Moufang planes with the Newton property

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Abstract

We prove that a Moufang plane with the Fano property satisfies the Newton property if and only if it is a Pappian projective plane.

1 Preliminaries

Throughout the paper we follow the conventions and notation of our earlier paper [7], where the necessary background material concerning projective planes is also summarized.

Let $\mathcal{R}$ be a set and $+, \cdot$ be binary operations on $\mathcal{R}$ such that

- $(\mathcal{R}, +)$ is a commutative group with zero element 0;
- $a \cdot 0 = 0 \cdot a = 0$ for all $a \in \mathcal{R}$;
- $(\mathcal{R} \setminus \{0\}, \cdot)$ is a loop;
- $a \cdot (b + c) = a \cdot b + a \cdot c$ $(a, b, c \in \mathcal{R})$;
- $(a + b) \cdot c = a \cdot c + b \cdot c$ $(a, b, c \in \mathcal{R})$;
- $a \cdot (a \cdot b) = (a \cdot a) \cdot b$ $(a, b \in \mathcal{R})$;
- $a \cdot (b \cdot b) = (a \cdot b) \cdot b$ $(a, b \in \mathcal{R})$.

Then $(\mathcal{R}, +, \cdot)$ is called an *alternative division ring*. In the following we will write simply ab instead of $a \cdot b$. We denote the unit of $(\mathcal{R} \setminus \{0\}, \cdot)$ by 1. In every alternative division ring for all $a \in \mathcal{R} \setminus \{0\}$ there are $a', a'' \in \mathcal{R}$ such that $aa' = 1$, $a''a = 1$, and $a' = a''$. This element is called the inverse of a and is denoted by a^{-1}.

By a difficult theorem of Bruck-Kleinfield [1] and Skornyakov [4], an alternative division ring either is associative or is a Cayley-Dickson algebra over some field. From this it follows that in every alternative division ring we have the *inverse property*

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$$a(a^{-1}b) = (ba^{-1})a = b \text{ for all } a \in \mathcal{R}\setminus\{0\}\,, b \in \mathcal{R},$$

since this holds in every Cayley-Dickson algebra.

The incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$, where

- $\mathcal{P} := \{[x, y, 1], [1, x, 0], [0, 1, 0] \mid x, y \in \mathcal{R}\}$;
- $\mathcal{L} := \{\langle a, 1, b\rangle\,, \langle 1, 0, a\rangle\,, \langle 0, 0, 1\rangle \mid a, b \in \mathcal{R}\}$;
- $([x, y, z], \langle a, b, c\rangle) \in \mathcal{I}$ if and only if $xa + yb + zc = 0$

is a projective plane called *the projective plane over the alternative division ring* $\mathcal{R}$.

A projective plane is called a *Moufang plane*, if it satisfies the Little Desargues Property, i.e., if two triangles ABC and $A'B'C'$ are centrally perspective from a point O, and the points $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $\overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$ and O are collinear, then the triangles are also axially perspective. A projective plane is a Moufang plane if and only if it can be coordinatized by an alternative division ring, i.e., it is isomorphic to a projective plane over an alternative division ring. For a proof see [3] or [6].

A Moufang plane is Desarguesian if and only if the coordinatizing alternative division ring $\mathcal{R}$ is associative, i.e., $a(bc) = (ab)c$ for all $a, b, c \in \mathcal{R}$, and hence $\mathcal{R}$ is a skewfield. A Desarguesian plane is Pappian if and only if the coordinatizing skewfield $\mathbb{K}$ is commutative: for all $a, b \in \mathbb{K}$ $ab = ba$ holds. By a theorem of Bruck (see [5]) if an alternative division ring is commutative, then it is associative. This implies that a Moufang plane is Pappian if and only if the coordinatizing alternative division ring $\mathcal{R}$ is commutative.

The Artin-Zorn theorem ([3]) states that every finite alternative division ring is a skewfield, so it follows that every finite Moufang plane is Desarguesian. Also, by Wedderburn's theorem, every finite skewfield is a field, therefore every finite Moufang plane is Pappian.

It can be shown (see e.g. [6]) that the collineation group of a Moufang plane acts transitively on four-points: if (A_1, A_2, A_3, A_4) and (A'_1, A'_2, A'_3, A'_4) are quadruples of points in general position (i.e., no three of them are collinear), then there is a collineation that sends A_i to A'_i for all $i \in \{1, 2, 3, 4\}$. From this it follows that for any Moufang plane the coordinatizing ring can be chosen such that the coordinates of four given points in general position are $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, $[1, 1, 1]$.

