



## ON TWO BIVARIATE KINDS OF $(p, q)$ -BERNOULLI POLYNOMIALS

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*Abstract.* The main aim of this paper is to introduce and investigate  $(p, q)$ -extensions of two bivariate kinds of Bernoulli polynomials and numbers. We firstly examine several  $(p, q)$ -analogues of the Taylor expansions of products of some trigonometric functions and determine their coefficients which are also analyzed in detail. Then, we introduce two bivariate kinds of  $(p, q)$ -Bernoulli polynomials and acquired multifarious formulas and relations including connection formulas, recurrence formulas, correlations with aforementioned coefficients, partial  $(p, q)$ -differential equations and  $(p, q)$ -integral representations.

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### 1. INTRODUCTION

The Appell polynomials  $A_n(x)$  defined by

$$f(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}, \quad (1.1)$$

where  $f$  is a formal power series in  $t$ , have found remarkable applications in different branches of mathematics, theoretical physics and chemistry, see [1, 3–13] and references cited therein. One of the most famous polynomials of the of Appell families is Bernoulli polynomials  $B_n(x)$ , generated by  $f(t) = \frac{t}{e^t - 1}$  in (1.1). Also, Bernoulli numbers, denoted by  $B_n := B_n(0)$  are of considerable importance in number theory, combinatorics and numerical analysis (cf. [1, 3, 5, 6, 8–10]). Further, they have the following exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi),$$

and can be generated by the following recurrence relation

$$\sum_{k=0}^n \binom{n+1}{k} B_k = 0 \text{ for } n \geq 1 \text{ and } B_0 = 1.$$

Bernoulli numbers are directly related to several combinatorial numbers such as Stirling, Cauchy and harmonic numbers. For example, except  $B_1$  we have

$$B_n = (-1)^n \sum_{m=0}^n \frac{(-1)^m m!}{m+1} S_2(n, m), \quad (1.2)$$

where

$$S_2(n, m) = \frac{1}{m!} \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n,$$

denote the second kind of Stirling numbers [9] with  $S_2(n, m) = 0$  for  $n < m$ . Very recently in [7], Jamei *et al.* introduced a new kind of bivariate Bernoulli polynomials and studied their main properties. As a valuable application of these extended polynomials, they introduced an extension of the well-known Euler-Maclaurin quadrature formula. Rahmani [8] defined a new family of  $p$ -Bernoulli numbers, which are derived from the Gaussian hypergeometric function, and established some basic properties. Based on a three-term recurrence relation, he gave an algorithm for computing Bernoulli numbers and presented a similar algorithm for Bernoulli polynomials.

The Bernoulli polynomials and numbers have found diverse extensions such as poly-Bernoulli numbers, which are somehow connected to multiple zeta values. The  $q$ -extension of Bernoulli numbers and polynomials has now found many applications in combinatorics, statistics and various branches of applied mathematics. Mahmudov [5, 6] introduced a class of generalized Bernoulli polynomials, Euler polynomials and Genocchi polynomials based on the  $q$ -integers and obtained the  $q$ -analogues of well-known formulas including  $q$ -analogue of the Srivastava-Pinter addition theorem and correlations with the  $q$ -Bernstein polynomials. Duran *et al.* [3] considered the new generating functions of the Bernoulli, the Euler and the Genocchi polynomials under post quantum calculus, denoted by  $(p, q)$ -calculus. From those generating functions, they analyzed their various behaviours and derived a relation between the new and old polynomials by making use of the fermionic  $p$ -adic integral over the  $p$ -adic number fields. Njionou [11] introduced the  $(p, q)$ -Appell polynomials which covers generalizations of some famous family of polynomials such as the  $(p, q)$ -Bernoulli, the  $(p, q)$ -Euler and the  $(p, q)$ -Genocchi polynomials. He provided several characterizations of these polynomials.

In this paper, we introduce a  $(p, q)$ -extension of the aforesaid bivariate Bernoulli polynomials and establish their properties. Multifarious connections and inversion formulas are stated and proved. In the following section, some preliminaries and

definitions are given and in Section 3, a bivariate kind of  $(p, q)$ -Bernoulli polynomials is introduced and some of its fundamental properties are stated and proved.

## 2. PRELIMINARIES AND DEFINITIONS

Let  $\mathbb{N}$  denotes the set of all natural numbers,  $\mathbb{R}$  denotes the set of all real numbers and  $\mathbb{C}$  denotes the set of all complex numbers. Let us introduce the following notation (see [4, 10, 12])

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad 0 < |q| < |p| \leq 1$$

for any positive integer.

The twin-basic number is a natural generalization of the  $q$ -number, that is

$$\lim_{p \rightarrow 1} [n]_{p,q} = [n]_q.$$

The  $(p, q)$ -factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^n [k]_{p,q}!, \quad n \geq 1, \quad [0]_{p,q}! = 1.$$

Let us introduce also the so-called  $(p, q)$ -binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}, \quad 0 \leq k \leq n.$$

Note that as  $p \rightarrow 1$ , the  $(p, q)$ -binomial coefficients reduce to the  $q$ -binomial coefficients.

It is clear by definition that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}.$$

Let us introduce also the so-called falling and raising  $(p, q)$ -powers respectively [4, 12]

$$\begin{aligned} (x \ominus a)_{p,q}^n &= (x-a)(px-aq) \cdots (xp^{n-1} - aq^{n-1}), \\ (x \oplus a)_{p,q}^n &= (x+a)(px+aq) \cdots (xp^{n-1} + aq^{n-1}). \end{aligned}$$

These definitions are extended to

$$\begin{aligned} (a \ominus b)_{p,q}^\infty &= \prod_{k=0}^{\infty} (ap^k - q^k b), \\ (a \oplus b)_{p,q}^\infty &= \prod_{k=0}^{\infty} (ap^k + q^k b), \end{aligned}$$

where the convergence is required.

