On minimal additive complements of integers

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Abstract

Let $C, W \subseteq \mathbb{Z}$. If $C + W = \mathbb{Z}$, then the set C is called an additive complement to W in $\mathbb{Z}$. If no proper subset of C is an additive complement to W, then C is called a minimal additive complement. Let $X \subseteq \mathbb{N}$. If there exists a positive integer T such that $x+T \in X$ for all sufficiently large integers $x \in X$, then we call X eventually periodic. In this paper, we study the existence of a minimal complement to W when W is eventually periodic or not. This partially answers a problem of Nathanson.

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1 Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}$ be the set of integers. For $A, B \subseteq \mathbb{Z}$ and $k \in \mathbb{Z}$, let $A + B = \{a + b : \ a \in A, \ b \in B\}$ and $kA = \{ka : a \in A\}$. If $A + B = \mathbb{Z}$, then A is called an additive complement to B in $\mathbb{Z}$. If no proper subset of A is a complement to B, then A is called a minimal complement to B in $\mathbb{Z}$.

It is easy to see that if $A \subseteq \mathbb{Z}$ is a (minimal) complement to $B \subseteq \mathbb{Z}$, then A is also a (minimal) complement to $B + d$, $d \in \mathbb{Z}$, where $B + d = \{b + d : b \in B\}$.

In 2011, Nathanson [4] proved the following theorem.

Nathanson's theorem (See [4, Theorem 8]). *Let W be a nonempty, finite set of integers. In $\mathbb{Z}$, every complement to W contains a minimal complement to W.*

In the same paper, Nathanson also posed the following problem.

Problem (See [4, Problem 11]). Let W be an infinite set of integers. Does there exist a minimal complement to W? Does there exist a complement to W that does not contain a minimal complement?

For the second part of the above problem, in 2012, Chen and Yang [2] gave two infinite sets W_1 and W_2 of integers such that there exists a complement to W_1 that does not contain a minimal complement and every complement to W_2 contains a minimal complement.

For the first part of the above problem, in 2012, Chen and Yang [2] proved the following results.

Theorem A (See [2, Theorem 1]). *Let W be a set of integers with* $\inf W = -\infty$ *and* $\sup W = +\infty$. *Then there exists a minimal complement to W.*

By Theorem A, now we only need to consider the cases $\inf W > -\infty$ or $\sup W < +\infty$. Without loss of generality, we may assume that $\inf W > -\infty$.

Theorem B (See [2, Theorem 2]). *Let $W = \{1 = w_1 < w_2 < \cdots\}$ be a set of integers and*

$$\overline{W} = (\mathbb{Z} \cap (0, +\infty)) \setminus W = \{\overline{w_1} < \overline{w_2} < \cdots\}.$$

(a) If $\limsup_{i\to+\infty}(w_{i+1} - w_i) = +\infty$, then there exists a minimal complement to W.

(b) If $\lim_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) = +\infty$, then there does not exist a minimal complement to W.

Let $W = \cup_{k=0}^{\infty}[10^k, 2\times 10^k]$. Then it is clear that both $\limsup_{i\to+\infty}(w_{i+1} - w_i) = +\infty$ and $\limsup_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) = +\infty$ hold. Hence $\lim_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) = +\infty$ in Theorem B (b) cannot be changed to $\limsup_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) = +\infty$.

In this paper, we will give further results on Nathanson's problem and deal with some sets W do not satisfy the conditions of Theorem B.

First we give some definitions. Let $S \subseteq \mathbb{N}$. Denote by $S \mod m$ the set of residues of S modulo m, i.e.,

$$S \mod m = \{r : r \in \{0, 1, \ldots, m-1\}, r \equiv s \pmod m \text{ for some } s \in S\}.$$

Let $X \subseteq \mathbb{N}$. If there exists a positive integer T such that $x + T \in X$ for all $x \in X$, then we call X *periodic with period* T. If $X \cup C$ is a periodic set for some finite set $C \subseteq \mathbb{N}$, then we call X *quasiperiodic*. If there exists a positive integer T such that $x + T \in X$ for all sufficiently large integers $x \in X$, then we call X *eventually periodic with period* T. Clearly, a periodic set must be quasiperiodic and a quasiperiodic set must be eventually periodic. If W is eventually periodic with $|\mathbb{N} \setminus W| = +\infty$, then both $\lim_{i\to+\infty}(w_{i+1} - w_i) < +\infty$ and $\lim_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) < +\infty$ hold. Hence W does not satisfy the conditions of Theorem B.