The concept of *cross-ratio* is generalized to the case of Moufang planes in [2]. If $A[a_1, a_2, 1]$ and $B[b_1, b_2, 1]$ are points in a projective plane over an alternative division ring $\mathcal{R}$, then the points of the line $\overleftrightarrow{AB}$ are of the form

$$[t(a_1, a_2, 1) + (1 - t)(b_1, b_2, 1)] \ (t \in \mathcal{R})$$

or $[1, x, 0]$. Thus the points of $\overleftrightarrow{AB}$ can be identified by the elements of $\mathcal{R} \cup \{\infty\}$, where $[1, x, 0]$ is identified with (∞), and any other point of $\overleftrightarrow{AB}$ is identified in an obvious way with (t).

If A, B, C, D are collinear points identified by $a, b, c, d \in \mathcal{R}$, then

$$(ABCD) := ((a-d)^{-1}(b-d))((b-c)^{-1}(a-c)).$$

If one of the points is identified by (∞), then the cross-ratio is defined as follows:

$$(\infty BCD) := (b-d)(b-c)^{-1} \ ; \ (A\infty CD) := (a-d)^{-1}(a-c);$$

$$(AB\infty D) := (a-d)^{-1}(b-d) \ ; \ (ABC\infty) := (b-c)^{-1}(a-c).$$

In [2] it is shown that in a Moufang plane four collinear points form a harmonic quadruple if and only if their cross-ratio is -1.

In a Desarguesian projective plane for any triple of distinct collinear points (A, B, C) there is a unique point D, called the harmonic conjugate of C with respect to (A, B), such that $(ABCD)$ is harmonic. We call this property the *uniqueness of the fourth harmonic point*. In [6] it is shown that it is not necessary to suppose the Desargues property: in a projective plane the uniqueness of the fourth harmonic point holds if and only if the plane is a Moufang plane with the Fano property.

We recall that a projective plane satisfies the *Newton property* if the following holds:

(N) Let *abcd* be a complete quadrilateral and l be a line. If P, Q and R are the intersections of l and the diagonals of *abcd*, then the harmonic conjugates of P, Q and R with respect to the corresponding vertices are collinear.

In our previous paper [7] we have proved that a Desarguesian plane satifying the Fano axiom has the Newton property if and only if it is Pappian. Since the uniqueness of the fourth harmonic point holds in every Moufang plane with the Fano property, the Newton property has meaning in Moufang planes with the Fano property. So it is natural to ask whether this theorem can be generalized to Moufang planes. We answer this question affirmatively and prove the following theorem.

Theorem 1.1 *A Moufang plane with the Fano property satisfies the Newton property if and only if it is Pappian.*

2 Proof of the main theorem

In this section we will use the notation of the figure above. We have to prove that P', Q' and R' are collinear if and only if the given Moufang plane is Pappian. The Moufang plane can be coordinatized by an alternative division ring $\mathcal{R}$. By our previous remarks it is enough to prove that P', Q' and R' are collinear if and only if $\mathcal{R}$ is commutative.

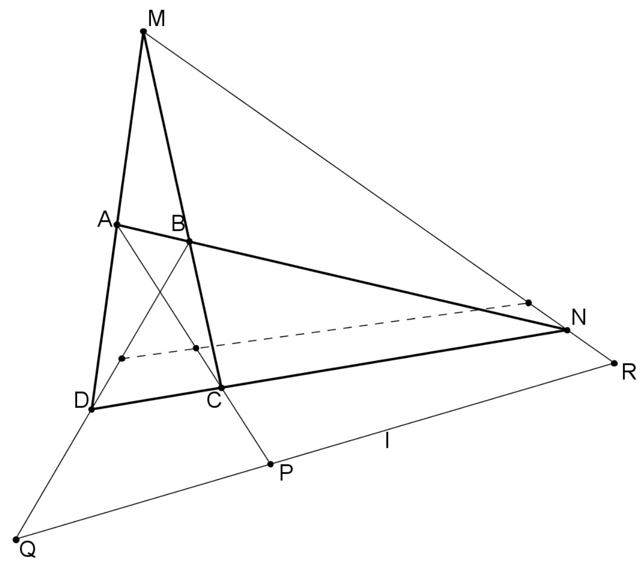

Figure 1: The Newton property

We choose the coordinates such that $A=[0,0,1]$, $B=[1,1,1]$, $C=[1,0,0]$, $D=[0,1,0]$. Then easy calculations show that $M=[0,1,1]$, $N=[1,1,0]$. The point P is $[k,0,1]$ and Q is $[1,l+1,1]$ for some $k,l\in\mathcal{R}\setminus\{0\}$.

