

# CONTACT STRUCTURES ON PRODUCT 5-MANIFOLDS AND FIBRE SUMS ALONG CIRCLES

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ABSTRACT. Two constructions of contact manifolds are presented: (i) products of  $S^1$  with manifolds admitting a suitable decomposition into two exact symplectic pieces and (ii) fibre connected sums along isotropic circles. Baykur has found a decomposition as required for (i) for all closed, oriented 4-manifolds. As a corollary, we can show that all closed, oriented 5-manifolds that are Cartesian products of lower-dimensional manifolds carry a contact structure. For symplectic 4-manifolds we exhibit an alternative construction of such a decomposition; this gives us control over the homotopy type of the corresponding contact structure. In particular, we prove that  $\mathbb{C}P^2 \times S^1$  admits a contact structure in every homotopy class of almost contact structures. The existence of contact structures is also established for a large class of 5-manifolds with fundamental group  $\mathbb{Z}_2$ .

## 1. INTRODUCTION

Contact structures, by virtue of their maximal non-integrability, appear to be better adapted to twisted products (i.e. fibre bundles) than to Cartesian products of manifolds. Classical examples supporting this dictum are the natural contact structures on unit cotangent bundles  $ST^*B$  (which are Cartesian products only if the base manifold  $B$  is parallelisable) and the Boothby–Wang [4] construction, where the contact form is the connection 1-form on an  $S^1$ -bundle, with curvature form equal to a symplectic form on the base representing the Euler class of the bundle. More recently, Lerman [26] has shown how to build contact structures on bundles whose curvature satisfies a certain non-degeneracy condition. Lerman’s work is a contact version of Sternberg’s minimal coupling construction and Weinstein’s fat bundles in symplectic geometry, and it generalised earlier results of Yamazaki.

By contrast, little is known about contact structures on Cartesian products. Some simple examples, such as products of spheres, can be obtained by contact surgery on a sphere. Manifolds of the form  $M^3 \times S^2$  admit a contact structure thanks to the parallelisability of (closed, oriented) 3-manifolds. Using a branched covering construction, one can then also put a contact structure on the product of  $M^3$  with an arbitrary (closed, oriented) surface, see [12].

But it took more than twenty years from Lutz’s discovery of a contact structure on the 5-torus to a construction by Bourgeois [6] of a contact structure on higher-dimensional tori and, more generally, products of contact manifolds with a surface of genus at least one.

In Section 2 of the present note we exhibit a method for putting contact structures on Cartesian products of even-dimensional manifolds  $V$  with  $S^1$ , provided the manifold  $V$  admits a suitable decomposition into two exact symplectic pieces (Theorem 1). For closed, oriented 4-manifolds, such a decomposition has been shown to

exist by Baykur [2]. As a corollary, we find that any closed, oriented 5-manifold that is the Cartesian product of lower-dimensional manifolds admits a contact structure (Corollary 2).

In Section 4 we prove an analogue of Baykur’s decomposition result for the special case of closed symplectic 4-manifolds (Theorem 4). Our proof rests on Donaldson’s theorem about codimension 2 symplectic submanifolds, as well as the classification of tight contact structures on circle bundles over surfaces due to Giroux and Honda. (By contrast, Baykur’s argument is based on the theory of Lefschetz fibrations on 4-manifolds and open book decompositions of 3-manifolds adapted to contact structures.) Signs are important for the discussion in Section 4, so in Section 3 we have inserted a brief description of Boothby–Wang contact forms and their convex resp. concave fillings.

In Section 5 we discuss some simple examples, notably  $V = \mathbb{C}\mathbb{P}^2$ . Here we need not invoke Donaldson’s theorem. Furthermore, the explicit nature of our decomposition theorem allows us to show, for instance, that  $\mathbb{C}\mathbb{P}^2 \times S^1$  admits a contact structure in every homotopy class of almost contact structures (Proposition 6).

A further construction of contact manifolds will be presented in Section 6: the connected sum along isotropic circles (Theorem 7), which may be regarded as a counterpart to the fibre connected sum along codimension 2 contact submanifolds described in [12], cf. [15, Chapter 7.4].

Finally, in Section 7 we use the connected sum along circles to produce contact structures on a class of 5-manifolds with fundamental group  $\mathbb{Z}_2$  described recently by Hambleton and Su [21] (Proposition 8). This class is in some respects more restricted than that discussed by Charles Thomas and the first author [16], but the Hambleton–Su description is more explicit, and it includes some additional cases. In fact, it was their classification result that prompted us to consider the questions discussed in the present note.

We close this introduction with a comment on notational and other conventions. We write  $M, N$  for closed odd-dimensional (typically: contact) manifolds. Our contact structures are always assumed to be cooriented, i.e. defined as  $\ker \alpha$  with  $\alpha$  a globally defined 1-form, unique up to multiplication by a positive function. A  $(2n + 1)$ -dimensional contact manifold  $(N, \ker \alpha)$  is given the orientation induced by the volume form  $\alpha \wedge (d\alpha)^n$ .

The notation  $V$  stands for a closed even-dimensional (typically: symplectic) manifold;  $W$  is used for compact even-dimensional manifolds with boundary (typically: a symplectic filling of a contact manifold). Given a  $2n$ -dimensional symplectic manifold  $(W, \omega)$ , we equip it with the orientation induced by the volume form  $\omega^n$ . We write  $\overline{W}$  for the manifold with the opposite orientation.

Unless stated otherwise, (co-)homology is understood with integer coefficients.

## 2. CONTACT STRUCTURES ON PRODUCTS WITH $S^1$

A compact manifold  $W$  with an exact symplectic form  $\omega = d\lambda$  is called an *exact symplectic filling* of the contact manifold  $(M, \xi)$  if the following conditions hold:

- $M = \partial W$  as oriented manifolds,
- the pull-back of  $\lambda$  to  $M$  under the inclusion map is a contact form defining the contact structure  $\xi$ .

These conditions imply that the Liouville vector field  $X$  on  $(W, \omega)$  defined by  $i_X \omega = \lambda$  is pointing outwards along the boundary  $M$ , so any exact filling is in particular a strong filling.

**Remark.** Every Stein filling is an exact filling. With a construction described in [8], any (weak or strong) symplectic filling with the property that the symplectic form is exact can be modified in a collar neighbourhood of the boundary so as to become an exact filling (of the same contact manifold); see also [14]. For more information about the various notions of filling see [15, Chapter 5] and [31, Chapter 12].

We can use the flow of  $X$  to define a collar neighbourhood  $(-\varepsilon, 0] \times M$  of  $M$  in  $W$ . Here  $\{0\} \times M$  is identified with  $M = \partial W$ , and with  $t$  denoting the parameter in  $(-\varepsilon, 0]$  we have  $\partial_t = X$ . Then, on this collar neighbourhood, the symplectic form  $\omega$  can be written as  $\omega = d(e^t \lambda)$ .

