

## ON THE DIOPHANTINE INEQUALITY

$$|X^2 - cXY^2 + Y^4| \leq c + 2$$

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ABSTRACT. Generalizing some earlier results, we find all the coprime integer solutions of the Diophantine inequality

$$|X^2 - cXY^2 + Y^4| \leq c + 2, \quad (X, Y) = 1,$$

except when  $c \equiv 2 \pmod{4}$ , in which case we bound the number of integer solutions. Our work is based on the results on the Diophantine equation

$$AX^4 - BY^2 = C,$$

where  $A, B$  are positive integers and  $C \in \pm\{1, 2, 4\}$ .

### 1. INTRODUCTION

In recent years, several results have been established concerning solutions of quartic Thue equations and inequalities, see [16], [17], [10] and the references given there. Let  $c$  be a positive integer. In [7], Cusick resolved an infinite family of Diophantine equations  $X^4 - cX^2Y^2 + Y^4 = 1$  and this result was generalized by Cohn [6] and Walsh [21] to the equations  $X^2 - cXY^2 + Y^4 = \pm 1, \pm 2, \pm 4$ , cf. Lemma 2. Further, here we would like to mention two other theorems related to the quartic Thue inequalities. In 1997, Wakabayashi [20, Theorem 2] proved that the only primitive solutions (i. e.  $(X, Y) = 1$ ) of the Thue inequality

$$|X^4 - a^2X^2Y^2 + Y^4| \leq a^2 - 2 \quad (a \geq 8)$$

are  $(X, Y) = (0, 0), (\pm 1, 0), (0, \pm 1), (\pm a, \pm 1), (\pm 1, \pm a), (\pm 1, \pm 1)$  with mixed signs. Dujella, Ibrahimpasić and Jadrijević [8] resolved the Thue inequality

$$|X^4 - 2(n^2 - 1)X^2Y^2 + Y^4| \leq 2n + 3,$$

where  $n \geq 0$  is an integer.

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As a common generalization of the previous inequalities, let us consider the Diophantine equation

$$(1.1) \quad X^2 - cXY^2 + Y^4 = \mu, \quad |\mu| \leq c + 2.$$

Determining completely the integer solutions of equation (1.1) depends on a conjecture on the number of solutions to Diophantine equation

$$(1.2) \quad aX^4 - bY^2 = 1.$$

In 2005, Togbé, Voutier and Walsh [18] made the following conjecture.

**Conjecture 1.** *Let  $t > 1$  denote a positive integer. Then the only positive integer solution to*

$$(t + 1)X^4 - tY^2 = 1$$

*is  $(X, Y) = (1, 1)$ , unless  $t = m^2 + m$  for some positive integer  $m$ , in which case there is also the solution  $(X, Y) = (2m + 1, 4m^2 + 4m + 3)$ .*

We note that using the function *SIntegralLjunggrenPoints* in the computer algebra program package MAGMA [5] that the conjecture above is true for  $1 < t \leq 1000$  apart from the possible values of

$$t \in \{388, 502, 634, 702, 744, 772, 820, 829, 878, 883, 953, 955, 956, 957, 993\}.$$

In these exceptional cases MAGMA failed to find all integer points on the quadric, because it was not able to compute the rank of the corresponding elliptic curve.

Our main result is the following.

**Theorem 1.** *For a positive integer  $c \geq 3$ , all of the coprime integer solutions  $(X, Y)$  of Diophantine equation (1.1) are given by*

$$(1.3) \quad (X, \pm Y) = (-1, 0), (1, 0), (0, 1), (-1, 1), (1, 1), (c-1, 1), (c, 1), (c+1, 1),$$