Suppose that W is an eventually periodic set and m is a positive period.

By shifting a number, we may assume that W has the following structure:

$$(1) \qquad W = (m\mathbb{N} + X_m) \cup Y^{(0)} \cup Y^{(1)},$$

where $X_m \subseteq \{0, 1, \ldots, m-1\}$, $Y^{(0)} \subseteq \mathbb{Z}^-$, $Y^{(1)}$ are finite sets with $Y^{(0)} \bmod m \subseteq X_m$ and $(Y^{(1)} \bmod m) \cap X_m = \emptyset$.

For example, if $W = \{2, 4, 7, 8, 9, 12, 13, 17, 18, 22, 23, 27, 28, \ldots\}$, then by shifting a number 5, we may assume that

$$W = \{-3, -1, 2, 3, 4, 7, 8, 12, 13, 17, 18, 22, 23, \ldots\}.$$

Hence $m = 5$, $X_m = \{2, 3\}$, $Y^{(0)} = \{-3\}$, $Y^{(1)} = \{-1, 4\}$.

In this paper, we study that what conditions are needed to ensure the existence of a minimal complement to W. First we prove a sufficient condition.

Theorem 1. *Let W be defined in (1). If there exists a minimal complement to W, then there exists $C \subseteq \{0, 1, \ldots, m-1\}$ such that the following two conditions hold:*

(a) $C + (X_m \cup Y^{(1)}) \mod m = \{0, 1, \ldots, m-1\}$;

(b) For any $c \in C$, there exists $y \in Y^{(1)}$ such that
$c + y \not\equiv c' + x \pmod m$, where $c' \in C$, $x \in X_m$.

Remark 1. *By the proof of Theorem 1, we know that Theorem 1 also holds when $Y^{(1)}$ is an infinite set with $|Y^{(1)} \cap \mathbb{Z}^-| < +\infty$.*

Let $m = 3$, $X_m = \{0\}$, $Y^{(1)} \subseteq 3\mathbb{N} + 1$. By Theorem 1, we have the following corollary.

Corollary 1. *Let $Y \subseteq 3\mathbb{N} + 1$ and $W = 3\mathbb{N} \cup Y$. Then there does not exist a minimal complement to W.*

Remark 2. *We can choose an infinite set Y in Corollary 1 such that W is not eventually periodic. Hence, there exists an infinite, not eventually periodic set $W \subseteq \mathbb{N}$ such that $w_{i+1} - w_i \in \{1, 2, 3\}$ for all i, and there does not exist a minimal complement to W.*

Remark 3. *If $W \subseteq \mathbb{N}$ is a quasiperiodic set, then $Y^{(1)} = \emptyset$ and the condition (b) in Theorem 1 does not hold. Hence there does not exist a minimal complement to W.*

In the next step we prove a necessary condition.

Theorem 2. *Let W be defined in* (1). *Suppose that there exists $C \subseteq \{0, 1, \ldots, m-1\}$ such that the following two conditions hold:*

(a) $C + (X_m \cup Y^{(1)}) \mod m = \{0, 1, \ldots, m-1\}$;

(b) For any $c \in C$, there exists $y \in Y^{(1)}$ such that $c+y \not\equiv c'+x \pmod m$, where $c' \in C \setminus \{c\}$, $x \in X_m \cup Y^{(1)}$.

Then there exists a minimal complement to W.

By Theorems 1 and 2, we have the following corollary.

Corollary 2. *Let $W = (m\mathbb{N}+X_m) \cup Y^{(0)} \cup \{a\}$, where $X_m \subseteq \{0, 1, \ldots, m-1\}$, $(Y^{(0)} \mod m) \subseteq X_m$ and $a \not\equiv x \pmod m$ if $x \in X_m$. Then there exists a minimal complement to W if and only if there exists a subset $C \subseteq \{0, 1, \ldots, m-1\}$ such that:*

(a) $C + (X_m \cup \{a\}) \mod m = \{0, 1, \ldots, m-1\}$;

(b) For any $c \in C$, $c + a \not\equiv c' + x \pmod m$, where $c' \in C \setminus \{c\}$ and $x \in X_m$.