This allows us to define a symplectic completion  $W \cup_M ([0, \infty) \times M)$ , where the symplectic form on  $W$  is  $\omega$ , and on  $[0, \infty) \times M$  it is  $d(e^t \lambda)$ .

Given any non-negative function  $h$  on  $M$ , we have an embedding

$$\begin{aligned} M &\longrightarrow [0, \infty) \times M \\ x &\longmapsto (h(x), x) \end{aligned}$$

under which the 1-form  $e^t \lambda$  pulls back to  $e^h \lambda$ . Moreover, two contact forms defining the same (cooriented) contact structure on a compact manifold  $M$  can be made to coincide after multiplying each of them with a suitable function  $f_i: M \rightarrow [1, \infty)$ ,  $i = 1, 2$ . Thus, given two exact symplectic fillings  $(W_i, d\lambda_i)$ ,  $i = 1, 2$ , of the same contact manifold  $(M, \xi)$ , we may assume without loss of generality that  $\lambda_1|_{TM} = \lambda_2|_{TM}$ .

**Theorem 1.** *Let  $(W_1, d\lambda_1)$  and  $(W_2, d\lambda_2)$  be two exact symplectic fillings of the same contact manifold  $(M, \xi)$ . Then the manifold  $(W_1 \cup_M \overline{W_2}) \times S^1$  admits a contact structure.*

*Proof.* As explained, we may assume that  $\lambda_1|_{TM} = \lambda_2|_{TM}$ . Write  $\beta$  for this contact form on  $M$ . We then find collar neighbourhoods  $(-1 - \varepsilon, -1] \times M$  of  $\{-1\} \times M \equiv M = \partial W_i$  in  $W_i$  where  $\lambda_i = e^{t+1} \beta$ ,  $i = 1, 2$ . (This shift in the collar parameter by 1 is made for notational convenience in the construction that follows.)

Now choose two smooth functions  $f$  and  $g$  on the interval  $(-1 - \varepsilon, 1 + \varepsilon)$  subject to the following conditions (see Figure 1):

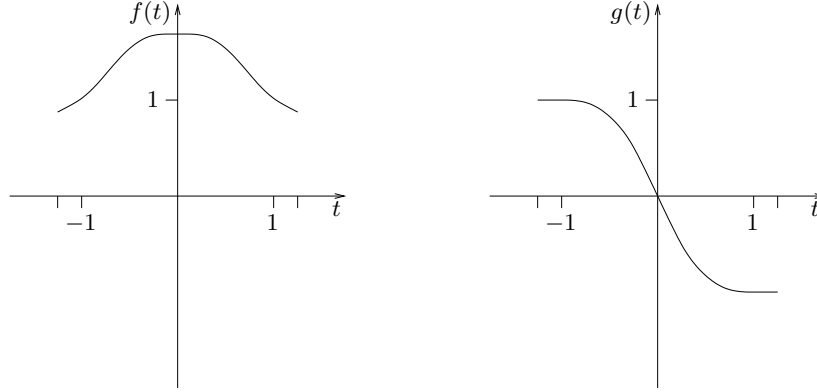
- $f$  is an even function with  $f(t) = e^{t+1}$  near  $(-1 - \varepsilon, -1]$ ,
- $g$  is an odd function with  $g(t) = 1$  near  $(-1 - \varepsilon, -1]$ ,
- $f'g - fg' > 0$ .

Now we consider the manifold

$$N := (W_1 \cup_M ([-1, 1] \times M) \cup_M \overline{W_2}) \times S^1,$$

which is a diffeomorphic copy of the manifold in the theorem. We write  $\theta$  for the  $S^1$ -coordinate. Define a smooth 1-form  $\alpha$  on  $N$  by

$$\alpha = \begin{cases} \lambda_1 + d\theta & \text{on } \text{Int}(W_1) \times S^1, \\ f\beta + g d\theta & \text{on } (-1 - \varepsilon, 1 + \varepsilon) \times M \times S^1, \\ \lambda_2 - d\theta & \text{on } \text{Int}(\overline{W_2}) \times S^1. \end{cases}$$

FIGURE 1. The functions  $f$  and  $g$ .

Then  $\alpha \wedge (d\alpha)^n$  (where  $\dim N = 2n + 1$ ) equals

$$\begin{cases} (d\lambda_1)^n \wedge d\theta & \text{on } \text{Int}(W_1) \times S^1, \\ n f^{n-1} (f'g - f g') dt \wedge \beta \wedge (d\beta)^{n-1} \wedge d\theta & \text{on } (-1 - \varepsilon, 1 + \varepsilon) \times M \times S^1, \\ -(d\lambda_2)^n \wedge d\theta & \text{on } \text{Int}(\overline{W}_2) \times S^1. \end{cases}$$

This shows that  $\alpha$  is a contact form on  $N$ .  $\square$

**Remark.** We discovered this construction by a slightly roundabout route. Given an exact symplectic filling  $(W, d\lambda)$ , one can form an open book decomposition of a manifold  $N_W$  with page  $W$  and monodromy the identity. As Giroux [19] has observed — generalising a 3-dimensional construction due to Thurston and Winkelnkemper — this manifold  $N_W$  carries an adapted contact form  $\alpha$ , cf. [15, Theorem 7.3.3]. Starting from such a contact form adapted to an open book, Bourgeois [6] constructed a  $T^2$ -invariant contact form on the product of  $N_W$  with a 2-torus  $T^2$ , cf. [15, Theorem 7.3.6]. The contact reduction (cf. [15, Chapter 7.7]) with respect to the  $S^1$ -action of one of the factors in  $T^2 = S^1 \times S^1$  yields a contact structure on  $(W \cup_{\partial W} \overline{W}) \times S^1$ . The contact structure near  $\partial W \times S^1$  is independent of the topology of the exact symplectic filling, so this construction generalises as described in the preceding proof.

**Example.** If one takes both  $W_i$  equal to the 2-disc, and  $\lambda_i = x dy - y dx$ , the above construction yields, up to isotopy, the standard tight contact structure  $\ker(z d\theta + x dy - y dx)$  on  $S^2 \times S^1$ .

As shown by Hirzebruch and Hopf [23], for any closed, oriented 4-manifold  $V$  the third integral Stiefel–Whitney class  $W_3(V)$  vanishes. Then we also have  $W_3(V \times S^1) = 0$ , which implies that  $V \times S^1$  admits an almost contact structure [15, Proposition 8.1.1]. The same is true for 5-manifolds of the form  $M^3 \times \Sigma^2$ , since all closed, oriented 3-manifolds are spin.

