

# ON A GENERALIZATION OF A PROBLEM OF ERDŐS AND GRAHAM

SZ. TENGELY AND N. VARGA

*Dedicated to Professor Lajos Tamássy on his 90th birthday*

ABSTRACT. In this paper we provide bounds for the size of the solutions of the Diophantine equation  $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$ , where  $a, b \in \mathbb{Z}, a \neq b$  are parameters. We also determine all integral solutions for  $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}$ .

## 1. INTRODUCTION

Let us define

$$f(x, k, d) = x(x + d) \cdots (x + (k - 1)d).$$

Erdős [12] and independently Rigge [26] proved that if  $x \geq 1$  and  $k \geq 2$ , then  $f(x, k, 1)$  is never a perfect square. A celebrated result of Erdős and Selfridge [13] states that  $f(x, k, 1)$  is never a perfect power of an integer, provided  $x \geq 1$  and  $k \geq 2$ . That is, they completely solved the Diophantine equation

$$(1) \quad f(x, k, d) = y^l$$

with  $d = 1$ . The literature of this type of Diophantine equations is very rich. First consider some results related to  $l = 2$ . Euler proved (see [10] pp. 440 and 635) that a product of four terms in arithmetic progression is never a square solving (1) with  $k = 4, l = 2$ . Obláth [25] obtained a similar statement for  $k = 5$ . Saradha and Shorey [30] proved that (1) has no solutions with  $k \geq 4$ , provided that  $d$  is a power of a prime number. Laishram and Shorey [23] extended this result to the case where either  $d \leq 10^{10}$ , or  $d$  has at most six prime divisors. Bennett, Bruin, Győry and Hajdu [3] solved (1) with  $6 \leq k \leq 11$  and  $l = 2$ . Hirata-Kohno, Laishram, Shorey and Tijdeman [22] completely solved (1) with  $3 \leq k < 110$ .

---

2000 *Mathematics Subject Classification.* Primary 11D61; Secondary 11Y50.  
*Key words and phrases.* Diophantine equations.

Now assume for this paragraph that  $l \geq 3$ . Many authors have considered the more general equation

$$(2) \quad f(x, k, d) = by^l,$$

where  $b > 0$  and the greatest prime factor of  $b$  does not exceed  $k$ . Saradha [29] proved that (2) has no solution with  $k \geq 4$ . Győry [16] studied the cases  $k = 2, 3$ , he determined all integral solutions. Győry, Hajdu and Saradha [18] proved that the product of four or five consecutive terms of an arithmetical progression of integers cannot be a perfect power, provided that the initial term is coprime to the difference. Hajdu, Tengely and Tijdeman [20] proved that the product of  $k$  coprime integers in arithmetic progression cannot be a cube when  $2 < k < 39$ . Hajdu and Kovács [19] proved that the product of  $k$  consecutive terms of a primitive arithmetic progression is never a fifth power when  $3 \leq k \leq 54$ . Győry, Hajdu and Pintér [17] proved that for any positive integers  $x, d$  and  $k$  with  $\gcd(x, d) = 1$  and  $3 < k < 35$ , the product  $x(x+d) \cdots (x+(k-1)d)$  cannot be a perfect power.

Erdős and Graham [11] asked if the Diophantine equation

$$\prod_{i=1}^r f(x_i, k_i, 1) = y^2$$

has, for fixed  $r \geq 1$  and  $\{k_1, k_2, \dots, k_r\}$  with  $k_i \geq 4$  for  $i = 1, 2, \dots, r$ , at most finitely many solutions in positive integers  $(x_1, x_2, \dots, x_r, y)$  with  $x_i + k_i \leq x_{i+1}$  for  $1 \leq i \leq r-1$ . Skalba [32] provided a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [35] answered the above question of Erdős and Graham in the negative when either  $r = k_i = 4$ , or  $r \geq 6$  and  $k_i = 4$ . Bauer and Bennett [2] extended this result to the cases  $r = 3$  and  $r = 5$ . Bennett and Van Luijk [4] constructed an infinite family of  $r \geq 5$  non-overlapping blocks of five consecutive integers such that their product is always a perfect square. Luca and Walsh [24] studied the case  $(r, k_i) = (2, 4)$  for all  $i = 1, \dots, r$ .