We see that Theorems 1 and 2 transfer Nathanson's problem into a finite modulo version when W is an eventually periodic set. In the next theorem, we give a sufficient and necessary condition, but we cannot bound the module.

Theorem 3. *Let W be defined in (1). There exists a minimal complement to W if and only if there exists $T \in \mathbb{Z}^+, m \mid T$, and $C \subseteq \{0, 1, \ldots, T-1\}$ such that*

(a) $C+(X_T \cup Y^{(1)}) \mod T = \{0, 1, \ldots, T-1\}$, where $X_T = \cup_{i=0}^{\frac{T}{m}-1}\{im+X_m\}$;

(b) for any $c \in C$, there exists $y \in Y^{(1)}$ for which $c+y \not\equiv c'+x \pmod T$, where $c' \in C \setminus \{c\}$ and $x \in X_T$.

Finally, as a complement to Remark 2, we give the following theorem.

Theorem 4. *There exists an infinite, not eventually periodic set* $W \subseteq \mathbb{N}$ *such that* $w_{i+1} - w_i \in \{1, 2\}$ *for all* i *and there exists a minimal complement to* W.

Now we pose two problems for further research.

Problem 1. *We know that Theorem 1 also holds when* $Y^{(1)}$ *is infinite. Is Theorem 2 also true when* $Y^{(1)}$ *is infinite?*

Problem 2. *Does there exist an upper bound for* T *in Theorem 3 using* $m, Y^{(0)}$ *and* $Y^{(1)}$*?*

2 Proofs

Proof of Theorem 1. Suppose that D is a minimal complement to W. For $i \in \{0, 1, \ldots, m-1\}$, let $D_i = \{d \in D : \ d \equiv i \pmod{m}\}$ and

$$C = \{j : \ 0 \leq j \leq m-1 \text{ and } |D_j \cap \mathbb{Z}^-| = +\infty\}.$$

For any $t \in \{0, 1, \ldots, m-1\} \setminus C$, the set $\{d \in D : \ d \equiv t \pmod{m}\} + W$ does not contain any sufficiently small negative integers. It follows from $D + W = \mathbb{Z}$ that $C + W \bmod m = \{0, 1, \ldots, m-1\}$. That is, $C + (X_m \cup Y^{(1)}) \bmod m = \{0, 1, \ldots, m-1\}$.

Next we shall prove (b). Suppose that there exists $c \in C$ such that for any $y \in Y^{(1)}$ there exist $c' \in C$ and $x \in X_m$ with $c + y \equiv c' + x \pmod{m}$. We take an integer $d \in D$ such that $d \equiv c \pmod{m}$ and we shall prove that $D \setminus \{d\}$ is also a complement to W. For any integer n, write $n = d' + w$, where $d' \in D$ and $w \in W$.

Case 1. $d' \neq d$. Then $n = d' + w \in (D \setminus \{d\}) + W$.

Case 2. $d' = d$.

Subcase 2.1. $(\{w\} \bmod m) \subseteq X_m$. In this case, there exists a positive integer k_0 such that $w + km \in W$ for all integers $k \geq k_0$. Since $|D_c \cap \mathbb{Z}^-| =$

$+\infty$, it follows that there exists an integer $k \geq k_0$ such that $d - km \in D$. Hence $n = (d-km)+(w+km)$, where $d-km \in D \setminus \{d\}$ and $w+km \in W$. That is, $n \in (D \setminus \{d\}) + W$.

Subcase 2.2. $w \in Y^{(1)}$. Since $(\{c+y\} \bmod m) \subseteq (C + X_m \bmod m)$ for any $y \in Y^{(1)}$ and $d \equiv c \pmod m$, $w \in Y^{(1)}$, it follows that $\{n\} \bmod m = \{d+w\} \bmod m\} \subseteq (C + X_m \bmod m)$. Hence there exist a $c' \in D$ with $c' \pmod m \in C$ and $x \in W$ with $x \bmod m \in X_m$ such that $n \equiv c' + x \pmod m$. We choose a sufficiently large integer k such that $c' - km \in D$, $c' - km \neq d$ and $x + km \in W$. Hence $n = (c'-km)+(x+km)$, where $c' - km \in D \setminus \{d\}$ and $x + km \in W$.