**Corollary 2.** *Any closed, oriented 5-manifold that is the Cartesian product of two lower-dimensional manifolds admits a contact structure.*

*Proof.* For manifolds of the form  $M^3 \times \Sigma^2$  this was proved in [12]. For manifolds of the form  $V^4 \times S^1$  we appeal to a result of Baykur [2], according to which any closed, oriented 4-manifold  $V$  admits a decomposition  $V = W_1 \cup_M \overline{W}_2$  as required

by Theorem 1. In fact,  $W_1$  and  $W_2$  may be taken to be *Stein* fillings of the same contact manifold  $(M, \xi)$ .  $\square$

It was first shown by Akbulut and Matveyev [1] that any closed, oriented 4-manifold can be decomposed into two Stein pieces, but their result did not give any information about the induced contact structures on the separating hypersurface. Baykur's result provides this additional information by an approach via the theory of Lefschetz fibrations on 4-manifolds and open book decompositions of 3-manifolds adapted to contact structures. In Section 4 we prove a similar decomposition result for the subclass of *symplectic* 4-manifolds, based on Donaldson's theorem about codimension 2 symplectic submanifolds and the classification of tight contact structures on circle bundles over surfaces due to Giroux and Honda. The following section serves as a preparation for Section 4.

### 3. CONVEX VS. CONCAVE FILLINGS OF BOOTHBY–WANG CONTACT FORMS

We briefly recall some facts about the Boothby–Wang construction; for more details see [15, Chapter 7.2].

Let  $\pi: M \rightarrow B$  be a principal  $S^1$ -bundle over a closed, oriented manifold  $B$ . By a connection 1-form on  $M$  we mean a differential 1-form  $\alpha$  which is invariant ( $\mathcal{L}_{\partial_\theta}\alpha \equiv 0$ ) and normalised ( $\alpha(\partial_\theta) \equiv 1$ ). Note that the relation with the usual  $i\mathbb{R}$ -valued connection 1-form  $A$  is  $\alpha = -iA$ . The differential  $d\alpha$  induces a well-defined closed 2-form  $\omega$  on  $B$ , i.e.  $d\alpha = \pi^*\omega$ . This  $\omega$  is called the curvature form of the connection 1-form  $\alpha$ . The Euler class of the bundle is given by  $e = -[\omega/2\pi]$ .

We also write  $\pi: W \rightarrow B$  for the projection map in the associated  $D^2$ -bundle. With  $r$  denoting the radial coordinate on  $D^2$ , we can extend  $\alpha$  to an  $r$ -invariant 1-form on  $W$  outside the zero section; the 1-form  $r^2\alpha$  is well defined and smooth on all of  $W$ .

We say that the  $S^1$ - or  $D^2$ -bundle of Euler class  $e$  is *positive* (resp. *negative*) if the base manifold  $B$  admits a symplectic form  $\omega$  (compatible with the orientation of  $B$ ) such that the cohomology class  $[\omega/2\pi]$  equals  $e$  (resp.  $-e$ ). Clearly, in either case the connection 1-form  $\alpha$  (on the  $S^1$ -bundle with Euler class  $e$ ) with curvature form  $\mp\omega$  — such an  $\alpha$  can always be found — will be a contact form on  $M$ . (Notice that for  $\dim B = 4m$ , a 2-form  $\omega$  on  $B$  is symplectic if and only if  $-\omega$  is symplectic, so any positive bundle will also be negative, and vice versa.)

The concave part of the following lemma is [28, Lemma 2.6]; the convex part is completely analogous. This has also been observed by Niederkrüger [30].

**Lemma 3.** *If  $e$  is positive (resp. negative), then there is a symplectic form  $\Omega$  on  $W$  which makes it a strong concave (resp. convex) filling of  $(M, \ker(\mp\alpha))$ .*

*Proof.* Depending on whether  $e$  is positive or negative, we have  $\pm e = [\omega/2\pi]$  for some symplectic form  $\omega$  on  $B$ . We find a connection 1-form  $\alpha$  with  $d\alpha = \mp\pi^*\omega$ . In both cases we set

$$\Omega = \frac{1}{2}d(r^2\alpha) + \pi^*\omega.$$

This is a symplectic form on  $W$ , compatible with its orientation, and a Liouville vector field defined near  $\partial W$  (in fact, everywhere outside the zero section) and pointing into (resp. out of)  $W$  along the boundary is given by  $X = ((r^2 \mp 2)/2r) \partial_r$ . This completes the proof.  $\square$

## 4. DECOMPOSITIONS OF SYMPLECTIC FOUR-MANIFOLDS

The following is a special case of Baykur's result (and with exact symplectic rather than Stein fillings), but it will be proved by entirely different methods. As we shall see in Section 5, the decomposition we construct of a symplectic 4-manifold  $V$  is explicit enough, in specific cases, to allow control over the homotopy type of the corresponding contact structure on  $V \times S^1$ .

**Theorem 4.** *Suppose that  $(V, \omega)$  is a closed symplectic 4-manifold. Then there are exact symplectic fillings  $W_1, W_2$  of a contact 3-manifold  $(M, \xi)$  with the property that  $W_1 \cup_M \overline{W_2}$  is diffeomorphic to  $V$ , where we glue using a contactomorphism of the boundaries  $\partial W_1$  and  $\partial W_2$ .*

Before turning to the proof of this theorem, we collect a few observations. By (possibly) perturbing and rescaling the symplectic form  $\omega$  we can assume that  $\omega/2\pi$  represents an integral cohomology class in  $H^2(V; \mathbb{R})$ . According to a famous result of Donaldson [7], for  $k \in \mathbb{N}$  sufficiently large the Poincaré dual of the cohomology class  $k[\omega/2\pi]$  can be represented by a 2-dimensional connected symplectic submanifold  $\Sigma \subset V$ .

**Proposition 5.** *Suppose that the connected symplectic surface  $\Sigma$  in the symplectic 4-manifold  $(V, \omega)$  represents the Poincaré dual of  $k[\omega/2\pi]$  in the second homology group  $H_2(V)$ . Then there is a closed tubular neighbourhood  $\nu\Sigma$  of  $\Sigma$  with  $\omega$ -concave boundary  $\partial(\nu\Sigma)$ . Write  $M$  for this boundary, oriented as the boundary of the closure  $W_1$  of the complement of  $\nu\Sigma$ . With  $\xi$  denoting the induced contact structure on  $M$ , the symplectic manifold  $(W_1, \omega)$  is — after a suitable modification of the symplectic form in a neighbourhood of  $M$  in  $W_1$  — an exact filling of the contact 3-manifold  $(M, \xi)$ .*

*Proof.* Since the self-intersection  $[\Sigma]^2 = k^2 \int_V \omega \wedge \omega/4\pi^2$  is positive, the stipulated  $\omega$ -concave neighbourhood  $\nu\Sigma$  exists by Lemma 3 and the symplectic neighbourhood theorem [29, Theorem 3.30], cf. [10, Corollary 6].