In this paper we study the Diophantine equation

$$(3) \quad \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2,$$

where  $a, b \in \mathbb{Z}, a \neq b$  are parameters. We provide bounds for the size of solutions and an algorithm to determine all solutions  $(x, y) \in \mathbb{Z}^2$ . The method of proof is based on Runge's method [15, 21, 27, 31, 34, 37]. In 2008, Sankaranarayanan and Saradha [28] established improved upper bounds for the size of the solutions of the Diophantine equations  $F(x) = y^m$  and  $F(x) = G(y)$ , for which Runge's method can be applied.

They generalized the method to obtain bounds for the solutions of equations of the form  $P(x)/Q(x) = y^m$ . Based on this latter result we provide bounds for the solutions of equation (3). We note that solutions of (3) in integers also correspond to integer solutions to the hyperelliptic equation

$$x(x+1)(x+2)(x+3)(x+a)(x+b) = Y^2,$$

where  $Y = (x+a)(x+b)y$ . Baker [1] applied his theory of lower bounds for linear forms in logarithms to obtain upper bound for the size of solutions of hyperelliptic equations. Many authors improved the bound see e.g. [5, 7, 8, 9, 33, 36]. Still these bounds remain astronomical. It is also possible to apply Runge's method to provide upper bound for the size of integral solutions of this hyperelliptic curve. Our method yields better bound, thus it is more efficient to determine all integral solutions.

**Theorem 1.** (I) *If  $(x, y) \in \mathbb{Z}^2$  is a solution of (3) with  $a \equiv b \pmod{2}$ , then*

$$|x| \leq \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}, |B_2|, |B_1|^{1/2}, |B_0|^{1/3}, |\frac{1}{4}(a+b-6)^2 ab|\},$$

where

$$\begin{aligned} A_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - 2a - 2b + 7 \\ A_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + 2a^2 - \frac{1}{4}b^3 + 2b^2 - 4a - 4b + 6 \\ A_0 &= -\frac{1}{4}(a+b-4)^2 ab \\ B_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - 4a - 4b - 5 \\ B_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + 4a^2 - \frac{1}{4}b^3 + 4b^2 - 16a - 16b + 6 \\ B_0 &= -\frac{1}{4}(a+b-8)^2 ab. \end{aligned}$$

(II) *If  $(x, y) \in \mathbb{Z}^2$  is a solution of (3) with  $a \not\equiv b \pmod{2}$ , then*

$$|x| \leq 2 \max\{|C_2|, |C_1|^{1/2}, |C_0|^{1/3}, |D_2|, |D_1|^{1/2}, |D_0|^{1/3}\},$$

where

$$\begin{aligned}
C_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{7}{2}a - \frac{7}{2}b - \frac{5}{4} \\
C_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{7}{2}a^2 - \frac{1}{4}b^3 + \frac{7}{2}b^2 - \frac{49}{4}a - \frac{49}{4}b + 6 \\
C_0 &= -\frac{1}{4}(a+b-7)^2ab \\
D_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{5}{2}a - \frac{5}{2}b + \frac{19}{4} \\
D_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{5}{2}a^2 - \frac{1}{4}b^3 + \frac{5}{2}b^2 - \frac{25}{4}a - \frac{25}{4}b + 6 \\
D_0 &= -\frac{1}{4}(a+b-5)^2ab.
\end{aligned}$$

We apply the above theorem to determine all integral solutions of (3) with  $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}$ ,  $a \neq b$ .

**Corollary 1.** *All solutions  $(x, y) \in \mathbb{Z}^2$ ,  $y \neq 0$  of (3) with  $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}$ ,  $a \neq b$  are as follows*

$$\begin{aligned}
a = -4, b = -3, & \quad (x, y) \in \{(-6, 2), (1, 2)\} \\
a = -4, b = 5, & \quad (x, y) \in \{(-6, 6)\} \\
a = -2, b = 7, & \quad (x, y) \in \{(3, 6)\} \\
a = 6, b = 7, & \quad (x, y) \in \{(-4, 2), (3, 2)\}.
\end{aligned}$$

## 2. PROOF OF THE RESULTS

In the proof we will use the following result of Fujiwara [14].

