# Two applications of the theorem of Carnot

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**Abstract**

Using the theorem of Carnot we give elementary proofs of two statements of C. Bradley. We prove his conjecture concerning the tangents to an arbitrary conic from the vertices of a triangle. We give a synthetic proof of his theorem concerning the "Cevian conic", and we also give a projective generalization of this result.

## 1 Preliminaries

Throughout this paper we work in the Euclidean plane and in its projective closure, the real projective plane. By $XY$ we denote the *signed* distance of points $X$, $Y$ of the Euclidean plane. This means that we suppose that on the line $\overleftrightarrow{XY}$ an orientation is given, and $XY = d(X,Y)$ or $XY = -d(X,Y)$ depending on the direction of the vector $\overrightarrow{XY}$. The *simple ratio* of the collinear points $X$, $Y$, $Z$ (where $Y \neq Z$ and $X \neq Y$) is defined by

$$(XYZ) := \frac{XZ}{ZY}$$

and it is independent of the choice of orientation on the line $\overleftrightarrow{XY}$, thus in our formulas we can use the notation $XY$ without mentioning the orientation.

We recall here the most important tools that we use in our paper. The proofs of these theorems can be found in [4].

**Theorem 1.1** *Let $ABC$ be an arbitrary triangle in the Euclidean plane, and let $A_1$, $B_1$, $C_1$ be points (different from the vertices) on the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$, respectively. Then*

- *(Menelaos) $A_1$, $B_1$, $C_1$ are collinear if and only if*

$$(ABC_1)(BCA_1)(CAB_1) = -1,$$

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- *(Ceva)* $\overleftrightarrow{AA_1}$, $\overleftrightarrow{BB_1}$, $\overleftrightarrow{CC_1}$ *are concurrent if and only if*

$$(ABC_1)(BCA_1)(CAB_1) = 1.$$

Referring to the theorem of Ceva, if $P$ is a point that is not incident to any side of the triangle, we call the lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, $\overleftrightarrow{CP}$ *Cevians*, and we call the points $\overleftrightarrow{AP} \cap \overleftrightarrow{BC}$, $\overleftrightarrow{BP} \cap \overleftrightarrow{AC}$, $\overleftrightarrow{CP} \cap \overleftrightarrow{AB}$ the *feet of the Cevians through $P$*.

Now we formulate the most important theorem on projective conics, the *theorem of Pascal* (together with its converse). We note that this theorem is valid not only in the real projective plane, but in any projective plane over a field (i.e. in any Pappian projective plane).

**Theorem 1.2** *(Pascal) Suppose that the points $A$, $B$, $C$, $D$, $E$, $F$ of the real projective plane are in general position (i.e. no three of them are collinear). Then there is a conic incident with these points if and only if the points $\overleftrightarrow{AB} \cap \overleftrightarrow{DE}$, $\overleftrightarrow{BC} \cap \overleftrightarrow{EF}$ and $\overleftrightarrow{CD} \cap \overleftrightarrow{FA}$ are collinear.*

## 2 The theorem of Carnot

The theorem of Menelaos gives a necessary and sufficient condition for points on the sides of a triangle to be collinear. The theorem of Carnot is a natural generalization of this theorem, and gives a necessary and sufficient condition for two points on each side of a triangle to form a conic. The proof ([4]) depends on the theorems of Menelaos and Pascal. For completeness we recall it here.

**Theorem 2.1** *(Carnot) Let $ABC$ be an arbitrary triangle in the Euclidean plane, and let $(A_1, A_2)$, $(B_1, B_2)$, $(C_1, C_2)$ be pairs points (different from the vertices) on the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AB}$, respectively. Then the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ és $C_2$ are on a conic if and only if*

$$(ABC_1)(ABC_2)(BCA_1)(BCA_2)(CAB_1)(CAB_2) = 1.$$

*Proof.* Let $A_3 := \overleftrightarrow{BC} \cap \overleftrightarrow{B_2C_1}$, $B_3 := \overleftrightarrow{AC} \cap \overleftrightarrow{A_1C_2}$ and $C_3 := \overleftrightarrow{AB} \cap \overleftrightarrow{A_2B_1}$. By the theorem of Pascal $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ are on a conic if and only if $A_3$, $B_3$, $C_3$ are collinear. Thus we have to prove that the collinearity of these points is equivalent to the condition above.