The class  $k[\omega/2\pi]$ , being Poincaré dual to the homology class represented by  $\Sigma$ , may be represented by a differential form with support in the interior of  $\nu\Sigma$  (this is known as the localisation principle, see [5]). It follows that, under the inclusion  $W_1 \subset V$ , the class  $[\omega/2\pi]$  pulls back to the zero class in  $H^2(W_1; \mathbb{R})$ . This means that  $\omega|_{W_1} = d\beta$  for some 1-form  $\beta$  on  $W_1$ .

A priori,  $(W_1, d\beta)$  is not an exact filling of  $(M, \xi)$ , because the Liouville vector field  $Y$  defined by  $i_Y \omega = \beta$  need not be transverse to  $M$ . However, as pointed out in the remark at the beginning of Section 2, we can modify the symplectic form in a collar neighbourhood of the boundary so as to obtain an exact filling.  $\square$

For the proof of the main result of this section we need a more precise understanding of the positive (and fillable, hence tight) contact structure  $\xi$  on the 3-manifold  $M$ . Notice that with our orientation convention for  $M$  (see Proposition 5), this manifold is an  $S^1$ -bundle over  $\Sigma$  with negative Euler number  $e = -[\Sigma]^2$ .

Write  $g$  for the genus of  $\Sigma$ . The adjunction equality  $2g - 2 = [\Sigma]^2 - \langle c_1(V), [\Sigma] \rangle$ , cf. [31, Theorem 3.1.9], shows that by choosing  $k$  (in the above proposition) sufficiently large, which Donaldson's theorem allows us to do, we may always assume that  $g > 1$ .

Recall that tight contact structures on  $S^1$ -bundles admit a numerical invariant, called the *maximal twisting*  $\mathfrak{t}$ , which measures the maximal contact framing of a

Legendrian knot smoothly isotopic to the fibre, when measured with respect to the framing the fibre inherits from the fibration. If the contact framing is non-negative with respect to the fibration framing, traditionally one declares  $\tau = 0$ . (For more on twisting see [25].)

According to the proof of Lemma 3, the contact structure  $\xi$  on  $M$  is *horizontal*, that is, the contact planes are transverse to the fibres of the  $S^1$ -fibration. This property implies that  $\xi$  has negative maximal twisting, see the proof of [25, Theorem 3.8]. On the other hand, according to the classification of tight contact structures on a circle bundle with negative Euler number over a surface of genus  $g > 1$ , there are exactly two contact structures (up to isotopy) which are horizontal [25, Theorem 2.11], and both are universally tight [25, Lemma 3.9].

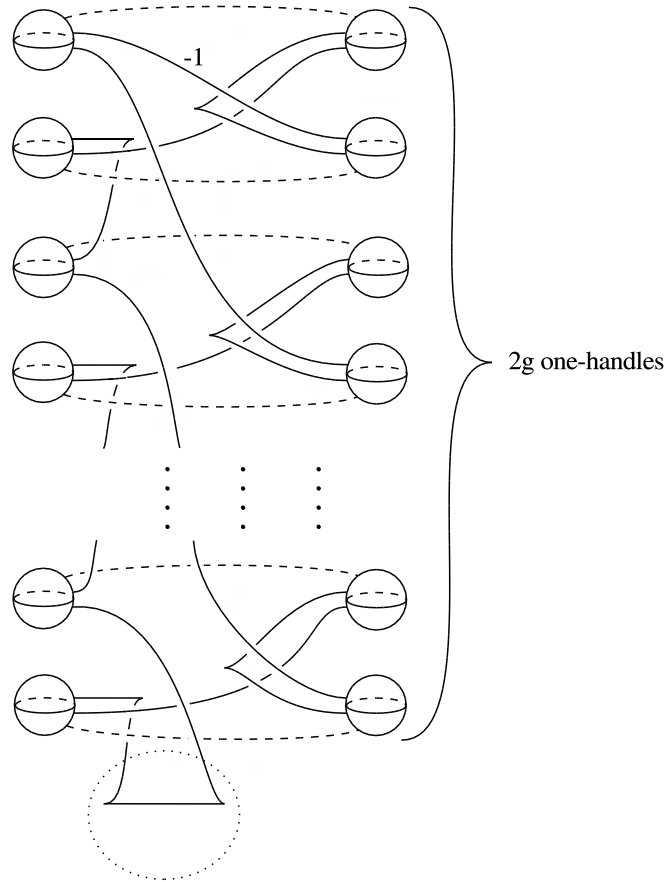


FIGURE 2. The surgery diagram for  $(M, \xi)$ .

We claim that these two contact structures can be described by the contact surgery diagram of Figure 2 (which is taken from [27]) after the surgery curve has been stabilised in one direction  $2g - 2 - e$  times, i.e., after the dotted ellipse in Figure 2 has been replaced by the diagram in Figure 3, which contains  $2g - 2 - e$  additional zigzags on the left, or by the mirror image of that diagram. Indeed,

the Legendrian knot  $K$  shown in the diagram has Thurston–Bennequin invariant  $\text{tb}(K) = 2g - 1$  (and rotation number  $\text{rot}(K) = 0$ ), so contact  $(-1)$ -surgery on it produces the circle bundle over the surface of genus  $g$  with Euler number  $2g - 2$ , cf. [20, Example 11.2.4]. Stabilising  $K$  before the surgery has the effect of reducing the Euler number. It turns out that each of the  $2g - 1 - e$  tight contact structures on  $M$  with negative twisting can be obtained via one of the  $2g - 1 - e$  different ways of performing  $2g - 2 - e$  stabilisations [25, Section 3.2]. The result will be universally tight, however, only in case the stabilisations are all of the same sign, see [20, p. 435].

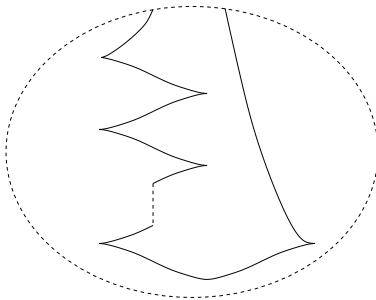


FIGURE 3. The stabilisations.

Notice that, since contact  $(-1)$ -surgery corresponds to attaching a Stein 2-handle, Figure 2 (with the appropriate stabilisations of  $K$ ) actually describes a Stein filling  $W_2$  of  $(M, \xi)$  by the disc bundle over  $\Sigma$  with Euler number  $e$ .