Next, we calculate the coordinates of the point R. This point lies on the line $\overleftrightarrow{PQ}$. The coordinates of $\overleftrightarrow{PQ}$ are $\langle a_1,1,c_1\rangle$ such that

$$ka_1+c_1=0$$

and

$$a_1+(l+1)+c_1=0.$$

If $k=1$, then for all $l\in\mathcal{R}$, $kl=lk=-1$. Otherwise, from these equations we get

$$a_1=(k-1)^{-1}(l+1)$$

and

$$c_1=-k[(k-1)^{-1}(l+1)],$$

thus $\overleftrightarrow{PQ}$ is the line $\big\langle(k-1)^{-1}(l+1),1,-k[(k-1)^{-1}(l+1)]\big\rangle$. A simple calculation shows that the line $\overleftrightarrow{MN}$ is $\langle -1,1,-1\rangle$. So for the coordinates $[r_1,r_2,1]$ of R we have the following system of equations:

$$\left.\begin{array}{r} -r_1+r_2-1=0 \\ r_1[(k-1)^{-1}(l+1)]+r_2-k[(k-1)^{-1}(l+1)]=0 \end{array}\right\}.$$

From this we find that $R = [r_1, r_1 + 1, 1]$, where

$$r_1 = (k[(k-1)^{-1}(l+1)] - 1)[(k-1)^{-1}(l+1) + 1]^{-1}.$$

If $(k-1)^{-1}(l+1) + 1 = 0$, then an easy calculation shows that $k = -l$, whence $kl = lk = -k^2$.

Now we want to calculate the coordinates of the point P'. On the line $\overleftrightarrow{AC}$ the points A, C and P are identified by (0), (∞) and (k), respectively. So P' is identified by (p') such that $(-p')^{-1}(-k) = -1$. Thus $p' = -k$, and therefore $P' = [-k, 0, 1]$. Similarly we get that $Q' = [1, 1-l, 1]$. And finally, on the line $\overleftrightarrow{MN}$ the points M, N, R, $R' = [r_1', r_1' + 1, 1]$ are identified by (0), (∞), (r_1) and (r_1'), so

$$r_1' = -r_1 = -(k[(k-1)^{-1}(l+1)] - 1)[(k-1)^{-1}(l+1) + 1]^{-1}.$$

The Newton property holds if and only if P', Q' and R' are collinear. We calculate the coordinates $\langle a_2, 1, c_2 \rangle$ of the line $\overleftrightarrow{P'Q'}$. Since P' is on this line,

$$-ka_2 + c_2 = 0,$$

and Q' is also on the line, so

$$a_2 + 1 - l + c_2 = 0.$$

If $k+1 = 0$, then $k = -1$, thus for any $l \in \mathcal{R}$, we have $kl = lk = -l$. Otherwise, from this system of equations we obtain that the coordinates of $\overleftrightarrow{P'Q'}$ are

$$\left\langle (k+1)^{-1}(l-1), 1, k[(k+1)^{-1}(l-1)] \right\rangle.$$

The point $R' = [r_1', r_1' + 1, 1]$ is on this line if and only if

$$r_1'[(k+1)^{-1}(l-1)] + r_1' + 1 + k[(k+1)^{-1}(l-1)] = 0.$$

If $1 + [(k+1)^{-1}(l-1)] = 0$, then easy calculation shows that $k = -l$, thus $kl = lk = -k^2$. Otherwise, we get that P', Q' and R' are collinear if and only if

$$r_1' = -(1 + k[(k+1)^{-1}(l-1)])(1 + [(k+1)^{-1}(l-1)])^{-1}.$$

It follows that in a Moufang plane the Newton property holds if and only if for any $k, l \in \mathcal{R} \setminus \{0\}$

$$[k[(k-1)^{-1}(l+1)] - 1][(k-1)^{-1}(l+1) + 1]^{-1} =$$
$$= [1 + k[(k+1)^{-1}(l-1)]][1 + [(k+1)^{-1}(l-1)]]^{-1} \tag{1}$$

holds. In the cases $k = -l$, $k = -1$, $l = -1$, and $k = 1$ mentioned above, the commutativity holds trivially.

Let

$$a := (k-1)^{-1}(l+1)$$

and

$$b := (k+1)^{-1}(l-1).$$

Then we get $l+1 = (k-1)a$ and $l-1 = (k+1)b$, so

$$(k-1)a - 1 = (k+1)b + 1. \tag{2}$$

If $a = b$, then

$$\begin{aligned}
(k-1)^{-1}(l+1) &= (k+1)^{-1}(l-1),\\
(k+1)(k-1)^{-1} &= (l-1)(l+1)^{-1},\\
(k-1+2)(k-1)^{-1} &= (l+1-2)(l+1)^{-1},\\
(k-1)(k-1)^{-1} + 2(k-1)^{-1} &= (l+1)^{-1} - 2(l+1)^{-1},\\
(k-1)^{-1} &= -(l+1)^{-1}
\end{aligned}$$