*except for the following cases:*

- (1) *If  $c = 338$ , then equation (1.1) has the additional integer solution  $(X, \pm Y) = (114243, 6214), (13051348805, 6214)$ .*
- (2) *If  $c = n^2$ , then equation (1.1) has the additional integer solution  $(X, \pm Y) = (1, n), (n^4 - 1, n)$ .*
- (3) *If  $c = n^2 - 1$ , then equation (1.1) has the additional integer solution  $(X, \pm Y) = (1, n), (n^4 - n^2 - 1, n)$ .*
- (4) *If  $c = n^2 + 1$ , then equation (1.1) has the additional integer solution  $(X, \pm Y) = (1, n), (n^4 + n^2 - 1, n)$ .*

- (5)  $c \equiv 2 \pmod{4}$  is not as in (4), with  $\mu = -c + 2$ . In this exceptional case, there are at most two additional solutions  $(X, Y)$  to equation (1.1). Under Conjecture 1, there is no other additional solution.

For  $c = 1$  and  $2$  we can give all the solutions by using elementary considerations. Indeed, if  $c = 2$  then we have

$$x^2 - 2xy^2 + y^4 = (x - y^2)^2 \leq 4$$

and  $x - y^2 = 0, \pm 1, \pm 2$ , so all the solutions are

$$(x, y) = (m^2, m), (m^2 \pm 1, m), (m^2 \pm 2, m),$$

where  $m$  is an integer. For  $c = 1$  we obtain

$$|x^2 - xy^2 + y^4| = \frac{x^2 + (x - y^2)^2 + y^4}{2} \leq 3,$$

and a straightforward calculation shows that all the solutions are

$$(x, y) = (\pm 1, \pm 1), (-1, 0), (1, 0), (0, \pm 1), (0, 0), (2, \pm 1).$$

The organization of this paper is as follows. In Section 2, we will recall some useful results, particularly results related to the Diophantine equation

$$AX^4 - BY^2 = C,$$

where  $A, B$  are positive integers and  $C \in \pm\{1, 2, 4\}$ . The proof of Theorem 1 will be done in Section 3 by the means of the results cited in Section 2.

## 2. PRELIMINARIES

In 1995, Cohn [6] considered the Diophantine equation

$$x^2 - cxy^2 + y^4 = 1,$$

where  $c$  is an odd positive integer. Walsh [21] solved the equation with  $c$  an even positive integer.

**Lemma 2** (Cohn 1995, Walsh 1999). *The only solutions to  $x^2 - cxy^2 + y^4 = 1$  in non-negative integers  $(x, y)$  are  $(c, 1), (1, 0), (0, 1)$ , unless either  $c$  is a perfect square, in which case there are also the solutions  $(1, \sqrt{c}), (c^2 - 1, \sqrt{c})$ , or  $c = 338$  in which there are the solutions  $(x, y) = (114243, 6214), (13051348805, 6214)$ .*

We will also recall some classical results on the quartic Diophantine equation

$$(2.1) \quad aX^2 - bY^4 = \pm 1, \pm 2, \pm 4.$$

We will consider the positive values of the constants in equation (2.1) and recall two results.

The result on the Diophantine equation

$$(2.2) \quad aX^2 - bY^4 = 4, \quad 2 \nmid ab,$$

was obtained by Luo and Yuan [15]. Let  $(x_1, y_1)$  be the minimal positive integer solution of equation  $ax^2 - by^2 = 4$ . We define  $x_k, y_k$  by

$$(2.3) \quad \left( \frac{x_1\sqrt{a} + y_1\sqrt{b}}{2} \right)^k = \frac{x_k\sqrt{a} + y_k\sqrt{b}}{2}, \quad 2 \nmid k.$$

**Lemma 3** (Luo and Yuan, 2007). (1) *If  $y_1$  is not a square, then the equation (2.2) has no positive integer solution except for the case  $y_1 = 3\Box$  and  $by_1^2 + 3 = 3\Box$ , when  $(X, Y) = (x_3, \sqrt{y_3})$  is the only solution of (2.2).*