**Remarks.** (1) Lemma 3 also provides a strong symplectic filling of  $(M, \xi)$  with the topology of  $W_2$ , but this contains the closed surface  $\Sigma$  (i.e. the zero section) as a symplectic submanifold, so this filling is far from being exact.

(2) An alternative argument for the existence of a Stein structure on  $W_2$ , which does not rely on the classification of contact structures, can be given by appealing to a result of Bogomolov and de Oliveira [3]. Consider the holomorphic  $\mathbb{C}\mathbb{P}^1$ -bundle over  $\Sigma$  associated to the  $S^1$ -bundle of Euler number  $-e > 0$ . This has a 0- and an  $\infty$ -section. The complement of a neighbourhood of the 0-section gives a holomorphic filling of  $(M, \xi)$  with the desired topology of  $W_2$ . The main result from [3] then tells us that a small deformation of that holomorphic filling will be a Stein filling (or potentially the blow-up of a Stein filling, but this can only occur for  $g = 0$  and  $e = -1$ .)

(3) For symplectic manifolds  $(V, \omega)$  of higher dimensions  $2n > 4$ , the manifold  $W_2$  will be a 2-disc bundle over a  $(2n - 2)$ -dimensional symplectic submanifold  $\Sigma \subset V$ , so it has homology in dimension  $2n - 2 > n$  and cannot possibly carry a Stein structure. Note, however, that in order to extend our result to higher dimensions it would be enough to find an exact symplectic form (filling the boundary contact structure). As first observed by McDuff [28] and discussed further in [13, Section 7], the homological restrictions for Stein fillings do not apply to exact symplectic fillings.

*Proof of Theorem 4.* Simply take  $W_1$  as in Proposition 5, and  $W_2$  as in the preceding discussion.  $\square$



## 5. EXAMPLES

Here are a couple of simple cases where we do not need to invoke Donaldson's theorem for proving Theorem 4. In the second example we also show how to obtain additional information about the homotopy classification of the contact structures obtained via Theorem 1.

(1) If the symplectic manifold  $V$  we want to decompose as  $W_1 \cup \overline{W}_2$  is a ruled surface, that is, if it admits an  $S^2$ -bundle structure over some surface  $\Sigma$ , then we can take  $W_1 = W_2$  to be any disc bundle over  $\Sigma$  that admits a Stein structure, with even (resp. odd) Euler number  $e$  if  $V$  is the trivial (resp. non-trivial)  $S^2$ -bundle. (The condition for the existence of a Stein structure is that the sum of the Euler number  $e$  and the Euler characteristic of  $\Sigma$  be non-positive; see [20, Exercise 11.2.5].)

(2) Now let  $V$  be the complex projective plane  $\mathbb{C}\mathbb{P}^2$ . Our aim is to show that the decomposition  $\mathbb{C}\mathbb{P}^2 = W_1 \cup \overline{W}_2$  can be chosen in such a way that we can realise every homotopy class of almost contact structures on  $\mathbb{C}\mathbb{P}^2 \times S^1$  by a contact structure.

Almost contact structures on  $N := \mathbb{C}\mathbb{P}^2 \times S^1$  are determined by their first Chern class, since  $H^2(N)$  has no 2-torsion. (In [15, Proposition 8.1.1] this assumption on 2-torsion was erroneously omitted; for a correct proof of the preceding statement see [22].) The map  $[z_0 : z_1 : z_2] \mapsto [\bar{z}_0 : \bar{z}_1 : \bar{z}_2]$  defines an orientation-preserving diffeomorphism of  $\mathbb{C}\mathbb{P}^2$  that acts as minus the identity on  $H^2(\mathbb{C}\mathbb{P}^2)$ . Therefore, since the second Stiefel–Whitney class  $w_2(N)$  equals  $1 \in \mathbb{Z}_2 = H^2(N; \mathbb{Z}_2)$ , it suffices to show that for any odd  $c \in \mathbb{N}$  we can realise one of  $\pm c \in \mathbb{Z} \cong H^2(N)$  as the first Chern class of some contact structure; in other words, we need not worry about signs.

Choose a smooth complex projective curve  $C_d \subset \mathbb{C}\mathbb{P}^2$  of degree  $d \geq 2$ . Define  $W_1$  to be the complement of an open tubular neighbourhood of  $C_d$ , with symplectic form given by the restriction of the standard Kähler form on  $\mathbb{C}\mathbb{P}^2$  to  $W_1$  (possibly adjusted as in the proof of Proposition 5).

The genus  $g = g(d)$  of  $C_d$  — determined by the adjunction equality — equals  $g = (d-1)(d-2)/2$ . The manifold  $W_2$  in our general construction now is the normal disc bundle of  $C_d$ , but with reversed orientation. So the Euler number of this bundle is  $e = -[C_d]^2 = -d^2$ . (The condition for this to admit a Stein structure is  $0 \geq 2 - 2g + e = 3d - 2d^2$ , which is why we have to rule out the case  $d = 1$ .)

**Proposition 6.** *The described decomposition of  $\mathbb{C}\mathbb{P}^2$  (for any  $d \geq 2$ ) leads to a contact structure on  $\mathbb{C}\mathbb{P}^2 \times S^1$  with first Chern class  $\pm(2d-3)$ . This means that every homotopy class of almost contact structures on  $\mathbb{C}\mathbb{P}^2 \times S^1$  can be realised by a contact structure.*

*Proof.* We begin with the case  $d \geq 4$ . Here  $g > 1$ , so we can use the Stein surface  $W_2$  described in Figure 2 of Section 4. That is, we attach a 2-handle along the  $(2g-2-e)$ -fold positive or negative stabilisation  $S_{\pm}^{2g-2-e}K$  of the Legendrian knot  $K$  shown in that figure. In the present situation we have

$$2g - 2 - e = (d-1)(d-2) - 2 + d^2 = 2d^2 - 3d.$$

Recall from [20, Section 11.3] that the generator  $[C_d]$  of  $H_2(W_2)$  corresponds to the 2-handle attached along  $S_{\pm}^{2d^2-3d}K$  (with framing  $-1$  relative to the contact framing). Moreover, the first Chern class  $c_1(W_2)$  evaluates on that generator as the rotation number  $\text{rot}(S_{\pm}^{2d^2-3d}K) = \pm 2d^2 - 3d$  of the attaching circle.