Hence, $(D\setminus\{d\})+W = \mathbb{Z}$ which contradicts the fact that D is a minimal complement. Therefore, (b) holds. □

Proof of Theorem 2. Let $C_1 = C + X_m \bmod m$, $C_2 = \{0, 1, \dots, m-1\} \setminus C_1$,

$$C' = \{d \in \mathbb{Z} : \ d \equiv c \pmod m \text{ for some } c \in C\},$$

$$C_1' = \{d \in \mathbb{Z} : \ d \equiv c \pmod m \text{ for some } c \in C_1\},$$

$$C_2' = \{d \in \mathbb{Z} : \ d \equiv c \pmod m \text{ for some } c \in C_2\}.$$

By (a), we have $C' + W = \mathbb{Z}$. Since $C + X_m \bmod m = C_1$, it follows that

$$C' + (W \setminus Y^{(1)}) \bmod m = C + X_m \bmod m = C_1,$$

and so $\left(C' + (W \setminus Y^{(1)})\right) \cap C_2' = \emptyset$. It follows from (b) that $C_2' \neq \emptyset$. Noting that $C' + W = \mathbb{Z}$, we have $(C' + Y^{(1)} \bmod m) \supseteq C_2'$. Since $Y^{(1)}$ is a finite set, by the proof of Nathanson's theorem (See [4, Theorem 4, page 2015]), there exists $D' \subseteq C'$ such that $D' + Y^{(1)} \supseteq C_2'$ and for any $d \in D'$,

$$(D' \setminus \{d\}) + Y^{(1)} \not\supseteq C_2'.$$

Next we shall prove that D' is a minimal complement to W.

For $i \in C$, let $D_i' = \{d \in D' : d \equiv i \pmod m\}$. First we prove that $|D_i' \cap \mathbb{Z}^-| = +\infty$ for all $i \in C$. Suppose that there exists a $j \in C$

such that $|D'_j \cap \mathbb{Z}^-| < +\infty$. By (b), there exists a $y \in Y^{(1)}$ such that $j + y \not\equiv c + x \pmod{m}$, where $c \in C \setminus \{j\}$, $x \in X_m \cup Y^{(1)}$ and so

$$D' + Y^{(1)} \not\supseteq \{d \in \mathbb{Z} : d \equiv j + y \pmod{m}\}.$$

Noting that $(\{j+y\} \bmod m) \not\subseteq C + X_m \bmod m = C_1$, we have $(\{j + y\} \bmod m) \subseteq C_2$. It follows that $D' + Y^{(1)} \not\supseteq C'_2$, a contradiction. Hence, $|D'_i \cap \mathbb{Z}^-| = +\infty$ for all $i \in C$.

Next we prove that D' is a complement. For any integer $n \in C'_1$, by $C + X_m \bmod m = C_1$, there exists $c \in C$ and $x \in X_m$ such that $n \equiv c + x \pmod{m}$. Since $|D'_c \cap \mathbb{Z}^-| = +\infty$, there exists a sufficiently small negative integer $d \in D'_c$ such that $n - d > 0$. The congruences $n \equiv c + x \pmod{m}$ and $d \equiv c \pmod{m}$ imply that $n - d \equiv x \pmod{m}$. Hence, $n - d \in m\mathbb{N} + X_m$ and so

$$n = d + (n - d) \in D'_c + (m\mathbb{N} + X_m) \subseteq D' + W.$$

Hence $C'_1 \subseteq D' + W$. On the other hand, $D' + W \supseteq D' + Y^{(1)} \supseteq C'_2$. Therefore, $D' + W = \mathbb{Z}$.

Finally, we prove that D' is a minimal complement. For any $d \in D'$, we have

$$\Big((D' \setminus \{d\}) + (W \setminus Y^{(1)}) \bmod m\Big) \subseteq C + X_m \bmod m = C_1.$$

It follows that

$$\Big((D' \setminus \{d\}) + (W \setminus Y^{(1)})\Big) \cap C'_2 = \emptyset,$$

and so $(D' \setminus \{d\}) + W \not\supseteq C'_2$. Hence $(D' \setminus \{d\}) + W \neq \mathbb{Z}$.