(2) *if  $y_1$  is a square, then equation (2.2) has at most one positive integer solution other than  $(X, Y) = (x_1, \sqrt{y_1})$ , which is either  $(X, Y) = (x_3, \sqrt{y_3})$  or  $(X, Y) = (x_2, \sqrt{y_2})$ , the latter occurring if and only if  $x_1$  and  $y_1$  are both squares and  $a = 1, b \neq 5$ .*

For the Diophantine equation

$$(2.4) \quad aX^2 - bY^4 = 2, \quad 2 \nmid ab,$$

Ljunggren [12] proved that the equation has at most two solutions in positive integers  $(X, Y)$ . Luca and Walsh [14] obtained a more precise result. In fact, let  $(u_1, v_1)$  be the minimal positive integer solution of equation  $ax^2 - by^2 = 2$ . We also define  $u_k, v_k$  by

$$\left( \frac{u_1\sqrt{a} + v_1\sqrt{b}}{\sqrt{2}} \right)^k = \frac{u_k\sqrt{a} + v_k\sqrt{b}}{\sqrt{2}}, \quad 2 \nmid k.$$

**Lemma 4** (Luca and Walsh, 2001). 1. *If  $v_1$  is not a square, then equation (2.4) has no solutions.*

2. *If  $v_1$  is a square and  $v_3$  is not a square, then  $(X, Y) = (u_1, \sqrt{v_1})$  is the only solution of equation (2.4).*

3. *If  $v_1$  and  $v_3$  are both squares, then  $(X, Y) = (u_1, \sqrt{v_1})$  and  $(u_3, \sqrt{v_3})$  are the only solutions of equation (2.4).*

We recall here another result obtained Ljunggren [12].

**Lemma 5** (Ljunggren, 1954). *The Diophantine equation  $aX^2 - bY^4 = 1$  has at most one positive integer solution  $(X, Y)$  with  $a > 1$ .*

For the negative values of the constants in equation (2.1), first we consider the equation

$$(2.5) \quad aX^4 - bY^2 = 4.$$

We proceed like for Lemma 3 by considering the definition of  $x_1, y_1$  and that of  $x_k, y_k$  given by equation (2.3). Ljunggren [13] proved the following result.

**Lemma 6** (Ljunggren, 1967). *If  $2 \nmid ab$ , then the Diophantine equation  $aX^4 - bY^2 = 4$  has at most two positive integer solutions  $(X, Y)$  with  $2 \nmid XY$ .*

- (1) *If  $x_1 = h^2$  and  $ax_1^2 - 3 = s^2$ , there are only two solutions, namely,  $(X, Y) = (\sqrt{x_1}, y_1), (\sqrt{x_3}, y_3)$ .*
- (2) *If  $x_1 = h^2$  and  $ax_1^2 - 3 \neq s^2$ , then  $(X, Y) = (\sqrt{x_1}, y_1)$  is the only solution.*
- (3) *If  $x_1 = 5h^2$  and  $a^2x_1^4 - 5ax_1^2 + 5 = 5s^2$ , then the only solution is  $(X, Y) = (\sqrt{x_5}, y_5)$ .*

In 2009, Yuan and Li [22] confirmed a conjecture posed by Akhtari, Togbé and Walsh [2], [3]. See also [19].

**Lemma 7** (Yuan and Li, 2009). *If  $2 \nmid ab$ , then the Diophantine equation  $aX^4 - bY^2 = 2$  has at most one positive integer solution  $(X, Y)$ .*

The most difficult equation in (2.1) is

$$(2.6) \quad aX^4 - bY^2 = 1.$$

The upper bound of the number of solutions to equation (2.6) is obtained by Akhtari [1] and Yuan-Zhang [23], independently. They showed the following result.

**Lemma 8** (Akhtari 2009, Yuan and Zhang 2010). *The Diophantine equation (2.6) has at most two positive integer solutions  $(X, Y)$ .*

A special case of the equation was studied by Bennett, Togbé and Walsh [4].