The second homology of  $\mathbb{C}\mathbb{P}^2$  (and of  $\mathbb{C}\mathbb{P}^2 \times S^1$ ) is generated by the class  $[\mathbb{C}\mathbb{P}^1] = [C_d]/d$ . Write  $\eta = \ker \alpha$  for the contact structure on  $\mathbb{C}\mathbb{P}^2 \times S^1$  obtained via the construction in the proof of Theorem 1. Since  $d\alpha = d\lambda_2$  over  $\text{Int}(\overline{W}_2) \times S^1$ , we can now compute (possibly up to signs)

$$\begin{aligned} \langle c_1(\eta), [\mathbb{C}\mathbb{P}^1] \rangle &= \langle c_1(\eta), [C_d] \rangle / d = \langle c_1(W_2), [C_d] \rangle / d \\ &= \text{rot}(S_{\pm}^{2d^2-3d}K) / d = \pm(2d-3). \end{aligned}$$

In the case  $d = 3$  we have  $g = 1$  and  $e = -9$ . Lemma 3.9 of [25] still applies to show that the horizontal contact structure  $\xi$  we want to realise on  $M = \partial W_2$  is universally tight. Since  $(M, \xi)$  is symplectically filled by  $W_1$ , it follows from a result of Gay [11, Corollary 3] that its torsion in the sense of Giroux [18] is zero. This characterises  $\xi$  up to diffeomorphism, see [18, p. 686]. Therefore, we may again use the Stein manifold  $W_2$  from Figure 2 (with  $g = 1$ ), and the calculation of the first Chern class of  $\eta$  goes through as in the case  $d \geq 4$ .

Finally, for  $d = 2$  we have  $g = 0$  and  $e = -4$ . So here the separating hypersurface  $M$  is the lens space  $L(4, 1)$  with a contact structure  $\xi$  that admits a symplectic (in fact, Stein) filling by  $W_1$ , whose interior is diffeomorphic to the complement of a quadric in  $\mathbb{C}\mathbb{P}^2$ ; the manifold  $W_2$  is the disc bundle over  $S^2$  with Euler class  $-4$ . From the Mayer–Vietoris sequence of the splitting  $\mathbb{C}\mathbb{P}^2 = W_1 \cup_M \overline{W}_2$  one finds  $H_1(W_1) = \mathbb{Z}_2$ . Moreover, by the Seifert–van Kampen theorem, the inclusion  $L(4, 1) \subset W_1$  must induce a surjective homomorphism on fundamental groups. It follows that  $\pi_1(W_1) = \mathbb{Z}_2$ .

When we pass to the double cover of  $W_1$ , we obtain a symplectic filling of the double cover  $(\mathbb{R}\mathbb{P}^3, \tilde{\xi})$  of  $(L(4, 1), \xi)$ . Since there is a unique tight contact structure on  $\mathbb{R}\mathbb{P}^3$ , *viz.* the one covered by the standard contact structure on  $S^3$ , we conclude that  $\xi$  is a universally tight contact structure on  $L(4, 1)$ .

From the classification of tight contact structures on  $L(4, 1)$ , see [18] and [24], we know that there are two (resp. one) such structures up to isotopy (resp. diffeomorphism), given by the surgery diagram in Figure 4 and its mirror image. (Here we appeal again to [20, p. 435] in order to see that only the stabilisations  $S_{\pm}^2 K_0$  of the Legendrian unknot  $K_0$  give rise to a universally tight contact structure on  $L(4, 1)$ , whereas contact  $(-1)$ -surgery on  $S_+ S_- K_0$  gives the unique tight but virtually overtwisted contact structure.)

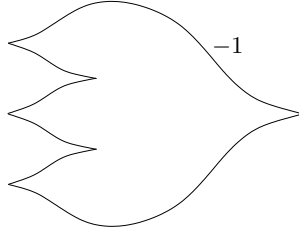


FIGURE 4. The universally tight contact structure on  $L(4, 1)$ .

Read as a Kirby diagram, that figure describes the desired Stein filling  $W_2$  of  $(L(4, 1), \xi)$  by the disc bundle over  $S^2$  with Euler class  $-4$ . Finally, we have  $\text{rot}(S_{\pm}^2 K_0)/2 = \pm 1$ , as desired.  $\square$

6. THE  $S^1$ -CONNECTED SUM

Let  $\iota_i: S^1 \times D^4 \rightarrow M_i$  ( $i = 1, 2$ ) be two orientation-preserving embeddings into oriented 5-dimensional manifolds. Choose an orientation-reversing diffeomorphism  $\phi$  of  $S^3 = \partial D^4$ . Then the  $S^1$ -connected sum of the two manifolds  $M_1, M_2$  is defined as

$$M_1 \#_{S^1} M_2 := (M_1 \setminus \iota_1(S^1 \times \text{Int}(D^4))) \cup_{\partial} (M_2 \setminus \iota_2(S^1 \times \text{Int}(D^4))),$$

where the boundaries are glued by

$$\iota_1(\theta, p) \sim \iota_2(\theta, \phi(p)) \text{ for } \theta \in S^1, p \in S^3.$$

**Remark.** Given an embedding of  $S^1 \equiv S^1 \times \{0\}$  into an oriented 5-manifold, there are two possible extensions (up to isotopy) to an embedding of  $S^1 \times D^4$ , since normal framings are classified by  $\pi_1(\text{SO}(4)) = \mathbb{Z}_2$ . So the notation  $\#_{S^1}$  is slightly ambiguous. In many cases discussed in [21] the diffeomorphism type of the resulting manifold does not actually depend on the choice of framing of the embedded circles. On the other hand, by Cerf's theorem there is a unique orientation-reversing diffeomorphism of  $S^3$  up to isotopy, so the particular choice of  $\phi$  is irrelevant. The construction in the following theorem goes through in all (odd) dimensions, but only with the specific choice of gluing described in the proof.

**Theorem 7.** *Let  $(M_1, \xi_1)$  and  $(M_2, \xi_2)$  be two 5-dimensional contact manifolds. Given two orientation-preserving embeddings  $S^1 \times D^4 \rightarrow M_i$  ( $i = 1, 2$ ), the corresponding  $S^1$ -connected sum  $M_1 \#_{S^1} M_2$  carries a contact structure. The embeddings of  $S^1 \times D^4$  can be isotoped in such a way that the contact structure on  $M_1 \#_{S^1} M_2$  coincides with  $\xi_i$  on  $M_i \setminus S^1 \times D^4$ .*

*Proof.* For the time being, we drop the subscript  $i$  and consider only a single embedding as in the theorem. The restriction of  $\xi$  to the embedded  $S^1 \equiv S^1 \times \{0\} \subset M$  is a trivial  $U(2)$ -bundle. Therefore, by the  $h$ -principle for isotropic immersions [9, Chapter 16], we may assume that  $S^1$  is tangent to  $\xi$ .

The conformal symplectic normal bundle (see [15]) of an isotropic  $S^1$  in a 5-dimensional contact manifold is a trivial  $U(1)$ -bundle. Since the natural homomorphism  $\pi_1(U(1)) \rightarrow \pi_1(\text{SO}(4))$  is surjective, the trivialisation of this conformal symplectic normal bundle can be chosen in such a way that the corresponding framing of  $S^1$  coincides with the one given by the embedding of  $S^1 \times D^4$  into  $M$ .

