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ON LIE IDEALS AND SYMMETRIC GENERALIZED (α,β)-BIDERIVATION IN PRIME RING

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Abstract. Let $\mathfrak{R}$ be a prime ring with char$(\mathfrak{R}) \neq 2$. A biadditive symmetric map $\Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is called symmetric (α,β)-biderivation if, for any fixed $y \in \mathfrak{R}$, the map $x \mapsto \Delta(x,y)$ is a (α,β)-derivation. A symmetric biadditive map $\Gamma : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric generalized (α,β)-biderivation if for any fixed $y \in \mathfrak{R}$, the map $x \mapsto \Gamma(x,y)$ is a generalized (α,β)-derivation of $\mathfrak{R}$ associated with the (α,β)-derivation $\Delta(.,y)$. In the present paper, we investigate the commutativity of a ring having a generalized (α,β)-biderivation satisfying certain algebraic conditions.

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1. INTRODUCTION

Throughout this article $\mathfrak{R}$ denotes an associative ring with center Z. An additive subgroup $\mathfrak{L}$ of R is said to be a Lie ideal of $\mathfrak{R}$ if $[l,r] \in \mathfrak{L}$, $\forall\ l \in \mathfrak{L}$ and $r \in \mathfrak{R}$. A Lie ideal $\mathfrak{L}$ is called square closed if $u^2 \in \mathfrak{L}\ \forall\ u \in \mathfrak{L}$. And it is easy to check that $2uv \in \mathfrak{L}\ \forall\ u,v \in \mathfrak{L}$.

A derivation $\mathfrak{d} : \mathfrak{R} \to \mathfrak{R}$ is an additive map such that $\mathfrak{d}(xy) = \mathfrak{d}(x)y + x\mathfrak{d}(y)\ \forall\ x,y \in R$. An additive map $\mathbb{F} : \mathfrak{R} \to \mathfrak{R}$ is a generalized derivation if there exists a derivation $\mathfrak{d} : \mathfrak{R} \to \mathfrak{R}$ such that $\mathbb{F}(xy) = \mathbb{F}(x)y + x\mathfrak{d}(y)$ holds $\forall\ x,y \in \mathfrak{R}$. If $\varphi : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric map ($\varphi(x,y) = \varphi(y,x)\ \forall\ x,y \in \mathfrak{R}$) the map $\tau : \mathfrak{R} \to \mathfrak{R}$ defined by $\tau(x) = \varphi(x,x)$ is the trace of φ. If φ is also biadditive (i.e., additive in both arguments), its trace τ satisfies $\tau(x+y) = \tau(x)+\tau(y)+2\varphi(x,y)$, $\forall\ x,y \in \mathfrak{R}$. A symmetric biadditive map $\varphi : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric biderivation if $\varphi(xy,z) = \varphi(x,z)y + x\varphi(y,z)\ \forall\ x,y,z \in \mathfrak{R}$. The concept of symmetric biderivation was introduced by G. Maksa [9]. A symmetric biadditive map $\zeta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric left bicentralizer if $\zeta(xy,z) = \zeta(x,z)y$ (and consequently $\zeta(x,yz) = \zeta(x,y)z)\ \forall\ x,y,z \in \mathfrak{R}$.

Symmetric biderivations were proved to be related to the general solution of some functional equations (see [11]). The maps $(u,v) \mapsto \Psi[u,v], \Psi \in \mathcal{C}$, are typical examples of biderivations and they were called inner biderivations. Here $\mathcal{C}$ is the extended centroid of $\mathfrak{R}$, that is, the center of the two-sided Martindale quotient ring $\mathcal{Q}$ (we refer the reader to [3] for more details). In [7], it is shown that every biderivation of a noncommutative prime ring $\mathfrak{R}$ is inner. In [5], this result is extended to semiprime. In [14], it is proved that if φ is a nonzero symmetric biderivation, where $\mathfrak{R}$ is a prime ring of char$(\mathfrak{R}) \neq 2$, with the property:

$$\varphi(x,x)x = x\varphi(x,x), x \in \mathfrak{R}. \tag{1.1}$$

then $\mathfrak{R}$ is commutative. He also proved that if φ_1, φ_2 are nonzero biderivations on $\mathfrak{R}$, $\mathfrak{D}$ is a symmetric biadditive map and $\tau_1(\tau_2(x)) = \mathfrak{d}(x)$ holds $\forall\ x \in \mathfrak{R}$, where τ_1, τ_2, and $\mathfrak{d}$ are the traces of φ_1, φ_2, and $\mathfrak{D}$, respectively, then either $\varphi_1 = 0$ or $\varphi_2 = 0$. Let's mention two results proved in [15]. The first one states that if φ_1 and φ_2 are symmetric biderivations on a prime ring $\mathfrak{R}$, char$(\mathfrak{R}) \neq 2, 3$, such that $\varphi_1(x,x)\varphi_2(x,x) = 0$ holds $\forall\ x \in \mathfrak{R}$, then either $\varphi_1 = 0$ or $\varphi_2 = 0$. The second result says that if $[[\varphi(x,x), x], x] \in Z\ \forall\ x \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative. In [16] the authors extended the results in [14] assuming condition (1.1) over a nonzero ideal and a nonzero Lie ideal of a prime ring respectively.

The notion of generalized biderivation was introduced in [6]. A biadditive map $\Gamma : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is a generalized biderivation associated with a biderivation $\Delta : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ if for every $x, y \in \mathfrak{R}$, the maps $y \mapsto \Gamma(x,y)$ and $y \mapsto \Gamma(y,x)$ are generalized derivations of $\mathfrak{R}$ associated with $\Delta(x,.)$ and $\Delta(.,x)$. That is, $\Gamma(xy,z) = \Gamma(x,z)y + x\Delta(y,z)$ and $\Gamma(x,yz) = \Gamma(x,y)z + y\Delta(x,z)$ hold $\forall\ x,y,z \in \mathfrak{R}$. Brešar shown that every generalized biderivation Γ of an ideal I ($\Gamma : I \times I \to \mathfrak{R}$) of a prime ring $\mathfrak{R}$ with char$(\mathfrak{R}) \neq 2$, is of the form $\Gamma(x,y) = xay + ybx$ for some $a, b \in \mathfrak{Q}$, where $\mathfrak{Q}$ is Martindale quotient ring of $\mathfrak{R}$ (see [8] and [10] for details). In [1] authors extended some results of [14, 15] to generalized biderivations on prime and semiprime rings. Recently, in [2], symmetric generalized (θ,ϕ)-biderivations of a prime ring $\mathfrak{R}$ with char$(\mathfrak{R}) \neq 2$ have been considered. Notice that a symmetric left bicentralizer is a symmetric generalized biderivation associated with the biderivation $T = 0$.

2. Preliminaries

Lemma 1 ([12, Lemma 3]). *If the prime ring $\mathfrak{R}$ contains a commutative nonzero right ideal I, then $\mathfrak{R}$ is commutative.*

Lemma 2 ([13, Lemma 2.6]). *Let $\mathfrak{R}$ be a prime ring with char$(\mathfrak{R}) \neq 2$. If $\mathfrak{L}$ is a commutative Lie ideal of $\mathfrak{R}$, then $\mathfrak{L} \subseteq Z$.*

Lemma 3 ([4, Lemma 4]). *Let $\mathfrak{R}$ be a prime ring with char$(\mathfrak{R}) \neq 2$. If $\mathfrak{L} \nsubseteq Z$ is a Lie ideal of $\mathfrak{R}$ and $a\mathfrak{L}b = (0)$, then either $a = 0$ or $b = 0$.*

Lemma 4. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a Lie ideal of $\mathfrak{R}$ and α is automorphism. If $T : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is a symmetric left α-centralizer such that $T(\alpha(x_1), \alpha(x_1)) = 0\ \forall\ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $T = 0$.*

Proof. Assume that $\mathfrak{L} \nsubseteq Z$. We have

$$T(x_1, x_1) = 0\ \forall\ x_1 \in \mathfrak{L}. \tag{2.1}$$

Linearizing (2.1), we get

$$T(\alpha(x_1),\alpha(x_2)) = 0 \;\forall\; x_1,x_2 \in \mathfrak{L}. \tag{2.2}$$

Let us replace x_1 by $x_1 s - s x_1 \;\forall\; s \in \mathfrak{R}$ in (2.2), to obtain

$$T(\alpha(s),\alpha(x_2))\alpha^2(x_1) = 0 \;\forall x_1,x_2 \in \mathfrak{L},\; s \in \mathfrak{R}.$$