Therefore, D' is a minimal complement to W. □

Proof of Theorem 3. Assume that the set W satisfies the conditions of Theorem 3. Applying Theorem 2 with $m = T$, it follows that W has a minimal complement.

Suppose that W has a minimal complement denoted by E. We will prove the existence of a positive integer T and a set $C \subseteq \{0, 1, \ldots, T - 1\}$ which satisfy the conditions of Theorem 3. We will show that there exist positive integers K and L with $L > K$ such that $T = L - K$. We will prove that this integer T and the set

$$C = \{l : K \leq l < L,\ \ l \in E\} \bmod L - K$$

are suitable.

For $0 \leq i < m$, let

$$E_i^- = \{e : e < 0, e \in E, e \equiv i \pmod{m}\}.$$

Let $0 \leq i_1 < i_2 < \ldots < i_t < m$ be the sequence of indices with $|E_{i_j}^-| = \infty$. It is clear that there exists an integer N_0 such that, $e \in E$ and $e \leq N_0$ imply that $e \in E_{i_j}$ for some i_j. It follows from Theorem 1 that $Y^{(0)} \cup Y^{(1)} \neq \emptyset$. Let

$$y_+ = \max\{y : y \in Y^{(0)} \cup Y^{(1)}\},$$
$$y_- = \min\{y : y \in Y^{(0)} \cup Y^{(1)}\},$$

and $y_0 = y_+ - y_- + 1$. Let $\chi_E(k)$ denote the characteristic function of the set E, i.e.,

$$\chi_E(k) = \begin{cases} 1, \text{ if } k \in E; \\ 0, \text{ if } k \notin E. \end{cases}$$

Define the positive integer A by $A = N_0 + \min\{0, y_-\}$. Consider the following vectors:

$$\begin{aligned}
\mathbf{v}_A &= \left(\chi_E(A + y_-), \chi_E(A + y_- + 1), \ldots, \chi_E(A + y_+)\right), \\
\mathbf{v}_{A-m} &= \left(\chi_E(A - m + y_-), \chi_E(A - m + y_- + 1), \ldots, \chi_E(A - m + y_+)\right), \\
&\vdots \\
\mathbf{v}_{A-im} &= \left(\chi_E(A - im + y_-), \chi_E(A - im + y_- + 1), \ldots, \chi_E(A - im + y_+)\right), \\
&\vdots
\end{aligned}$$

It is clear that there are infinitely many vectors $\mathbf{v}_A, \mathbf{v}_{A-m}, \ldots, \mathbf{v}_{A-im}, \ldots$, each of them has y_0 coordinates, which are 0 or 1. Since there are at most 2^{y_0} different vectors, by the pigeon hole principle, there exists a vector $\mathbf{v}$

among them which occurs infinitely many times. In other words there exists an infinite sequence $0 \le k_1 < k_2 < \cdots$ such that $\mathbf{v}_{A-k_i m} = \mathbf{v}$. Define L by $L = A - k_1 m$. Obviously it can be chosen a k_i large enough such that

$$[A - k_i m, A - k_1 m[\cap E_{i_j} \neq \emptyset$$

and $k_i m - k_1 m \ge \max\{y_0, y_+, -y_-\}$ hold for every index i_j. In view of this fact we define K by $K = A - k_i m$. and $T = L - K$. It follows from the definition that K and L have the following properties.

(2) $$L \le N_0 + \min\{0, y_-\},$$

(3) $$K \le L - y_0,$$

(4) $$m \mid L - K,$$

(5) $$\chi_E(K + i) = \chi_E(L + i), \;\; for \;\; y_- \le i \le y_+,$$

(6) $$[K, L[\cap E_{i_j} \neq \emptyset \text{ for all } i_j.$$

In the next step we show that the positive integer T and set C defined above satisfy the conditions of Theorem 3.