The neighbourhood theorem for isotropic embeddings [15, Theorem 2.5.8] then implies that in a neighbourhood of  $S^1$  the contact structure  $\xi$  can be given by the contact form

$$\alpha := x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 + x_3 dy_3 - y_3 dx_3,$$

where the normal framing of  $S^1$  given by  $\partial_{y_1}, \partial_{y_2}, \partial_{x_3}, \partial_{y_3}$  corresponds to that of the given embedding. Here  $S^1$  is being identified with

$$S^1 = \{x_1^2 + x_2^2 = 1, y_1 = y_2 = x_3 = y_3 = 0\};$$

the conformal symplectic normal bundle is the trivial bundle spanned by  $\partial_{x_3}, \partial_{y_3}$ .

The vector field

$$X := y_1 \partial_{y_1} + y_2 \partial_{y_2} + \frac{1}{2} x_3 \partial_{x_3} + \frac{1}{2} y_3 \partial_{y_3}$$

is a contact vector field transverse to the boundary  $\Sigma := S^1 \times S^3$  of a small tubular neighbourhood of  $S^1 \subset M$ , i.e. the flow of  $X$  preserves  $\xi$ . (So  $\Sigma$  is a convex

hypersurface in the sense of Giroux [17].) Our goal will be to use this information in order to define a contactomorphism of a neighbourhood of  $\Sigma$  that sends  $\Sigma$  to itself but reverses the normal direction. Such a contactomorphism can then be used to effect the desired gluing in the theorem.

Use the flow of  $X$  to identify a neighbourhood of  $\Sigma$  in  $M$  with a neighbourhood of  $\Sigma \equiv \Sigma \times \{0\}$  in  $\Sigma \times \mathbb{R}$ ; the vector field  $X$  is then identified with  $\partial_t$ , where  $t$  denotes the  $\mathbb{R}$ -coordinate. Write

$$(x_1, x_2) = (\cos \theta, \sin \theta)$$

and

$$(y_1, y_2, x_3, y_3) = (e^t u_1, e^t u_2, e^{t/2} v_3, e^{t/2} w_3).$$

Thus,  $\theta$  is the coordinate on the  $S^1$ -factor, and  $(u_1, u_2, v_3, w_3)$  may be interpreted as coordinates on the  $S^3$ -factor of  $\Sigma$ .

In these coordinates, the  $X$ -invariant contact form  $\alpha_0 := e^{-t}\alpha$  is given by

$$\begin{aligned} \alpha_0 &= (u_1 \cos \theta + u_2 \sin \theta) dt + (u_1 \sin \theta - u_2 \cos \theta) d\theta \\ &\quad + \cos \theta du_1 + \sin \theta du_2 + v_3 dw_3 - w_3 dv_3. \end{aligned}$$

Replace the coordinates  $(u_1, u_2)$  by

$$(v_1, v_2) := (u_1 \cos \theta + u_2 \sin \theta, u_1 \sin \theta - u_2 \cos \theta).$$

Then  $\alpha_0$  is written as

$$\alpha_0 = dv_1 + v_1 dt + 2v_2 d\theta + v_3 dw_3 - w_3 dv_3.$$

As explained in [15, p. 180], we may in fact identify a neighbourhood of  $\Sigma$  in  $M$  contactomorphically with *all of*  $\Sigma \times \mathbb{R}$  with this  $\mathbb{R}$ -invariant contact form.

Now consider the following family of  $\mathbb{R}$ -invariant 1-forms on  $\mathbb{R} \times \Sigma$ :

$$\begin{aligned} \alpha_\varphi &:= dv_1 + (v_1 \cos \varphi - v_2 \sin \varphi) dt \\ &\quad + 2(v_1 \sin \varphi + v_2 \cos \varphi) d\theta + v_3 dw_3 - w_3 dv_3, \quad \varphi \in [0, \pi/2]. \end{aligned}$$

This is a contact form for each  $\varphi \in [0, \pi/2]$  since, on the universal cover  $\mathbb{R} \times \mathbb{R} \times S^3$  of  $\mathbb{R} \times \Sigma$ , this is simply the pull-back of  $\alpha_0$  under the coordinate transformation

$$(t, \theta) \longmapsto (t \cos \varphi + 2\theta \sin \varphi, -\frac{t}{2} \sin \varphi + \theta \cos \varphi).$$

Therefore, Gray stability (which applies on the non-compact manifold  $\mathbb{R} \times \Sigma$  because of the  $\mathbb{R}$ -invariance of our contact forms) gives us an isotopy whose time- $(\pi/2)$  map  $\phi$  pulls back  $\alpha_0$  to

$$\alpha_{\pi/2} = dv_1 - v_2 dt + 2v_1 d\theta + v_3 dw_3 - w_3 dv_3,$$

up to multiplication by some positive function.

The map

$$\psi: (t, \theta, v_1, v_2, v_3, w_3) \longmapsto (-t, \theta, v_1, -v_2, v_3, w_3)$$

is a contactomorphism for  $\ker \alpha_{\pi/2}$  that sends  $\Sigma$  to itself, reversing both the orientation of  $\Sigma$  and its normal orientation. So  $\phi \circ \psi \circ \phi^{-1}$  is a contactomorphism for  $\xi = \ker \alpha_0$  that preserves the isotopic copy  $\phi(\Sigma)$  of  $\Sigma$  while reversing its normal orientation. As explained, such a contactomorphism allows us to perform the  $S^1$ -connected sum of two contact manifolds.  $\square$

## 7. FIVE-MANIFOLDS WITH FUNDAMENTAL GROUP OF ORDER TWO

In [16] it was shown that every closed, orientable 5-manifold  $M$  with fundamental group  $\mathbb{Z}_2$  and second Stiefel–Whitney class  $w_2(M)$  equal to zero on homology admits a contact structure. The basis of this result was a structure theorem, also proved in [16], according to which any such manifold can be obtained from one of ten explicit model manifolds by surgery along 2-spheres.

An orientable 5-manifold is said to be of *fibred type* if the second homotopy group  $\pi_2(M)$  is a trivial  $\mathbb{Z}[\pi_1(M)]$ -module. In [21] Hambleton and Su give an explicit description of the fibred type 5-manifolds with fundamental group  $\mathbb{Z}_2$  and torsion-free second homology. Not all the manifolds discussed in [16] are of fibred type: one of the ten model manifolds fails to be so, and surgery along 2-spheres will also destroy that property, in general. On the other hand, the list in [21] contains manifolds where  $w_2(M)$  does not vanish on homology.