That is, $\alpha^{-2}(T(\alpha(s),\alpha(x_2)))\mathfrak{L}\alpha^{-2}(T(\alpha(r),\alpha(x_2))) = (0) \;\forall\; x_2 \in \mathfrak{L},\; s \in \mathfrak{R}$. By Lemma 3, $\alpha^{-2}(T(\alpha(r),\alpha(x_2))) = 0 \;\forall\; x_2 \in \mathfrak{L},\; r \in \mathfrak{R}$ that is, $T(\alpha(s),\alpha(x_2)) = 0$. Now we replace x_2 by $[x_2,r] \;\forall\; r \in \mathfrak{R}$, we get $T(\alpha(s),\alpha(r))\alpha^2(x_2) = 0$, that is $\alpha^{-2}(T(\alpha(s),\alpha(r)))\mathfrak{L}\alpha^{-2}(T(\alpha(s),\alpha(r))) = (0)$ and hence again by Lemma 3 gives that $\alpha^{-2}(T(\alpha(s),\alpha(r))) = 0 \;\forall\; r,s \in \mathfrak{R}$ i.e., $T(\alpha(s),\alpha(r)) = 0 \;\forall\; r,s \in \mathfrak{R}$. Now, replacing s by $\alpha^{-1}(t_1)$ and r by $\alpha^{-1}(t_2)$, we get $T(t_1,t_2) = 0 \;\forall\; t_1,t_2 \in \mathfrak{R}$, that is $T = 0$. □

Lemma 5. *Let $\mathfrak{R}$ be a ring and α,β automorphisms. If Γ is a symmetric generalized (α,β)-biderivation associated with a symmetric (α,β)-biderivation Δ, then the map $\Gamma - \Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ is a symmetric left α-bicentralizer.*

Proof. The map $\Omega = \Gamma - \Delta$, is clearly biadditive. For all $x_1,x_2,x_3 \in \mathfrak{R}$,

$$\begin{aligned}\Gamma(x_1x_2,x_3) &= (\Gamma-\Delta)(x_1x_2,x_3)\\ &= \Gamma(x_1x_2,x_3) - \Delta(x_1x_2,x_3)\\ &= \Gamma(x_1,x_3)\alpha(x_2) + \beta(x_1)\Delta(x_2,x_3)\\ &\quad - \Delta(x_1,x_3)\alpha(x_2) - \beta(x_1)\Delta(x_2,x_3)\\ &= \Gamma(x_1,x_3)\alpha(x_2) - \Delta(x_1,x_3)\alpha(x_2)\\ &= \Omega(x_1,x_3)\alpha(x_2).\end{aligned}$$

Therefore, Ω is a symmetric left α-centralizer of $\mathfrak{R}$. □

3. Main results

Proposition 1. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$, α,β automorphisms, and $\Delta : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ a symmetric (α,β)-biderivation with trace δ. If $\delta(x_1) = 0 \;\forall\; x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $\Delta = 0$.*

Proof. By our hypothesis

$$\delta(u) = 0 \;\forall\; x_1 \in \mathfrak{L}. \tag{3.1}$$

Linearizing (3.1) and using (3.1), we get

$$2\Delta(x_1,x_2) = 0 \;\forall\; x_1,x_2 \in \mathfrak{L}. \tag{3.2}$$

If we replace x_1 by $x_1 r - r x_1$ in (3.2), to get

$$\beta(x_1)\Delta(r,x_2) - \delta(r,x_2)\alpha(x_1) = 0. \tag{3.3}$$

Again replacing r by rx_3 and using (3.2), we find that

$$\beta(x_1)\Delta(r,x_2)\alpha(x_3)-\Delta(r,x_2)\alpha(x_3x_1)=0. \tag{3.4}$$

Multiplying (3.3) from left by x_3, we get

$$\beta(x_1)\Delta(r,x_2)\alpha(x_3)-\Delta(r,x_2)\alpha(x_1x_3)=0. \tag{3.5}$$

From (3.4), (3.5) it follows that $\Delta(r,x_2)\alpha([x_1,x_3])=0\ \forall\ x_1,x_2,x_3\in\mathfrak{L}$ and $r\in\mathfrak{R}$. If we replace r by rs, $s\in\mathfrak{R}$, we obtain $\Delta(r,x_2)\alpha(s)\alpha([x_1,x_3])=0$ and hence $\alpha^{-1}(\Delta(r,x_2))\mathfrak{R}[x_1,x_3]=(0)$ Thus by primeness of $\mathfrak{R}$ it follows that either $[x_1,x_3]=0$ for $x_1,x_3\in\mathfrak{L}$ or $\alpha^{-1}(\Delta(r,x_2))=0$. If $[x_1,x_3]=0\ \forall\ x_1,x_3\in\mathfrak{L}$, then $\mathfrak{L}\subseteq Z$ by Lemma 2. In other case, $\alpha^{-1}(\Delta(r,x_2))=0\ \forall\ x_2\in\mathfrak{L}$, $r\in\mathfrak{R}$, that is, $\Delta(r,x_2)=0$. Replacing x_2 by x_2s-sx_2, we get