We know from the conditions of Theorem 3 that $W = (T\mathbb{N} + X_T) \cup Y^{(0)} \cup Y^{(1)}$ and $X_T \subseteq \{0, 1 \ldots, T-1\}$. First we prove that $C + (X_T \cup Y^{(1)}) \bmod T = \{0, 1, \ldots T-1\}$. Let $K \le l < L$. It follows that $l = e + w$, where $e \in E$ and $w \in W$. As $w \ge \min\{0, y_-\}$, it follows from (2) that $e = l - w < L - \min\{0, y_-\} \le N_0$, thus we have $e \in E^-_{i_j}$.

Suppose that $w \in Y^{(0)} \cup Y^{(1)}$. Then we have $y_- \le w \le y_+$, which implies that $e = l - w$, where $K - y_+ \le e < L - y_-$. We have three cases.

Case 1. $K - y_+ \le e < K$. By (5) we have $e + (L - K) \in E$ and $K \le L - y_+ \le e + L - K < L$. Thus we have $l \equiv c + w \pmod T$, where $c \in C$ and $w \in X_T \cup (Y^{(1)} \bmod T)$.

Case 2. $K \leq e < L$. It follows that $l \equiv c + w \pmod{T}$, where $c \in C$ and $w \in X_T \cup (Y^{(1)} \bmod T)$.

Case 3. $L \leq e < L - y_-$. By (5) we have $e - (L - K) \in E$ and $K \leq e - (L - K) < K - y_- < L$, thus we have $l \equiv c + w \pmod{T}$, where $c \in C$ and $w \in X_T \cup (Y^{(1)} \bmod T)$.

Suppose that $w \in T\mathbb{N} + X_T$. Since $w \geq 0$ and $e = l - w < L \leq N_0$, we have $e \in E_{i_j}^-$, which implies that $e \equiv i_j \pmod{m}$. It follows from (6) that there exists an $e^{'}$ such that $e^{'} \in E_{i_j}^-$ and $K \leq e^{'} < L$, thus we have $e^{'} \equiv i_j \pmod{m}$. Let $w \equiv x \pmod{m}$, where $0 \leq x < m$, $x \in X_m$. Obviously, $l \equiv e + w \equiv i_j + x \equiv e^{'} + x \pmod{m}$. By using (4) it follows that there exists a u with $0 \leq u < \frac{L-K}{m}$ such that $l \equiv e^{'} + um + x \pmod{L-K}$. Therefore, $c \equiv e^{'} \pmod{L-K}$, where $0 \leq c < L - K$, $c \in C$ and $0 \leq um + x < L - K$ and $um + x \in X_T$ are suitable.

In the next step we show that the second condition of Theorem 3 holds. For every $K \leq e < L \leq N_0$ and $e \in E$, there exists a $w \in W$ such that $e + w \neq e^{'} + w^{'}$, when $e \neq e^{'}$, $e' \in E$ and $w' \in W$. If $w \in (m\mathbb{N} + X_m) \cup Y^{(0)}$, then there exists a positive integer s such that $e + w = (e - sm) + (w + sm)$, where $e - sm \in E$ and $w + sm \in W$, which is absurd. Then we may assume that $w \in Y^{(1)}$. It follows that $K + y_- \leq e + w \leq L + y_+$. Obviously, it is enough to prove that $e + w \not\equiv e^{'} + w^{'} \pmod{L-K}$, where $K \leq e^{'} < L$, $e^{'} \in E$ and $w^{'} \in X_T \cup (Y^{(1)} \bmod T)$. Suppose that for $w^{'} \in X_T$ we have $e + w = e^{'} + w^{'} + t(L-K)$ for some integer t. Then we have $e + w = e^{'} + w^{'} + t(L-K) = (e^{'} - sm) + (w^{'} + sm + t(L-K))$, where $e^{'} - sm \in E$ and $w^{'} + sm + t(L-K) \in W$ for some positive integer s, which is absurd.

For any $w^{'} \in Y^{(1)}$, clearly we have $K + y_- \leq e + w, e^{'} + w^{'} \leq L + y_+$. Assume that $e + w \equiv e^{'} + w^{'} \pmod{L-K}$. It follows that either $e + w = e^{'} + w^{'}$ or $e + w = e^{'} + w^{'} + (L-K)$ or $e + w = e^{'} + w^{'} - (L-K)$.

Case 1. $e + w = e^{'} + w^{'}$. Then we have $e = e^{'}$ and $w = w^{'}$, which is impossible.