**Lemma 1.** *Put  $p(z) = \sum_{i=0}^n a_i z^i$ ,  $a_n \neq 0$ , where  $a_i \in \mathbb{R}$  for all  $i = 0, 1, \dots, n$ . Then*

$$\max\{|\zeta| : p(\zeta) = 0\} \leq 2 \max \left\{ \left| \frac{a_{n-1}}{a_n} \right|, \left| \frac{a_{n-2}}{a_n} \right|^{1/2}, \dots, \left| \frac{a_0}{a_n} \right|^{1/n} \right\}.$$

*Proof of Theorem 1.* The polynomial part of the Puiseux expansion of

$$\left( \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} \right)^{1/2}$$

is  $x + 3 - \frac{a+b}{2}$ .

(I) First we deal with the case  $a \equiv b \pmod{2}$  that is, when  $\frac{a+b}{2}$  is an integer. We have that

$$\begin{aligned} x(x+1)(x+2)(x+3) - (x+a)(x+b) \left(x+2 - \frac{a+b}{2}\right)^2 = \\ 2x^3 + A_2x^2 + A_1x + A_0 =: f_A(x) \end{aligned}$$

and

$$\begin{aligned} x(x+1)(x+2)(x+3) - (x+a)(x+b) \left(x+4 - \frac{a+b}{2}\right)^2 = \\ -2x^3 + B_2x^2 + B_1x + B_0 =: f_B(x). \end{aligned}$$

It follows from Lemma 1 that  $f_A(x) \neq 0$  if

$$|x| > \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}\} =: r_A.$$

Similarly, one has that  $f_B(x) \neq 0$  if

$$|x| > \max\{|B_2|, |B_1|^{1/2}, |B_0|^{1/3}\} =: r_B.$$

Therefore  $f_A(x)f_B(x) < 0$ , if  $|x| > \max\{r_A, r_B\}$ . We obtain that either

$$\left(x+4 - \frac{a+b}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+2 - \frac{a+b}{2}\right)^2$$

or

$$\left(x+2 - \frac{a+b}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+4 - \frac{a+b}{2}\right)^2.$$

Since  $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$ , we get that  $y^2 = \left(x+3 - \frac{a+b}{2}\right)^2$  in both cases. Thus  $x$  is a root of the quadratic polynomial  $x(x+1)(x+2)(x+3) - (x+a)(x+b) \left(x+3 - \frac{a+b}{2}\right)^2$ . The constant term of this quadratic polynomial is  $-\frac{1}{4}(a+b-6)^2 ab$ , hence

$$|x| \leq |(a+b-6)^2 ab|.$$

(II) Now we consider the case  $a \not\equiv b \pmod{2}$ . We have that

$$\begin{aligned} x(x+1)(x+2)(x+3) - (x+a)(x+b) \left(x+3 - \frac{a+b-1}{2}\right)^2 = \\ -x^3 + C_2x^2 + C_1x + C_0 =: f_C(x) \end{aligned}$$

and

$$\begin{aligned} x(x+1)(x+2)(x+3) - (x+a)(x+b) \left(x+3 - \frac{a+b+1}{2}\right)^2 = \\ x^3 + D_2x^2 + D_1x + D_0 =: f_D(x). \end{aligned}$$

Lemma 1 implies that  $f_C(x) \neq 0$  if

$$|x| > 2 \max\{|C_2|, |C_1|^{1/2}, |C_0|^{1/3}\} =: r_C$$

and  $f_D(x) \neq 0$  if

$$|x| > 2 \max\{|D_2|, |D_1|^{1/2}, |D_0|^{1/3}\} =: r_D.$$

It is clear that  $f_C(x)f_D(x) < 0$ , if  $|x| > \max\{r_C, r_D\}$ . One gets that either

$$\left(x + 3 - \frac{a + b - 1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x + 3 - \frac{a + b + 1}{2}\right)^2$$

or

$$\left(x + 3 - \frac{a + b + 1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x + 3 - \frac{a + b - 1}{2}\right)^2.$$