$$\beta(x_2)\Delta(r,s)-\Delta(r,s)\alpha(x_2)=0. \tag{3.6}$$

Now, replacing s by sx_3 in (3.6), we get

$$\beta(x_2)\Delta(r,s)\alpha(x_3)-\Delta(r,s)\alpha(x_3x_2)=0. \tag{3.7}$$

If we multiply (3.6) by x_3 to the right, and subtract (3.7) we get

$$\Delta(r,s)\alpha([x_2,x_3])=0\ \forall x_2,x_3\in\mathfrak{L},r,s\in\mathfrak{R}.$$

Replace r by tr, to get $\Delta(t,s)\alpha(t)\alpha([x_2,x_3])=0$, that is $\alpha^{-1}(\Delta(t,s))T[x_2,x_3]=(0)$. So by primeness of $\mathfrak{R}$ either $\Delta=0$ or $\mathfrak{L}$ is commutative. Hence, Lemma 2 gives the required conclusion. □

Theorem 1. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$, α,β automorphisms and $\Delta:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ a symmetric (α,β)-biderivation with trace δ. If $\delta(x_1)\in Z\ \forall\ x_1\in\mathfrak{L}$, then either $\mathfrak{L}\subseteq Z$ or $\Delta=0$.*

Proof. By Assumption we have

$$\delta(x_1)\in Z\ \forall\ x_1\in\mathfrak{L}. \tag{3.8}$$

The linearizing (3.8), we find that

$$2\Delta(x_1,x_2)\in Z\ \forall\ x_1,x_2\in\mathfrak{L}. \tag{3.9}$$

Replace x_1 by $2x_1^2$ in (3.9), to get

$$4\Delta(x_1,x_2)\alpha(x_1)+\beta(x_1)\Delta(x_1,x_2)\in Z. \tag{3.10}$$

Therefore, in particular $\alpha(x_1)\delta(x_1)+\beta(x_1)\delta(x_1)\in Z\ \forall\ x_1\in\mathfrak{L}$. Then we have

$$0=[\alpha(x_1)\delta(x_1)+\beta(x_1)\delta(x_1),r]=[\alpha(x_1)+\beta(x_1),r]\delta(x_1). \tag{3.11}$$

For every $r,s\in\mathfrak{R}$, we have

$$[\alpha(x_1)+\beta(x_1),rs]\delta(x_1)=0=[\alpha(x_1)+\beta(x_1),r]s\delta(x_1)=0.$$

By primeness of $\mathfrak{R}$, given an arbitrary element $x_1\in\mathfrak{L}$, we have either $\delta(x_1)=0$ or $\alpha(x_1)+\beta(x_1)\in Z$. If $(\alpha+\beta)(x_1)\in Z$ then $x_1\in Z$. If $Z\cap\mathfrak{L}=0$, then $\delta(x_1)=0$

$\forall\ x_1 \in \mathfrak{L}$. Assume that $Z \cap \mathfrak{L} \neq 0$. If $\mathfrak{L} \nsubseteq Z$, then there exists $x_2 \in \mathfrak{L} \setminus Z$. Then $\forall\ x_1 \in Z \cap \mathfrak{L}$, the element $x_1 + x_2, x_1 - x_2 \in \mathfrak{L} \setminus Z$. Hence $\delta(x_1 + x_2) = 0$ and $\delta(x_1 - x_2) = 0$ and hence $\delta(x_1) = 0$. In conclusion we prove that $\delta(x_1) = 0\ \forall\ x_1 \in Z \cap \mathfrak{L}$ and above we already know that $\delta(x_1) = 0\ \forall\ x_1 \in \mathfrak{L} \setminus Z$. That is, $\delta(x_1) = 0$ $\forall\ x_1 \in \mathfrak{L}$. □

Theorem 2. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R}) \neq 2$, $\mathfrak{L}$ a nonzero square closed Lie ideal of $\mathfrak{R}$, α, β automorphisms and $\Gamma_1, \Gamma_2 : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ be two symmetric generalized (α, β)-biderivations associated with symmetric (α, β)-biderivations $\Delta_1, \Delta_2 : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, respectively. If $\Gamma_1(x_1, x_1)\alpha(x_1) = \beta(x_1)\Gamma_2(x_1, x_1)\ \forall\ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $\Delta_2 = 0$.*