Since $A_3$, $B_2$, $C_1$ are collinear, by the theorem of Menelaos

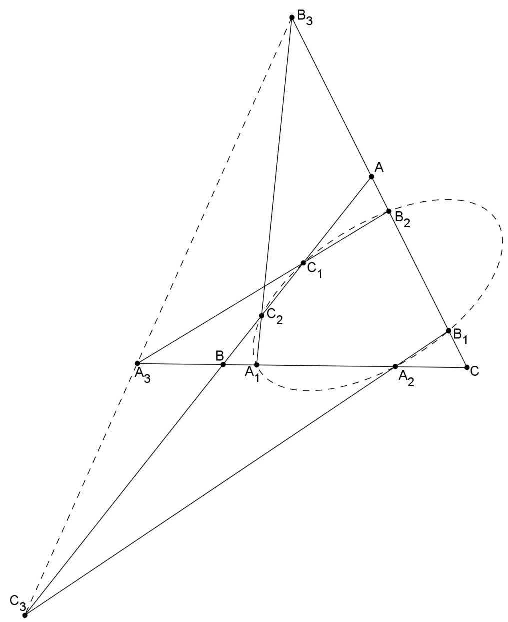
$$(ABC_1)(BCA_3)(CAB_2) = -1.$$

Similarly,

$$(ABC_2)(BCA_1)(CAB_3) = -1$$

and

$$(ABC_3)(BCA_2)(CAB_1) = -1.$$




Multiplying these equalities we get

$$(ABC_1)(ABC_2)(ABC_3)(BCA_1)(BCA_2)(BCA_3)(CAB_1)(CAB_2)(CAB_3) = -1.$$

Using the theorem of Menelaos again, $A_3$, $B_3$, $C_3$ are collinear if and only if

$$(ABC_3)(BCA_3)(CAB_3) = -1.$$

By our previous relation this holds if and only if

$$(ABC_1)(ABC_2)(BCA_1)(BCA_2)(CAB_1)(CAB_2) = 1.$$

■

A similar generalization of the theorem of Menelaos can be formulated not only for curves of second order (i.e., for conics), but also for the more general class of algebraic curves of order $n$. By the general theorem, if we consider $n$ points on each side of a triangle (different from the vertices), these $3n$ points are on an algebraic curve of order $n$ if and only if the product of the $3n$ simple ratios as above is $(-1)^n$. The most general version of this theorem has been obtained by B. Segre, cf. [5].

# 3 The theorem of Carnot from the point of view of barycentric coordinates

In this section we work in the real projective plane and represent its points by homogeneous coordinates. It is well-known that any four points $A$, $B$, $C$,

$D$ of general position (no three of the points are collinear) can be transformed by collineation to the points $A'[1,0,0]$, $B'[0,1,0]$, $C'[0,0,1]$, $D'[1,1,1]$. Thus working with the images under this collineation instead of the original points, we may assume for any four points of general position that their coordinates are $[1,0,0]$, $[0,1,0]$, $[0,0,1]$, $[1,1,1]$, respectively.

Let us choose the four-point above such that $D$ is the centroid of the triangle $ABC$. Then, using the mentioned collineation we call the coordinates of the image of any point $P$ the *barycentric coordinates* of $P$ with respect to the triangle $ABC$.

Then $[0,1,\alpha]$, $[\beta,0,1]$ and $[1,\gamma,0]$ are the barycentric coordinates of the points $A_1$, $B_1$, $C_1$ such that $(BCA_1)=\alpha$, $(CAB_1)=\beta$ és $(ABC_1)=\gamma$.

We prove this claim for the point $A_1$ of barycentric coordinates $[0,1,\alpha]$. Let $A_M$ be the midpoint of $\overline{BC}$. Since $D$ is the centroid of $ABC$, $A_M=\overleftrightarrow{AD}\cap\overleftrightarrow{BC}$, so an easy calculation shows that the barycentric coordinates of $A_M$ are $[0,1,1]$. Since the original points are sent to the points determined by the barycentric coordinates by a collineation, and collineations preserve cross-ratio, it means that $(BCA_1A_F)=\alpha$. Otherwise, since $A_M$ is the midpoint of $\overline{BC}$, $(BCA_M)=1$, so

$$(BCA_1A_M)=\frac{(BCA_1)}{(BCA_M)}=(BCA_1).$$

Thus we indeed have $(BCA_1)=\alpha$.

In terms of barycentric coordinates the theorem of Menelaos states that the points $[0,1,\alpha]$, $[\beta,0,1]$, $[1,\gamma,0]$ are collinear if and only if $\alpha\beta\gamma=-1$. Similarly, we have the following reformulation of the theorem of Ceva: the lines of $[0,1,\alpha]$ and $[1,0,0]$, $[\beta,0,1]$ and $[0,1,0]$, $[1,\gamma,0]$ and $[0,0,1]$ are concurrent if and only if $\alpha\beta\gamma=1$.