In both cases we get a contradiction, since  $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$  and there cannot be a square between consecutive squares. Thus  $|x| \leq \max\{r_C, r_D\}$ .  $\square$

*Proof of Corollary 1.* We wrote a Magma [6] code to solve equation (3). If  $a \equiv b \pmod{2}$ , then we used the bound

$$|x| \leq \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}, |B_2|, |B_1|^{1/2}, |B_0|^{1/3}\}$$

and we determined the roots of the quadratic equation  $x(x+1)(x+2)(x+3) - (x+a)(x+b)\left(x+3-\frac{a+b}{2}\right)^2$ . Some details of the computations are given in the following table. We only indicate those cases where there is a solution with  $y \neq 0$ .

$a$	$b$	bound for $ x $
-4	-3	96
-4	5	46
-2	7	50
6	7	114

$\square$

**Acknowledgement.** Nóra Varga is supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 "National Excellence Program". Szabolcs Tengely is supported in part by the OTKA grants NK104208 and K100339.

## REFERENCES

- [1] A. Baker. Bounds for the solutions of the hyperelliptic equation. *Proc. Cambridge Philos. Soc.*, 65:439–444, 1969.
- [2] M. Bauer and M. A. Bennett. On a question of Erdős and Graham. *Enseign. Math. (2)*, 53(3-4):259–264, 2007.
- [3] M. A. Bennett, N. Bruin, K. Győry, and L. Hajdu. Powers from products of consecutive terms in arithmetic progression. *Proc. London Math. Soc. (3)*, 92(2):273–306, 2006.
- [4] M. A. Bennett and R. Van Luijk. Squares from blocks of consecutive integers: a problem of Erdős and Graham. *Indag. Math., New Ser.*, 23(1-2):123–127, 2012.
- [5] Yu. Bilu. Effective analysis of integral points on algebraic curves. *Israel J. Math.*, 90(1-3):235–252, 1995.
- [6] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).
- [7] B. Brindza. On  $S$ -integral solutions of the equation  $y^m = f(x)$ . *Acta Math. Hungar.*, 44(1-2):133–139, 1984.
- [8] Y. Bugeaud, M. Mignotte, S. Siksek, M. Stoll, and Sz. Tengely. Integral points on hyperelliptic curves. *Algebra Number Theory*, 2(8):859–885, 2008.
- [9] Yann Bugeaud. Bounds for the solutions of superelliptic equations. *Compositio Math.*, 107(2):187–219, 1997.
- [10] L.E. Dickson. *History of the theory of numbers. Vol II: Diophantine analysis.* Chelsea Publishing Co., New York, 1966.
- [11] P. Erdős and R. L. Graham. *Old and new problems and results in combinatorial number theory.* , 1980.
- [12] P. Erdős. Note on the product of consecutive integers (II). *J. London Math. Soc.*, 14:245–249, 1939.
- [13] P. Erdős and J. L. Selfridge. The product of consecutive integers is never a power. *Illinois J. Math.*, 19:292–301, 1975.
- [14] M. Fujiwara. Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung. *Tôhoku Math. J.*, 10:167–171, 1916.
- [15] A. Grytczuk and A. Schinzel. On Runge’s theorem about Diophantine equations. In *Sets, graphs and numbers (Budapest, 1991)*, volume 60 of *Colloq. Math. Soc. János Bolyai*, pages 329–356. North-Holland, Amsterdam, 1992.
- [16] K. Győry. On the diophantine equation  $n(n + 1) \dots (n + k - 1) = bx^\ell$ . *Acta Arith.*, 83(1):87–92, 1998.
- [17] K. Győry, L. Hajdu, and Á. Pintér. Perfect powers from products of consecutive terms in arithmetic progression. *Compos. Math.*, 145(4):845–864, 2009.
- [18] K. Győry, L. Hajdu, and N. Saradha. On the Diophantine equation  $n(n + d) \dots (n + (k - 1)d) = by^l$ . *Canad. Math. Bull.*, 47(3):373–388, 2004.
- [19] L. Hajdu and T. Kovács. Almost fifth powers in arithmetic progression. *J. Number Theory*, 131(10):1912–1923, 2011.
- [20] L. Hajdu, Sz. Tengely, and R. Tijdeman. Cubes in products of terms in arithmetic progression. *Publ. Math. Debrecen*, 74(1-2):215–232, 2009.
- [21] D. L. Hilliker and E. G. Straus. Determination of bounds for the solutions to those binary Diophantine equations that satisfy the hypotheses of Runge’s theorem. *Trans. Amer. Math. Soc.*, 280(2):637–657, 1983.