shows that $k = -l$, so $kl = lk$ holds. Otherwise, (2) gives

$$k = ((a+1) + (b+1))(a-b)^{-1}. \tag{3}$$

With our notation, equation (1) takes the form

$$(ka-1)(a+1)^{-1} = (kb+1)(b+1)^{-1}.$$

From this we obtain

$$(ka)(a+1)^{-1} - (a+1)^{-1} = (b+1)^{-1} + (kb)(b+1)^{-1}. \tag{4}$$

Here, using the inverse property,

$$\begin{aligned}
(ka)(a+1)^{-1} &= (ka+k-k)(a+1)^{-1} = [k(a+1)-k](a+1)^{-1}\\
&= [k(a+1)](a+1)^{-1} - k(a+1)^{-1} = k - k(a+1)^{-1}.
\end{aligned}$$

Similarly,

$$(kb)(b+1)^{-1} = k - k(b+1)^{-1},$$

so equation (4) is equivalent to the following:

$$k[(b+1)^{-1} - (a+1)^{-1}] = (a+1)^{-1} + (b+1)^{-1}.$$

If $(b+1)^{-1} - (a+1)^{-1} = 0$, then $a = b$. As we have already shown, in this case $k = -l$. Otherwise we get

$$k = [(a+1)^{-1} + (b+1)^{-1}][(b+1)^{-1} - (a+1)^{-1}]^{-1}.$$

Taking (3) into account we find that

$$((a+1)+(b+1))(a-b)^{-1} = [(a+1)^{-1} + (b+1)^{-1}][(b+1)^{-1} - (a+1)^{-1}]^{-1}.$$

Introduce the notation $s := a + 1$ and $t := b + 1$. Then the Newton property is equivalent to

$$(s + t)(s - t)^{-1} = (s^{-1} + t^{-1})(t^{-1} - s^{-1})^{-1}, \tag{5}$$

for all $s, t \in \mathcal{R}$.

Taking into account that

$$(s + t)(s - t)^{-1} = (s - t + 2t)(s - t)^{-1} = 1 + 2t(s - t)^{-1},$$

and, similarly,

$$(s^{-1} + t^{-1})(t^{-1} - s^{-1})^{-1} = 1 + 2s^{-1}(t^{-1} - s^{-1})^{-1},$$

it follows that (5) is equivalent to

$$t(s - t)^{-1} = s^{-1}(t^{-1} - s^{-1})^{-1}.$$

Multiplying both sides by s from the left and by $(s-t)$ from the right we obtain

$$st = (t^{-1} - s^{-1})^{-1}(s - t).$$

Multiplying both sides by $(t^{-1} - s^{-1})$ from the right we get

$$(t^{-1} - s^{-1})(st) = s - t,$$

$$t^{-1}(st) - s^{-1}(st) = s - t.$$

By the inverse property $s^{-1}(st) = t$, our last relation can be written equivalently as follows:

$$t^{-1}(st) - t = s - t,$$

$$t^{-1}(st) = s,$$

and finally,

$$st = ts.$$

For all $s, t \in \mathcal{R}$, there are $a, b \in \mathcal{R}$ such that $s = a + 1$ and $t = b + 1$: $a = s - 1$ and $b = t - 1$. We verify that for all $a, b \in \mathcal{R}$ ($a \neq b$) there are $k, l \in \mathcal{R}$ such that

$$a = (k - 1)^{-1}(l + 1)$$

and

$$b = (k + 1)^{-1}(l - 1).$$

Indeed, our equations give

$$(k - 1)a = l + 1 \tag{6}$$

and

$$(k + 1)b = l - 1. \tag{7}$$

Expressing l from both equations we get

$$(k-1)a-1=(k+1)b+1,$$

$$k(a-b)=a+b+2,$$

$$k=(a+b+2)(a-b)^{-1}.$$

Substituting k into (6) we find that

$$l=[(a+b+2)(a-b)^{-1}-1]a-1.$$

We verify that the pair (k,l) so obtained also satisfies (7). Indeed,

$$[(a+b+2)(a-b)^{-1}+1]b=[(a+b+2)(a-b)^{-1}-1]a-2,$$

$$[(a+b+2)(a-b)^{-1}](a-b)=a+b+2,$$

and this holds by the inverse property.

Thus we conclude that every $(k,l)\in\mathcal{R}\times\mathcal{R}$ can be written in the form (s,t), so from the proved equality $st=ts$, it follows that $kl=lk$, for all $k,l\in\mathcal{R}$. Therefore, we have shown that the Newton property is equivalent to the commutativity of $\mathcal{R}$, and hence, by our previous remarks, to the Pappos property.

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