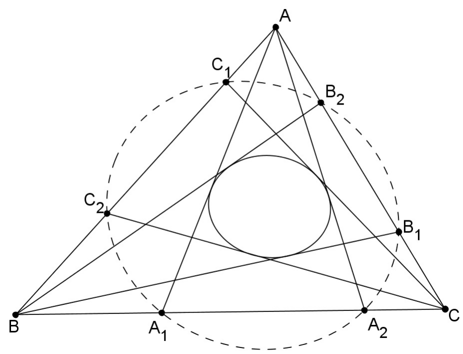
Finally, the theorem of Carnot takes the following form: $[0,1,\alpha_1]$, $[0,1,\alpha_2]$, $[\beta_1,0,1]$, $[\beta_2,0,1]$, $[1,\gamma_1,0]$ and $[1,\gamma_2,0]$ *are on a conic if and only if*

$$\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2=1.$$

# 4 Tangents to a conic from the vertices of a triangle

The next result was formulated by C. Bradley [1] as a conjecture. In this section we prove Bradley's conjecture applying the theorem of Carnot and using barycentric coordinates. We note that our proof remains valid in any projective plane coordinatized by a field, so we may state our theorem in any Pappian projective plane.

**Theorem 4.1** *Let a triangle $ABC$ and a conic $\mathcal{C}$ in the real projective plane be given. The tangent lines from the vertices of $ABC$ to $\mathcal{C}$ intersect the opposite sides of the triangle in six points that are incident to a conic.*




*Proof.* Let the vertices of the triangle be $A = [1,0,0]$, $B = [0,1,0]$ and $C = [0,0,1]$. Suppose that the tangents of $\mathcal{C}$ incident to $A$ intersect $\overleftrightarrow{BC}$ in $A_1[0,1,\alpha_1]$ and $A_2[0,1,\alpha_2]$; the tangents incident to $B$ intersect $\overleftrightarrow{AC}$ in $B_1[\beta_1,0,1]$ and $B_2[\beta_2,0,1]$, the tangents incident to $C$ intersect $\overleftrightarrow{AB}$ in $C_1[1,\gamma_1,0]$ and $C_2[1,\gamma_2,0]$.

If $[0,1,\alpha]$ is an arbitrary point of $\overleftrightarrow{BC}$, then the points of the line of $A$ and $[0,1,\alpha]$ have coordinates of the form $[1,\lambda,\alpha\lambda]$, where $\lambda \in \mathbb{R}$. If this line is a tangent of $c$, then there is exactly one $\lambda$ such that $[1,\lambda,\alpha\lambda]$ satisfies the equation

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0$$

of $\mathcal{C}$.

This condition implies that the equation

$$\lambda^2(a_{22} + 2a_{23}\alpha + a_{33}\alpha^2) + \lambda(2a_{12} + 2a_{23}\alpha) + a_{11} = 0$$

has exactly one solution $\lambda$. This holds if and only if the discriminant of this quadratic equation vanishes, i.e.,

$$4(a_{12} + \alpha a_{13})^2 - 4a_{11}(a_{22} + 2a_{23}\alpha + a_{33}\alpha^2) = 0.$$

From this an easy calculation leads to the following equation:

$$\alpha^2(a_{13}^2 - a_{11}a_{33}) + \alpha(2a_{12}a_{13} - 2a_{11}a_{23}) + (a_{12}^2 - a_{11}a_{22}) = 0.$$

The solutions of this equations are the $\alpha_1$ and $\alpha_2$ coordinates of $A_1$ and $A_2$. The product of the roots of this quadratic equation is the quotient of the constant and the coefficient of the second order term, i.e.,

$$\alpha_1\alpha_2 = \frac{a_{12}^2 - a_{11}a_{22}}{a_{13}^2 - a_{11}a_{33}}.$$

By similar calculations we find that

$$\beta_1\beta_2 = \frac{a_{23}^2 - a_{22}a_{33}}{a_{12}^2 - a_{11}a_{22}}$$

and

$$\gamma_1\gamma_2 = \frac{a_{13}^2 - a_{11}a_{33}}{a_{23}^2 - a_{22}a_{33}}.$$

Thus

$$\alpha_1\alpha_2\beta_1\beta_2\gamma_1\gamma_2 = \frac{a_{12}^2 - a_{11}a_{22}}{a_{13}^2 - a_{11}a_{33}} \cdot \frac{a_{23}^2 - a_{22}a_{33}}{a_{12}^2 - a_{11}a_{22}} \cdot \frac{a_{13}^2 - a_{11}a_{33}}{a_{23}^2 - a_{22}a_{33}} = 1,$$

and by theorem of Carnot, this implies our claim. ∎

# 5 The Cevian conic

In [2] C. Bradley proved the following theorem using barycentric coordinates. We give here a purely synthetic proof, applying again the theorem of Carnot.