- [22] N. Hirata-Kohno, S. Laishram, T. N. Shorey, and R. Tijdeman. An extension of a theorem of Euler. *Acta Arith.*, 129(1):71–102, 2007.
- [23] S. Laishram and T. N. Shorey. The equation  $n(n+d)\cdots(n+(k-1)d) = by^2$  with  $\omega(d) \leq 6$  or  $d \leq 10^{10}$ . *Acta Arith.*, 129(3):249–305, 2007.
- [24] F. Luca and P.G. Walsh. On a diophantine equation related to a conjecture of Erdős and Graham. *Glas. Mat., III. Ser.*, 42(2):281–289, 2007.
- [25] R. Obláth. Über das Produkt fünf aufeinander folgender Zahlen in einer arithmetischen Reihe. *Publ. Math. Debrecen*, 1:222–226, 1950.
- [26] O. Rigge. über ein diophantisches problem. In *9th Congress Math. Scand.*, pages 155–160. Mercator 1939, Helsingfors 1938.
- [27] C. Runge. Über ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen. *J. Reine Angew. Math.*, 100:425–435, 1887.
- [28] A. Sankaranarayanan and N. Saradha. Estimates for the solutions of certain Diophantine equations by Runge’s method. *Int. J. Number Theory*, 4(3):475–493, 2008.
- [29] N. Saradha. On perfect powers in products with terms from arithmetic progressions. *Acta Arith.*, 82(2):147–172, 1997.
- [30] N. Saradha and T. N. Shorey. Almost squares in arithmetic progression. *Compositio Math.*, 138(1):73–111, 2003.
- [31] A. Schinzel. An improvement of Runge’s theorem on Diophantine equations. *Comment. Pontificia Acad. Sci.*, 2(20):1–9, 1969.
- [32] M. Skalba. Products of disjoint blocks of consecutive integers which are powers. *Colloq. Math.*, 98(1):1–3, 2003.
- [33] V. G. Sprindžuk. The arithmetic structure of integer polynomials and class numbers. *Trudy Mat. Inst. Steklov.*, 143:152–174, 210, 1977. Analytic number theory, mathematical analysis and their applications (dedicated to I. M. Vinogradov on his 85th birthday).
- [34] Sz. Tengely. On the Diophantine equation  $F(x) = G(y)$ . *Acta Arith.*, 110(2):185–200, 2003.
- [35] M. Ulas. On products of disjoint blocks of consecutive integers. *Enseign. Math. (2)*, 51(3-4):331–334, 2005.
- [36] Paul M. Voutier. An upper bound for the size of integral solutions to  $Y^m = f(X)$ . *J. Number Theory*, 53(2):247–271, 1995.
- [37] P. G. Walsh. A quantitative version of Runge’s theorem on Diophantine equations. *Acta Arith.*, 62(2):157–172, 1992.



MATHEMATICAL INSTITUTE  
UNIVERSITY OF DEBRECEN  
P.O.Box 12  
4010 DEBRECEN  
HUNGARY  
*E-mail address:* [tengely@science.unideb.hu](mailto:tengely@science.unideb.hu)

MATHEMATICAL INSTITUTE  
MTA-DE RESEARCH GROUP "EQUATIONS, FUNCTIONS AND CURVES"  
HUNGARIAN ACADEMY OF SCIENCES AND UNIVERSITY OF DEBRECEN  
P.O.Box 12  
4010 DEBRECEN  
HUNGARY  
*E-mail address:* [nvarga@science.unideb.hu](mailto:nvarga@science.unideb.hu)