As Hambleton and Su point out, all the manifolds in their list have vanishing third integral Stiefel–Whitney class and thus admit an almost contact structure.

**Proposition 8.** *Every closed, orientable, fibred type 5-manifold with fundamental group  $\mathbb{Z}_2$  and torsion-free second homology admits a contact structure.*

*Proof.* According to [21, Theorem 3.6], every such manifold is an  $S^1$ -connected sum of some of the following manifolds:

- (i) one of the nine model manifolds from [16] with  $w_2 \neq 0$ ,
- (ii)  $S^2 \times \mathbb{R}\mathbb{P}^3$ ,
- (iii)  $(\#_k S^2 \times S^2) \times S^1$ ,
- (iv)  $\mathbb{C}\mathbb{P}^2 \times S^1$ .

By Theorem 7 it suffices to show that these individual manifolds carry a contact structure. (There is no orientation issue, because all the manifolds in this list admit an orientation-reversing diffeomorphism.) The manifolds in (i) are covered by [16]. The manifold in (ii) is the unit cotangent bundle of  $\mathbb{R}\mathbb{P}^3$ . The manifolds in (iii) and (iv) are covered by Corollary 2. (For (iii) one may alternatively think of  $S^2 \times S^2 \times S^1$  as the unit cotangent bundle of  $S^2 \times S^1$  and then take  $S^1$ -connected sums.) This completes the proof.  $\square$

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## REFERENCES

- [1] S. AKBULUT AND R. MATVEYEV, A convex decomposition theorem for 4-manifolds, *Int. Math. Res. Not.* **1998**, no. 7, 371–381.
- [2] R. İ. BAYKUR, Kähler decompositions of 4-manifolds, *Algebr. Geom. Topol.* **6** (2006), 1239–1265.

- [3] F. A. BOGOMOLOV AND B. DE OLIVEIRA, Stein small deformations of strictly pseudoconvex surfaces, in: *Birational Algebraic Geometry* (Baltimore, 1996), Contemp. Math. **207**, American Mathematical Society, Providence, RI (1997), 25–41.
- [4] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, *Ann. of Math. (2)* **68** (1958), 721–734.
- [5] R. BOTT AND L. W. TU, *Differential Forms in Algebraic Topology*, Grad. Texts in Math. **82**, Springer-Verlag, Berlin (1982).
- [6] F. BOURGEOIS, Odd dimensional tori are contact manifolds, *Int. Math. Res. Not.* **2002**, no. 30, 1571–1574.
- [7] S. K. DONALDSON, Symplectic submanifolds and almost-complex geometry, *J. Differential Geom.* **44** (1996), 666–705.
- [8] YA. ELIASHBERG, On symplectic manifolds with some contact properties, *J. Differential Geom.* **33** (1991), 233–238.
- [9] YA. ELIASHBERG AND N. MISHACHEV, *Introduction to the h-Principle*, Grad. Stud. Math. **48**, American Mathematical Society, Providence, RI (2002).
- [10] J. B. ETNYRE, Symplectic convexity in low-dimensional topology, *Topology Appl.* **88** (1998), 3–25.
- [11] D. T. GAY, Four-dimensional symplectic cobordisms containing three-handles, *Geom. Topol.* **10** (2006), 1749–1759.
- [12] H. GEIGES, Constructions of contact manifolds, *Math. Proc. Cambridge Philos. Soc.* **121** (1997), 455–464.
- [13] H. GEIGES, Applications of contact surgery, *Topology* **36** (1997), 1193–1220.
- [14] H. GEIGES, Contact Dehn surgery, symplectic fillings, and property P for knots, *Expo. Math.* **24** (2006), 273–280.
- [15] H. GEIGES, *An Introduction to Contact Topology*, Cambridge Stud. Adv. Math. **109**, Cambridge University Press, Cambridge (2008).
- [16] H. GEIGES AND C. B. THOMAS, Contact topology and the structure of 5-manifolds with  $\pi_1 = \mathbb{Z}_2$ , *Ann. Inst. Fourier (Grenoble)* **48** (1998), 1167–1188.
- [17] E. GIROUX, Convexité en topologie de contact, *Comment. Math. Helv.* **66** (1991), 637–677.
- [18] E. GIROUX, Structures de contact en dimension trois et bifurcations des feuilletages de surfaces, *Invent. Math.* **141** (2000), 615–689.
- [19] E. GIROUX, Géométrie de contact: de la dimension trois vers les dimensions supérieures, in: *Proceedings of the International Congress of Mathematicians*, Vol. 2 (Beijing, 2002), Higher Ed. Press, Beijing (2002), 405–414.
- [20] R. E. GOMPFF AND A. I. STIPSICZ, *4-Manifolds and Kirby Calculus*, Grad. Stud. in Math. **20**, American Mathematical Society, Providence, RI (1999).
- [21] I. HAMBLETON AND Y. SU, On certain 5-manifolds with fundamental group of order 2, preprint, [arXiv:0903.5244v3](https://arxiv.org/abs/0903.5244v3) (2009).
- [22] M. HAMILTON, On symplectic 4-manifolds and contact 5-manifolds, Dissertation, Ludwig-Maximilians-Universität, München (2008).
- [23] F. HIRZEBRUCH AND H. HOPF, Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten, *Math. Ann.* **136** (1958), 156–172.
- [24] K. HONDA, On the classification of tight contact structures I, *Geom. Topol.* **4** (2000), 309–368; erratum: Factoring nonrotative  $T^2 \times I$  layers, *Geom. Topol.* **5** (2001), 925–938.
- [25] K. HONDA, On the classification of tight contact structures II, *J. Differential Geom.* **55** (2000), 83–143.
- [26] E. LERMAN, Contact fiber bundles, *J. Geom. Phys.* **49** (2004), 52–66.
- [27] P. LISCA AND A. I. STIPSICZ, Tight, not semi-fillable contact circle bundles, *Math. Ann.* **328** (2004), 285–298.
- [28] D. MCDUFF, Symplectic manifolds with contact type boundaries, *Invent. Math.* **103** (1991), 651–671.
- [29] D. MCDUFF AND D. SALAMON, *Introduction to Symplectic Topology*, 2nd edition, Oxford Math. Monogr., Oxford University Press, Oxford (1998).
- [30] K. NIEDERKRÜGER, Compact Lie group actions on contact manifolds, Dissertation, Universität zu Köln (2005).
- [31] B. ÖZBAĞCI AND A. I. STIPSICZ, *Surgery on Contact 3-Manifolds and Stein Surfaces*, Bolyai Soc. Math. Stud. **13**, Springer-Verlag, Berlin (2004).

- [32] A. WEINSTEIN, Contact surgery and symplectic handlebodies, *Hokkaido Math. J.* **20** (1991), 241–251.

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