Proof. Assume that $\mathfrak{L} \nsubseteq Z$. Suppose that $\gamma_1, \gamma_2, \delta_1, \delta_2$ are traces of $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$, respectively. We have

$$\gamma_1(x_1)\alpha(x_1) = \beta(x_1)\gamma_2(x_1)\ \forall\ x_1 \in \mathfrak{L}. \tag{3.12}$$

Replacing x_1 by $x_1 + x_2$ in (3.12), we get

$$\begin{aligned}&\gamma_1(x_1)\alpha(x_2) + \gamma_1(x_2)\alpha(x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_2)\\ &= \beta(x_1)\gamma_2(x_2) + \beta(x_2)\gamma_2(x_1) + 2\beta(x_1)\Gamma_1(x_1, x_2) + 2\beta(x_2)\Gamma_2(x_1, x_2).\end{aligned} \tag{3.13}$$

Substituting x_1 by $-x_1$ in (3.13), we get

$$\begin{aligned}&\gamma_1(x_1)\alpha(x_2) - \gamma_1(x_2)\alpha(x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_1) - 2\Gamma_1(x_1, x_2)\alpha(x_2)\\ &= -\beta(x_1)\gamma_2(x_2) + \beta(x_2)\gamma_2(x_1) + 2\beta(x_1)\Gamma_1(x_1, x_2) - 2\beta(x_2)\Gamma_2(x_1, x_2).\end{aligned} \tag{3.14}$$

Comparing (3.13) and (3.14) and using the fact that char$(\mathfrak{R}) \neq 2$, we obtain

$$\gamma_1(x_1)\alpha(x_2) + 2\Gamma_1(x_1, x_2)\alpha(x_1) = \beta(x_2)\gamma_2(x_1) + 2\beta(x_1)\Gamma_2(x_1, x_2). \tag{3.15}$$

Replacing x_2 by $2x_2x_1$ in (3.15), we have

$$\begin{aligned}&\gamma_1(x_1)\alpha(x_2x_1) + 2\Gamma_1(x_1, x_2)\alpha(x_1^2) + 2\beta(x_2)\delta_1(x_1)\alpha(x_1)\\ &= \beta(x_2x_1)\gamma_2(x_1) + 2\beta(x_1)\Gamma_2(x_1, x_2)\alpha(x_1) + 2\beta(x_1)\beta(x_2)\delta_2(x_1).\end{aligned} \tag{3.16}$$

That is,

$$\begin{aligned}&(\gamma_1(x_1)\alpha(x_2) + 2\Gamma_1(x_1, x_2)\alpha(x_1) - 2\beta(x_1)\Gamma_2(x_1, x_2))\alpha(x_1) + 2\beta(x_2)\delta_1(x_1)\alpha(x_1)\\ &= \beta(x_2x_1)\gamma_2(x_1) + 2\beta(x_1)\beta(x_2)\delta_2(x_1).\end{aligned} \tag{3.17}$$

Using (3.15) in (3.17), we find that

$$\beta(x_2)\gamma_2(x_1)\alpha(x_1) + 2\beta(x_2)\delta(x_1)\alpha(x_1) = \beta(x_2x_1)\gamma_2(x_1) + 2\beta(x_1x_2)\delta_2(x_1). \tag{3.18}$$

Replacing x_2 by $2x_3x_2$ in (3.18) and using char$(R) \neq 2$, we get

$$\begin{aligned}&\beta(x_2x_3)\gamma_2(x_1)\alpha(x_1) + 2\beta(x_3x_2)\delta(x_1)\alpha(x_1)\\ &= \beta(x_3x_2x_1)\gamma_2(x_1) + 2\beta(x_1x_3x_2)\delta_2(x_1).\end{aligned} \tag{3.19}$$

Now, subtracting (3.19) from (3.18) multiplied by x_3 to the left, we get

$$\beta([x_3,x_1])\beta(x_2)\delta_2(x_1)\ \forall\ \ x_1,x_2,x_3\in\mathfrak{L}. \tag{3.20}$$

