with the choice  $\mathbf{n}$  and  $k$  as above. As a result, we get a horizontal open book decompositions compatible with  $\xi_C$  having  $kn_v$  binding components at each vertex  $v$ . Therefore the two contact structures  $\xi_M$  and  $\xi_C$  are compatible with horizontal open book decompositions with equal (and positive) binings, hence by Theorem 2.3 the structures are contactomorphic, concluding the proof.  $\square$

With these results at hand, now we can turn to the proof of the main result of the paper:

*Proof of Theorem 1.1.* Let  $C = (C_1, \dots, C_m)$  be the given set of symplectic surfaces in  $(X, \omega)$ , and  $W_C$  a smoothing of a singularity with resolution graph  $\Gamma_C$  given by the configuration  $C$ . Let  $U_C$  be an  $\omega$ -convex neighbourhood of  $C$  in  $X$  (the existence of which is proved in [11, Theorem 1.2]). According to Corollary 3.3 the contact structure  $\xi_C$  induced on  $\partial U_C$  is contactomorphic to the Milnor fillable contact structure  $\xi_M$  on  $\partial W_C$ , hence by the symplectic gluing theorem described in [7] (see also [18, Theorem 7.1.9]), the manifold  $X_C = (X - \text{int} U_C) \cup_{\phi} W_C$  admits a symplectic structure  $\omega_C$  which on  $X - U_C$  coincides with the given symplectic structure  $\omega$ . The proof is complete.  $\square$

#### 4. AN EXAMPLE

In this section we show an example of a family of singularities with resolution involving high genus curves, and for which the topological data of smoothings can



be computed. We will perform the symplectic surgery on symplectic 4-manifolds using these singularities and the smoothings.

**The singularity.** Let  $s, t$  and  $N$  be positive integers such that  $N - 1$  divides  $s + t$  and  $\gcd(N - 1, t) = 1$ . Consider the hypersurface singularity  $(S, 0) = (S_{s,t,N}, 0)$  given by the equation

$$(3) \quad (x^s + y^s)(x^t + y^{Nt}) + z^{N-1} = 0.$$

Repeatedly blow up the singular curve  $(x^s + y^s)(x^t + y^{Nt}) = 0$  and then considering the ramified  $(N - 1)$ -fold cover. After normalization and desingularization, and finally blowing down the rational  $(-1)$ -curves, we get the minimal resolution of the singularity  $(S, 0)$  with the following properties (cf. [16, 17]): The resolution consists of the union of two curves  $A$  and  $B$ , intersecting each other transversally once,  $A^2 = -N$  and  $g(A) = (s - 1)(N - 2)/2$  while  $B^2 = -1$  and  $g(B) = (t - 1)(N - 2)/2$ .

Specializing to  $s = 3$  and  $t = 30N - 33$ , the curve  $A$  is of genus  $N - 2$  and  $B$  is of genus  $15N^2 - 47N + 34$ . The condition  $\gcd(N - 1, t) = 1$  is satisfied when  $N - 1$  is not divisible by 3.

**Topological data of the smoothing.** We start with a short generic discussion about the computation of topological data of the Milnor fiber of a hypersurface singularity. Suppose therefore that  $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  defines the isolated singularity  $(S, 0)$  and  $p : (\tilde{S}, E) \rightarrow (S, 0) = (f^{-1}(0), 0)$  is its minimal good resolution. We write the exceptional divisor  $p^{-1}(0) = E$  as the union of irreducible components:  $E = E_1 \cup \dots \cup E_m$ . Let  $h = \text{rank } H_1(E)$  and  $p_g = \dim_{\mathbb{C}} H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$ . The canonical class  $K$  of  $\tilde{S}$  can be written as  $\sum r_i E_i$ , where the  $r_i$  are rational numbers, determined by adjunction formula  $2g(E_i) - 2 = E_i^2 + K \cdot E_i$ . The Milnor number and the signature of the Milnor fiber of the singularity of  $f^{-1}(0)$  can be computed as follows:

**Proposition 4.1** ([5]). *The Milnor number  $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x, y, z\} / (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$  is equal to  $\mu = K^2 - h + m + 12p_g$ . The signature of the Milnor fiber is equal to  $\sigma = -\frac{1}{3}(2\mu + K^2 + m + 2h)$ .  $\square$*

The singularity given by Equation (3) is given as a ramified cover along a singular plane curve. For an isolated plane curve singularity the Milnor number satisfies the following equation.

**Proposition 4.2** ([15]). *For an isolated plane curve singularity  $(C, 0) \subset (\mathbb{C}^2, 0)$*

$$\mu(C, 0) = d(d - 1) + \sum_{x \in \text{Sing}(\tilde{C})} \mu(\tilde{C}, x) + 1 - r,$$

where  $r$  is the number of different tangent lines of  $(C, 0)$ ,  $d$  is the multiplicity of  $C$  at 0 and  $\tilde{C}$  is the proper transform of  $C$  after one blow-up at singular point 0.  $\square$

Regarding the first Betti number of an isolated singularity we have

**Proposition 4.3** ([14]). *Let  $X_t$  be the Milnor fiber of a smoothing of a pure-dimensional isolated normal surface singularity  $(X_0, 0)$ , then  $b_1(X_t) = 0$ .  $\square$*

Using the above formulae, for the singularity  $(S, 0)$  specified by the function of Equation (3) we have

- The Milnor number  $\mu = b_2(X_t) = (N - 2)((s + t)(s + t - 1) + (N - 1)t(t - 1) + 1 - s - t)$ .

- The signature of the Milnor fiber is equal to  $\sigma = -\frac{1}{3}(2\mu + K^2 + m + 2h)$ .

For the specialization  $s = 3, t = 30N - 33$  (and  $N - 1$  is not divisible by 3) these data become

- $\mu(f + z^{N-1}) = 900N^4 - 3810N^3 + 5292N^2 - 2705N + 322$ ,
- $\sigma = -300N^4 + 960N^3 - \frac{2348}{3}N^2 + \frac{379}{3}N - 2$ .

**Symplectic surgery and the singularity.** Next, we construct a symplectic manifold which contains the curve configuration  $(A, B)$  described above. According to [2, Theorem 1] there is a surface bundle  $X \rightarrow \Sigma_{N-2}$  with fiber genus  $15N^2 - 47N + 34$  over the surface with genus  $N - 2$  such that there is a section with self-intersection  $-N$ . Let  $M$  denote the blow-up of  $X$  in a fiber. The fiber passing through the blown-up point, together with a section now provides the configuration of two curves  $(A, B)$  with intersection patterns as in the resolution graph of the singularity given by Equation (3).

Applying the symplectic surgery operation of replacing the neighbourhood  $\nu(A \cup B)$  with the smoothing  $W$  of the corresponding singularity, we get a symplectic 4-manifold  $M_W$ . Since the embedding map  $A \cup B \rightarrow X$  is onto on the first homology, Proposition 4.3 implies that  $b_1(M_W) = 0$ .

**Remark 4.4.** *Symplectic 4-manifolds containing similar configurations of symplectic submanifolds can be found near the Bogomolov-Miyaoka-Yau (BMY) line  $c_1^2 = 9\chi_h$ . (For 4-manifold on the BMY line, see [21].) We hope that using the symplectic surgery operation discussed in this paper, one will be able to construct symplectic manifolds with  $b_1 = 0$  (or even with  $\pi_1 = 0$ ) near the BMY line. We hope to return to this question in a future project.*

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