**Lemma 9** (Bennett, Togbé, and Walsh, 2006). *Let  $m$  be a positive integer. Then the only positive integral solutions to the equation*

$$(m^2 + m + 1)X^4 - (m^2 + m)Y^2 = 1$$

*are given by  $(X, Y) = (1, 1)$  and  $(X, Y) = (2m + 1, 4m^2 + 4m + 3)$ .*

### 3. PROOF OF THEOREM 1

If one of the variables  $X$  and  $Y$  is zero, under the condition  $(X, Y) = 1$ , then the other variable is  $\pm 1$ . So we will assume that  $XY \neq 0$ .

From (1.1) we have

$$(2X - cY^2)^2 - (c^2 - 4)Y^4 = 4\mu.$$

So we will consider the equation

$$(3.1) \quad x^2 - (c^2 - 4)y^2 = 4\mu, \quad |\mu| \leq c + 2,$$

where  $x = |2X - cY^2|$ ,  $y = Y^2$ .

Using the same argument as in Section 3 of [9], if equation (3.1) has a solution  $(x, y)$ , then we have

$$\mu = 1, -c + 2 \text{ or } c + 2.$$

**3.1. The case  $\mu = 1$ .** By the work of Cohn and Walsh, stated as Lemma 2, the only solutions are  $(X, \pm Y) = (0, 1), (1, 0), (c, 1)$ , and the exceptional solutions in the cases (1) and (2) of Theorem 1.

**3.2. The case  $\mu = c + 2$ .** From the equation

$$x^2 - (c^2 - 4)y^2 = 4(c + 2),$$

we see that  $(c + 2) | x^2$ . Put  $c + 2 = gh^2$  with  $g$  square-free. Then we have  $x = ghz$  with integers  $g, h$ , and  $z$ . Therefore, the above equation can be rewritten into

$$(3.2) \quad gz^2 - (c - 2)y^2 = 4.$$

One can see that the minimal solution is  $(z, y_1) = (h, 1)$ . By some results on Pellian equations, for any possible common divisor of  $gz$  and  $(c - 2)y$ , all of positive integer solutions to equation (3.2) are given by

$$\frac{z\sqrt{g} + y\sqrt{c-2}}{2} = \left( \frac{z_1\sqrt{g} + y_1\sqrt{c-2}}{2} \right)^k = \frac{z_k\sqrt{g} + y_k\sqrt{c-2}}{2}, \quad 2 \nmid k.$$

Using the above equality or classical properties of Lucas-Lehmer numbers [11], we have  $z_1 | z_k$ , i.e.  $h | z$ . Thus, equation (3.2) becomes

$$(c + 2)x'^2 - (c - 2)y^2 = 4.$$

Combining this and equation (3.1), we will study the equation

$$(3.3) \quad (c + 2)x'^2 - (c - 2)Y^4 = 4,$$

where  $x' = \frac{|2X - cY^2|}{c+2}$ .

It is easy to see that  $(x', \pm Y) = (1, 1)$  is a positive integer solution of the above equation. From

$$2X = cY^2 \pm (c + 2) = c \pm (c + 2),$$

we get the solutions  $(X, \pm Y) = (-1, 1), (c + 1, 1)$  listed in Theorem 1. We will try to find the solutions of equation (3.3) satisfying  $(x', \pm Y) \neq (1, 1)$ .

- If  $2 \nmid c$ , then the minimal solution of the equation

$$(c + 2)x^2 - (c - 2)y^2 = 4$$

is  $(x_1, y_1) = (1, 1)$ . We will apply Lemma 3. As

$$\left(\frac{\sqrt{c+2} + \sqrt{c-2}}{2}\right)^3 = \frac{(c-1)\sqrt{c+2} + (c+1)\sqrt{c-2}}{2},$$

we get  $(x', Y) = (c-1, \sqrt{c+1})$ , if  $c+1$  is a square. Put  $c+1 = (2s)^2, s \in \mathbb{N}$ . Therefore, we have the solutions in the exceptional case (3) of Theorem 1, with  $n$  even.