This implies that $[x_3,x_1]\mathfrak{L}\beta^{-1}(\delta_2(x_1))=(0)$. By Lemma 3, gives that for an arbitrary element $x_1\in\mathfrak{L}$ either $x_1\in Z(\mathfrak{L})$ or $\delta_2(x_1)=0$. If $Z(\mathfrak{L})=0$, then $\delta_2(x_1)=0$. If $\mathfrak{L}=Z(\mathfrak{L})$, then $\mathfrak{L}\subseteq Z$, by Lemma 2, a contradiction. Let us assume that $\mathfrak{L}\neq Z(\mathfrak{L})\neq 0$. Then there exists $x_1\in\mathfrak{L}\setminus Z(\mathfrak{L})$. So $\delta_2(x_1)=0$ since $\delta_2(x_2)=0$ $\forall\ x_2\in\mathfrak{L}\setminus Z(\mathfrak{L})$. Take $0\neq x_3\in Z(\mathfrak{L})$. Then $x_1+x_3,x_1-x_3\in\mathfrak{L}\setminus Z(\mathfrak{L})$ and so $\Delta(x_1+x_3,x_1+x_3)=0=\Delta(x_1-x_3,x_1-x_3)$, that is

$$\Delta_2(x_1,x_1)+2\Delta_2(x_1,x_3)+\Delta_2(x_3,x_3)=0$$

and

$$\Delta_2(x_1,x_1)-2\Delta_2(x_1,x_3)+\Delta_2(x_3,x_3)=0.$$

Adding the above two expression, we find that $2\delta_2(x_3)=0$. Since char$(\mathfrak{R})\neq 2$, we have $\delta_2(x_3)=0\ \forall\ x_3\in\mathfrak{L}$. Using Proposition 1, we get the required result. □

Using the same technique with necessary variation one can prove the following theorem.

Theorem 3. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, $\mathfrak{L}$ a nonzero square closed Lie ideal of R, α,β automorphisms and $\Gamma_1,\Gamma_2:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ be two symmetric generalized (α,β)-biderivations associated with symmetric (α,β)-biderivations $\Delta_1,\Delta_2:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$, respectively. If $\Gamma_1(x_1,x_1)\alpha(x_1)+\beta(x_1)\Gamma_2(x_1,x_1)=0\ \forall\ x_1\in\mathfrak{L}$, then either $\mathfrak{L}\subseteq Z$ or $\Delta_2=0$.*

It is immediately to get the following corollaries from Theorems 2, 3 and Lemma 1.

Corollary 1. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, I a nonzero ideal of $\mathfrak{R}$, and $\Gamma_1,\Gamma_2:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ be two symmetric generalized (α,β)-biderivations associated with symmetric (α,β)-biderivations $\Delta_1,\Delta_2:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$, respectively. If $\Gamma_1(x,x)\alpha(x)=\pm\beta(x)\Gamma_2(x,x)\ \forall\ x\in I$, then $\mathfrak{R}$ is commutative.*

Corollary 2. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, $\mathfrak{L}$ a nonzero square closed Lie ideal of $\mathfrak{R}$, α,β automorphisms and $\Gamma:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ be a symmetric generalized (α,β)-biderivation associated with a symmetric (α,β)-biderivation $\Delta:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$. If $[\Gamma(x,x),x]_{\alpha,\beta}=0\ \forall\ x\in I$, then either $\mathfrak{L}\subseteq Z$ or $\Delta=0$.*

Corollary 3. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, I a nonzero ideal of $\mathfrak{R}$, α,β automorphisms and $\Gamma:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ be a symmetric generalized (α,β)-biderivation associated with a symmetric (α,β)-biderivation $\Delta:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$. If $\Gamma(x,x)\alpha(x)=\pm\beta(x)\Delta(x,x)\ \forall\ x\in I$, then $\mathfrak{R}$ is commutative or Γ is a left α-bicentralizer.*

Theorem 4. *Let R be a prime ring, char$(\mathfrak{R}) \neq 2,3$, $\mathfrak{L}$ a nonzero Lie ideal of $\mathfrak{R}$, $\Gamma : \mathfrak{R}\times\mathfrak{R} \to \mathfrak{R}$ a symmetric generalized biderivation associated with a symmetric biderivation Δ. If*

$$\Gamma(\Gamma(x_1,x_1),\Gamma(x_1,x_1)) - \Delta(\Delta(x_1,x_1),\Delta(x_1,x_1)) = \Delta(\Gamma(x_1,x_1),\Gamma(x_1,x_1)) - \Gamma(\Delta(x_1,x_1),\Delta(x_1,x_1))$$

$\forall\ x_1 \in \mathfrak{L}$, then either $\mathfrak{L} \subseteq Z$ or $\Delta = \Gamma$ or $\Delta = 0$.