**Theorem 5.1** *Let $ABC$ be an arbitrary triangle in the Euclidean plane, and let $P$ be an arbitrary point not incident to any of the sides of $ABC$. Denote the feet of the Cevians through $P$ by $A_0$, $B_0$ and $C_0$. Suppose that the circle through $A_0$, $B_0$ and $P$ intersect $\overleftrightarrow{BC}$ in $A_1$ and $\overleftrightarrow{AC}$ in $B_2$; the circle through $B_0$, $C_0$ and $P$ intersect $\overleftrightarrow{AB}$ in $C_1$ and $\overleftrightarrow{AC}$ in $B_1$; the circle through $A_0$, $C_0$ and $P$ intersect $\overleftrightarrow{BC}$ in $A_2$ and $\overleftrightarrow{AB}$ in $C_2$. Then $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ are on a conic (called the* Cevian conic of $P$ with respect to *$ABC$).*

*Proof.* Let the circle through $B_0$, $C_0$ and $P$ be $c_a$; the circle through $A_0$, $C_0$ and $P$ be $c_b$; and the circle through $A_0$, $B_0$ and $P$ be $c_c$. The power of the point $A$ with respect to the circle $c_a$ is

$$AC_1 \cdot AC_0 = AB_1 \cdot AB_0,$$

whence

$$AC_1 = AB_1 \cdot \frac{AB_0}{AC_0}. \tag{1}$$

Similarly we get

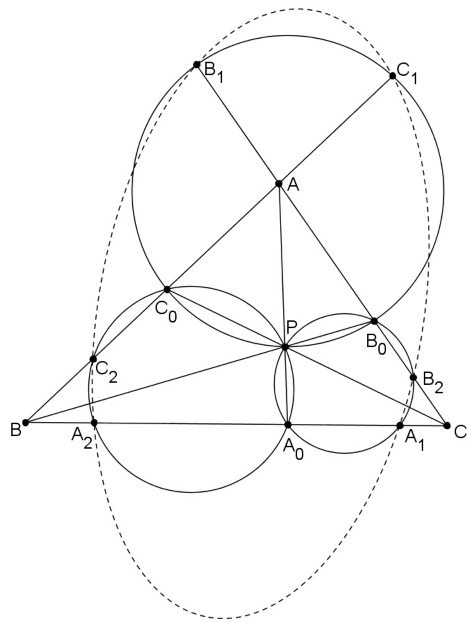
$$BA_2 = BC_2 \cdot \frac{BC_0}{BA_0}$$

and

$$CB_2 = CA_1 \cdot \frac{CA_0}{CB_0}.$$

The point $A$ is on the power line of $c_b$ and $c_c$, thus

$$AC_2 \cdot AC_0 = AB_2 \cdot AB_0.$$




Hence

$$AC_2 = AB_2 \cdot \frac{AB_0}{AC_0}.$$

Similarly we get

$$BA_1 = BC_1 \cdot \frac{BC_0}{BA_0}$$

and

$$CB_1 = CA_2 \cdot \frac{CA_0}{CB_0}.$$

Using these results,

$$AC_1 \cdot AC_2 \cdot BA_1 \cdot BA_2 \cdot CB_1 \cdot CB_2 =$$

$$= \frac{(AB_0)^2 \cdot (AB_1) \cdot (AB_2) \cdot (BC_0)^2 \cdot BC_1 \cdot BC_2 \cdot (CA_0)^2 \cdot (CA_1) \cdot (CA_2)}{(AC_0)^2 \cdot (BA_0)^2 \cdot (CB_0)^2}.$$

Applying the theorem of Ceva to the Cevians through $P$, we get

$$\frac{(C_0B)^2}{(AC_0)^2} \cdot \frac{(A_0C)^2}{(BA_0)^2} \cdot \frac{(B_0A)^2}{(CB_0)^2} = 1,$$

thus

$$AC_1 \cdot AC_2 \cdot BA_1 \cdot BA_2 \cdot CB_1 \cdot CB_2 = C_1B \cdot C_2B \cdot A_1C \cdot A_2C \cdot B_1A \cdot B_2A,$$

$$\frac{AC_1}{C_1B} \cdot \frac{AC_2}{C_2B} \cdot \frac{BA_1}{A_1C} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_1}{B_1A} \cdot \frac{CB_2}{B_2A} = 1,$$
$$(ABC_1)(ABC_2)(BCA_1)(BCA_2)(CAB_1)(CAB_2) = 1.$$