- If  $4|c$ , then equation (3.3) becomes

$$(c/2 + 1)x'^2 - (c/2 - 1)Y^4 = 2.$$

Using Lemma 4 and an argument similar to that of the above case, we obtain the solutions in the exceptional case (3) of Theorem 1, with  $n$  odd.

- If  $c \equiv 2 \pmod{4}$ , then from equation (3.3) we deduce

$$((c/2 + 1)/2)x'^2 - ((c/2 - 1)/2)Y^4 = 1$$

or

$$(c_1 + 1)x'^2 - c_1Y^4 = 1,$$

where  $c = 4c_1 + 2$ . By Lemma 5, there is no solution satisfying  $(x', \pm Y) \neq (1, 1)$ .

**3.3. The case  $\mu = -c + 2$ .** From the equation

$$x^2 - (c^2 - 4)y^2 = 4(-c + 2),$$

as in the case  $\mu = c + 2$ , we can see that  $(c-2)|x$ . So, there exists  $x'' \in \mathbb{N}$  such that  $x = (c-2)x''$ . Then we have

$$(c+2)y^2 - (c-2)x''^2 = 4.$$

Now, since  $y = Y^2$ , the equation becomes

$$(3.4) \quad (c+2)Y^4 - (c-2)x''^2 = 4,$$

where  $x'' = \frac{|2X - cY^2|}{c-2}$ . The trivial solution  $(x'', \pm Y) = (1, 1)$  provides

$$2X = cY^2 \pm (c-2) = c \pm (c+2).$$

We obtain the solutions  $(X, \pm Y) = (1, 1), (c-1, 1)$  listed in Theorem 1. Also, let us assume  $(x'', \pm Y) \neq (1, 1)$  in the following discussion.

- Case  $2 \nmid c$ : We need to consider the equation

$$(c+2)Y^4 - (c-2)x''^2 = 4.$$

To apply Lemma 6, we notice that  $(x_1, y_1) = (1, 1)$  is the minimal solution to

$$(c+2)y^2 - (c-2)x''^2 = 4.$$

Moreover, in general it is not true that  $ax_1^2 - 3 = x_3$ . Using an argument similar to that of the case  $\mu = c + 2$  and  $2 \nmid c$ , we have  $x_3 = c - 1$ . So the additional solution comes from  $ax_1^2 - 3 = c - 1$ , where  $c - 1$  is a square. Put  $c = (2s)^2 + 1$ ,  $s \in \mathbb{N}$ . Then we get the solution  $(Y, x'') = (2s, (2s)^2 + 2)$  of equation (3.4). This is corresponding to the solutions  $(X, \pm Y) = ((2s)^4 + (2s)^2 - 1, 2s)$  and  $(X, \pm Y) = (1, 2s)$  seen in the case (4) of Theorem 1, with  $n$  even.

- Case 4|c: Equation (3.4) becomes

$$(c/2 + 1)Y^4 - (c/2 - 1)x''^2 = 2.$$

By Lemma 7, there is no additional solution.

- The most difficult case in the paper is  $c \equiv 2 \pmod{4}$ . From equation (3.4), we deduce

$$((c/2 + 1)/2)Y^4 - ((c/2 - 1)/2)x''^2 = 1.$$

We cannot explicitly solve it. But we can use Lemma 8 to see that there is at most one non-trivial integer solution  $(Y, x'')$ . This will possibly provide at most two integer solutions to equation (1.1). This corresponds to case (5) of Theorem 1.

Finally, when  $(c/2 + 1)/2 = s^2 + s + 1$ , by Lemma 9, we obtain the case (4) of Theorem 1, with  $n$  odd. This completes the proof of Theorem 1.

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