Proof. Let γ, δ be the traces of Γ and Δ, respectively. By our hypothesis we have

$$\gamma^2(x_1) - \delta^2(x_1) = \delta(\gamma(x_1)) - \gamma(\delta(x_1)). \tag{3.21}$$

The substitution of x_1 by $x_1 + x_2$ in (3.21), gives that

$$\begin{aligned}
&4\gamma(\Gamma(x_1,x_2)) + 2\Gamma(\gamma(x_1),\gamma(x_2)) + 4\Gamma(\gamma(x_1),\Gamma(x_1,x_2)) \\
&\quad + 4\Gamma(\gamma(x_2),\Gamma(x_1,x_2)) - \delta(\Delta(x_1,x_2)) - 2\Delta(\delta(x_1),\delta(x_2)) \\
&\quad - 4\Delta(\delta(x_1),\Delta(x_1,x_2)) - 4\Delta(\delta(x_2),\Delta(x_1,x_2)) \\
&= 4\delta(\Gamma(x_1,x_2)) + 2\Delta(\gamma(x_1),\gamma(x_2)) + 4\Delta(\gamma(x_1),\Gamma(x_1,x_2)) \\
&\quad + 4\Delta(\gamma(x_2),\Gamma(x_1,x_2)) - 4\gamma(\Delta(x_1,x_2)) - 2\Gamma(\delta(x_1),\delta(x_2)) \\
&\quad - 4\Gamma(\delta(x_1),\Delta(x_1,x_2)) - 4\Gamma(\delta(x_2),\Delta(x_1,x_2)).
\end{aligned} \tag{3.22}$$

Now, substituting x_2 by $-x_2$ in the above expression, we find that

$$\begin{aligned}
&4\gamma(\Gamma(x_1,x_2)) + 2\Gamma(\gamma(x_1),\gamma(x_2)) - 4\Gamma(\gamma(x_1),\Gamma(x_1,x_2)) \\
&\quad - 4\Gamma(\gamma(x_2),\Gamma(x_1,x_2)) - \delta(\Delta(x_1,x_2)) - 2\delta(\delta(x_1),\delta(x_2)) \\
&\quad + 4\Delta(\delta(x_1),\Delta(x_1,x_2)) + 4\Delta(\delta(x_2),\Delta(x_1,x_2)) \\
&= 4\delta(\Gamma(x_1,x_2)) + 2\Delta(\gamma(x_1),\gamma(x_2)) - 4\Delta(\gamma(x_1),\Gamma(x_1,x_2)) \\
&\quad - 4\Delta(\gamma(x_2),\Gamma(x_1,x_2)) - 4\gamma(\Delta(x_1,x_2)) - 2\Gamma(\delta(x_1),\delta(x_2)) \\
&\quad + 4\Gamma(\delta(x_1),\Delta(x_1,x_2)) + 4\Gamma(\delta(x_2),\Delta(x_1,x_2)).
\end{aligned} \tag{3.23}$$

Comparing (3.22) and (3.23) and using the fact that char$(\mathfrak{R}) \neq 2$, we get

$$\begin{aligned}
&2\gamma(\Gamma(x_1,x_2)) + \Gamma(\gamma(x_1),\gamma(x_2)) - 2\delta(\Delta(x_1,x_2)) - \Delta(\delta(x_1),\delta(x_2)) \\
&= 2\delta(\Delta(x_1,x_2)) + \Delta(\gamma(x_1),\gamma(x_2)) - 2\gamma(\Delta(x_1,x_2)) - \Gamma(\delta(x_1),\delta(x_2)).
\end{aligned} \tag{3.24}$$

Now, substituting x_2 by $x_2 + x_3$ in(3.24), gives that

$$\begin{aligned}
&2\Gamma(\Gamma(x_1,x_2),\Gamma(x_1,x_3)) + \Gamma(\gamma(x_1),\Gamma(x_2,x_3)) \\
&\quad - 2\Delta(\Delta(x_1,x_2),\Delta(x_1,x_3)) - \Delta(\delta(x_1),\Delta(x_2,x_3)) \\
&= 2\delta(\Delta(x_1,x_2),\Delta(x_2,x_3)) + \Delta(\gamma(x_1),\Gamma(x_2,x_3)) \\
&\quad - 2\gamma(\Delta(x_1,x_2)) - \Gamma(\delta(x_1),\Delta(x_2,x_3)).
\end{aligned} \tag{3.25}$$