By the theorem of Carnot this proves our claim. ■

**Remark.** It is well known that for any triangle the lines connecting the vertices to the point of contact of the incircle on the opposite sides are concurrent. (This statement can easily be proved using the theorem of Ceva, or the theorem of Brianchon, which is the dual of the theorem of Pascal.) The point of concurrency is called the *Gergonne point* of the triangle. In [3] Bradley proved, using lengthy calculations, that *the Cevian conic of the Gergonne point with respect to a triangle is a circle, whose centre is the incentre of the triangle.* We give an easy elementary proof of his result.

We use the notations of the previous proof and we suppose that $P$ is the Gergonne point of $ABC$. In this case $AB_0 = AC_0$, so from (1) we get $AC_1 = AB_1$. So $B_1C_1A$ is an isosceles triangle, thus the perpendicular bisector of $\overline{B_1C_1}$ is the bisector of the angle $\angle BAC$. Similarly we can prove that the perpendicular bisector of $\overline{B_2C_2}$ is the bisector of $\angle BAC$, the perpendicular bisector of $\overline{A_1C_1}$ and $\overline{A_2C_2}$ is the bisector of $\angle ABC$, and the perpendicular bisector of $\overline{A_2B_1}$ and $\overline{A_1B_2}$ is the bisector of $\angle BCA$. Thus the perpendicular bisectors of the sides of the hexagon $A_1B_2C_2A_2B_1C_1$ pass through the incentre of $ABC$, so the vertices of the hexagon are on a circle whose centre is the incentre of $ABC$.

The following result is a projective generalization of the previous theorem.

**Corollary 5.2** *Let $ABC$ be an arbitrary triangle in the real projective plane, and let $P$, $I$, $J$ be arbitrary points not incident to any of the sides of $ABC$. Denote the feet of the Cevians through $P$ by $A_0$, $B_0$ and $C_0$. Suppose that the conic through $I$, $J$, $A_0$, $B_0$ and $P$ intersect $\overleftrightarrow{BC}$ in $A_1$ and $\overleftrightarrow{AC}$ in $B_2$; the conic through $I$, $J$, $B_0$, $C_0$ and $P$ intersect $\overleftrightarrow{AB}$ in $C_1$ and $\overleftrightarrow{AC}$ in $B_1$; the conic through $I$, $J$, $A_0$, $C_0$ and $P$ intersect $\overleftrightarrow{BC}$ in $A_2$ and $\overleftrightarrow{AB}$ in $C_2$. Then $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ are on a conic (called the* Cevian conic of $P$ with respect to $ABC$ and *$(IJ)$).*

*Proof.* The real projective plane is a subplane of the complex projective plane, so we may consider our configuration in the complex projective plane. Apply a projective collineation of the complex projective plane that sends $I$ and $J$ to $[1, i, 0]$ and $[1, -i, 0]$ (i.e., to the *circular points at infinity*), respectively. It is well known (see e.g. [6]) that a conic of the extended euclidean plane is a circle if and only if (after embedding to the complex projective plane) it is incident with the circular points at infinity. Thus applying our collineation we get the same configuration as in our previous theorem. ■

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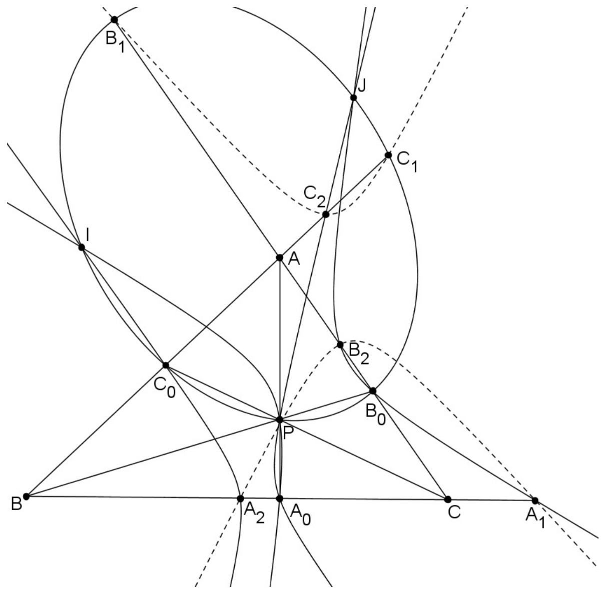



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