Case 2. $e + w = e^{'} + w^{'} + (L-K)$. Then we have $K + y_- \leq e^{'} + w^{'} \leq K + y_+$.

Thus we have $K + y_- - y_+ \le e' \le K + y_+ - y_-$. It follows from (5) that $e' + (L-K) \in E$ which implies that $e+w = ((e' + (L-K)) + w'$. Therefore we have $w = w'$ and $e = e' + (L-K)$ which is absurd because $K \le e, e' < L$.

Case 3. $e+w = e'+w'-(L-K)$. Then we have $L+y_- \le e'+w' \le L+y_+$. Thus we have $L + y_- - y_+ \le e' \le L + y_+ - y_-$. It follows from (5) that $e' - (L-K) \in E$ which implies that $e+w = ((e' - (L-K)) + w'$. Therefore we have $w = w'$ and $e = e' - (L-K)$, which is a contradiction because $K \le e, e' < L$.

The proof of Theorem 3 is completed. □

Proof of Theorem 4. By induction we can construct $\{d_i\}_{i=1}^{\infty}$, $\{W_i\}_{i=1}^{\infty}$ and $\{c_i\}_{i=1}^{\infty}$ such that

(i) $d_1 = -1,\ W_1 = \{1, 2, \ldots, 12\},\ c_1 = -3$;

(ii) d_i is the largest negative integer $\notin W_{i-1} + \{c_1, c_2, \ldots, c_{i-1}\}$ for $i \ge 2$;

(iii) $c_i < d_i + 2c_{i-1}$ for all $i \ge 2$;

(iv) for $i \ge 2$, let $W_i = W_{i-1} \cup \left([-2c_{i-1}, -2c_i - 1] \setminus \cup_{j=1}^{i-1}\{-c_i + d_j\}\right)$.

Let $W = \cup_{i=1}^{\infty} W_i$ and $C = \{c_i\}_{i=1}^{\infty}$.

Now we prove that C is a minimal complement to W.

First we prove $d_{i+1} - d_i \le -2$ for all integers $i \ge 1$. Clearly $d_2 = -3,\ d_2 - d_1 = -2$. Suppose that $d_{i+1} - d_i \le -2$ for all integers $i < k$ $(k \ge 2)$. Since

$$d_k = (d_k - c_k) + c_k, \quad d_k - 1 = (d_k - 1 - c_k) + c_k,$$

$$-2c_{k-1} \le d_k - 1 - c_k < d_k - c_k < -c_k + d_{k-1},$$

it follows that $d_k - c_k,\ d_k - 1 - c_k \in W_k$ and then $d_k, d_k - 1 \in W_k + \{c_k\}$. Hence $d_{k+1} \le d_k - 2$. By (iv), we have $w_{j+1} - w_j \in \{1, 2\}$. Since $d_k \to -\infty$, by (ii) we have $(-\infty, 9] \subseteq W + C$. For any integer $n \ge 10$, there exists an i such that $-c_{i-1} \le n < -c_i$. Hence

$$-c_i + d_1 < -c_{i-1} - c_i \le n - c_i < -2c_i,$$

and so $n - c_i \in W_i$, that is, $n \in W_i + \{c_i\}$. Therefore, $W + C = \mathbb{Z}$.

Next, we prove that the complement C is minimal. For any positive integer i, we write $d_i = c + w$ with $c \in C$ and $w \in W$. Now we shall prove that $c = c_i$. By (iv), we have $d_i - c_j \notin W$ for all integers $j > i$. Hence $c \neq c_j$ for all integers $j > i$. Since $-2c_{i-1}$ is the minimal value of $W \setminus W_{i-1}$ and for any positive integers $j \leq i-1$, $d_i - c_j \leq d_i - c_{i-1} < -2c_{i-1}$, it follows that $d_i - c_j \notin W \setminus W_{i-1}$ for all positive integer $j \leq i-1$. Noting that $d_i \notin W_{i-1} + \{c_1, \ldots, c_{i-1}\}$, we have $d_i \notin W + \{c_1, c_2, \ldots, c_{i-1}\}$. Hence $c = c_i$.

Therefore, C is a minimal complement to W. Furthermore, by (iii), we can choose suitable c_i such that W is infinite and not eventually periodic. □

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