Let us take $T = \Gamma - \Delta$, and denote $k = \gamma + \delta$ the trace of $K = \Gamma + \Delta$. By Lemma 5 T is a left bicentralizer of $\mathfrak{R}$. Then (3.24), reduces to

$$2T(\Gamma(x_1,x_2)\Gamma(x_1,x_3))+T(\gamma(x_1),\Gamma(x_2,x_3)) \\ +2T(\Delta(x_1,x_2),\Delta(x_1,x_3))+T(\delta(x_1),\Delta(x_2,x_3))=0. \quad (3.26)$$

Replacing x_3 by $2x_3z$ in (3.26)and using (3.26), we obtain

$$2T(\Gamma(x_1,x_2),x_3)\Delta(x_1,z)+T(\gamma(x_1),x_3)\Delta(x_2,z) \\ +2T(\Delta(x_1,x_2),x_3)\Delta(x_1,z)+T(\delta(x_1),x_3)\Delta(x_2,z)=0. \quad (3.27)$$

That is,

$$2T(K(x_1,x_2),x_3)\Delta(x_1,z)+T(k(x_1),x_3)\Delta(x_2,z)=0. \quad (3.28)$$

Choosing $x_2=x_1$ in (3.28), and using char$(\mathfrak{R})\neq 3$, we get

$$T(k(x_1),x_3)\Delta(x_1,z)=0. \quad (3.29)$$

Choosing $z=x_1$ in (3.28), we obtain

$$2T(K(x_1,x_2),x_3)\delta(x_1)+T(k(x_1),x_3)\Delta(x_2,x_1)=0. \quad (3.30)$$

Comparing (3.29) and (3.30) and using the fact that char$(\mathfrak{R})\neq 2$, gives that

$$T(K(x_1,x_2),x_3)\delta(x_1)=0. \quad (3.31)$$

Replacing x_2 by $2x_2x$ in (3.31), we find that

$$T(K(x_1,x_2),x_3)x\delta(x_1)+2T(x_2,x_3)\Delta(x_1,x)\delta(x_1)=0. \quad (3.32)$$

By replacing x_3 by $2x_3x$ in (3.31), we find that

$$T(K(x_1,x_2),x_3)x\delta(x_1)=0. \quad (3.33)$$

From (3.32) and (3.33) and using the fact that char$(R)\neq 2$, we find that

$$T(x_2,x_3)\Delta(x_1,x)\delta(x_1)=0. \quad (3.34)$$

Replace x_3 by $2x_3w_1$, to get $(T(x_2,x_3)\mathfrak{L}\Delta(x_1,x)\Delta(x_1)=0\ \forall\ x_1,x_2,x,x_3\in\mathfrak{L}$. Lemma 3, gives that either $(T(x_2,x_3)=0$ or $\Delta(x_1,x)\delta(x_1)=0$. In the first case $T(x_2,x_3)=0\ \forall\ x_2,x_3\in\mathfrak{L}$ and hence by Lemma 4, proves that $T=0$. Therefore $\Gamma=\Delta$. On the other hand, if $\Delta(x_1,x)\delta(x_1)=0\ \forall\ x_1,x\in\mathfrak{L}$, then replacing x by $2x_1x$, we have $\delta(x_1)x\delta(x_1)=0$, that is, $\delta(x_1)\mathfrak{L}\delta(x_1)=0$. Again, by Lemma 3, $\delta(x_1)=0$ and hence $\delta(x_1)=0$ for al $x_1\in\mathfrak{L}$. Proposition 1, gives that $\mathfrak{L}\subseteq Z$ or $\Delta=0$. □

Corollary 4. *Let $\mathfrak{R}$ be a prime ring, char$(\mathfrak{R})\neq 2$, I a nonzero ideal of $\mathfrak{R}$, α an automorphism, and $\Gamma:\mathfrak{R}\times\mathfrak{R}\to\mathfrak{R}$ a symmetric generalized (α,α)-biderivation associated with a symmetric (α,α)-biderivation Δ. If*

$$\Gamma(\Gamma(x_1,x_1),\Gamma(x_1,x_1))-\Delta(\Delta(x_1,x_1),\Delta(x_1,x_1)) \\ =\Delta(\Gamma(x_1,x_1),\Gamma(x_1,x_1))-\Gamma(\Delta(x_1,x_1),\Delta(x_1,x_1))$$

$\forall\ x_1\in I$, then either R is commutative or $\Delta=\Gamma$ or Δ is a left α-bicentralizer.

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