**Definition 1** ([4, 12]). Let  $f$  be an arbitrary function and  $a$  be a real number, then the  $(p, q)$ -integral of  $f$  is defined by

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right) \quad \text{if } 0 < |q| < |p| \leq 1.$$

Let  $f$  be a function defined on the set of the complex numbers.

**Definition 2** ([4, 12]). The  $(p, q)$ -derivative of the function  $f$  is defined as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and  $(D_{p,q}f)(0) = f'(0)$ , provided that  $f$  is differentiable at 0.

**Proposition 1.** *The  $(p, q)$ -derivative operator fulfills the following product and quotient rules*

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x), \\ D_{p,q}(f(x)g(x)) &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x). \end{aligned}$$

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}, \\ D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \end{aligned}$$

**Proposition 2.** *If  $F(x)$  is a  $(p, q)$ -antiderivative of  $f(x)$  and  $F(x)$  is continuous at  $x = 0$ , we have*

$$\int_a^b f(x) d_{p,q}x = F(b) - F(a), \quad 0 \leq a < b \leq \infty.$$

**Corollary 1.** *If  $f'(x)$  exists in a neighbourhood of  $x = 0$  and is continuous at  $x = 0$ , where  $f'(x)$  denotes the ordinary derivative of  $f(x)$ , we have*

$$\int_a^b D_{p,q}f(x) d_{p,q}x = f(b) - f(a).$$

**Proposition 3.** *Suppose that  $f(x)$  and  $g(x)$  are two functions whose ordinary derivatives exist in a neighbourhood of  $x = 0$ .  $a$  and  $b$  are two real numbers such that  $a < b$ , then*

$$\int_a^b f(px)(D_{p,q}g(x)) d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_{p,q}f(x)) d_{p,q}x. \quad (2.1)$$

As in the  $q$ -case, there are many definitions of the  $(p, q)$ -exponential function. The following two  $(p, q)$ -analogues of the exponential function (see [3, 4, 10, 11]) will be frequently used throughout this paper:

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n, \quad (2.2)$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n. \quad (2.3)$$

From the definitions (2.2) and (2.3) of the  $(p, q)$ -exponential functions, it is easy to see that [10]

$$e_{p,q}(x)E_{p,q}(-x) = 1. \quad (2.4)$$

From (2.2) we can derive

$$e_{p,q}(iz) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} (iz)^n = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \quad (2.5)$$

By (2.5), the  $(p, q)$ -cosine and the  $(p, q)$ -sine functions are defined (see [4, 10]) as follows:

$$\cos_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n}, \quad (2.6)$$

$$\sin_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}. \quad (2.7)$$

In the following definition, we generalize the notion of  $q$ -addition introduced by Jackson and studied later by Ward and Al-Salam, see [1, 2] for more details. Our  $(p, q)$ -addition reduces to the  $q$ -addition defined by Euler and recalled in [13]

**Definition 3.** Let  $x$  and  $y$  be two complex numbers.

(1) The  $(p, q)$ -addition of  $x$  and  $y$  which we denote by  $x \oplus_{p,q} y$  is defined by

$$(x \oplus_{p,q} y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-n)} x^k y^{n-k}. \quad (2.8)$$

(2) The  $(p, q)$ -subtraction of  $x$  and  $y$  which we denote by  $x \ominus_{p,q} y$  is defined by

$$(x \ominus_{p,q} y)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} p^{k(k-n)} x^k y^{n-k}. \quad (2.9)$$

**Theorem 1.** *The following relation holds true for any  $x, y \in \mathbb{R}$ .*

$$e_{p,q}(x)e_{p,q}(y) = e_{p,q}(x \oplus_{p,q} y) \tag{2.10}$$

$$e_{p,q}(x)e_{p,q}(-y) = e_{p,q}(x \ominus_{p,q} y). \tag{2.11}$$

*Proof.* By the definition of the  $(p, q)$ -addition and the Cauchy product we can readily see that

$$\begin{aligned} e_{p,q}(x)e_{p,q}(y) &= \sum_{k=0}^{\infty} \frac{p^{\binom{k}{2}} x^k}{[k]_{p,q}!} \sum_{\ell=0}^{\infty} \frac{p^{\binom{\ell}{2}} y^\ell}{[\ell]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{p^{\binom{k}{2} + \binom{n-k}{2}} x^k y^{n-k}}{[k]_{p,q}! [n-k]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-n)} x^k y^{n-k} \right) \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} = e_{p,q}(x \oplus_{p,q} y). \end{aligned}$$

The second assertion is proved in the same way. □

### 3. A BIVARIATE KIND OF $(p, q)$ -BERNOULLI POLYNOMIALS

Let  $x, y \in \mathbb{R}$ . It is well-known that the Taylor expansion of the two functions  $e^{xt} \cos yt$  and  $e^{xt} \sin yt$  are as follows [7]

$$e^{xt} \cos yt = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!}, \tag{3.1}$$

and

$$e^{xt} \sin yt = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}, \tag{3.2}$$

where

$$C_n(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}, \tag{3.3}$$

and

$$S_n(x, y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \tag{3.4}$$

Here we introduce a  $(p, q)$ -extension of the two above polynomials  $C_n(x, y)$  and  $S_n(x, y)$  by the following generating functions:

$$e_{p,q}(xt) \cos_{p,q}(yt) = \sum_{k=0}^{\infty} C_{k,p,q}(x, y) \frac{t^k}{[k]_{p,q}!}, \tag{3.5}$$

and

$$e_{p,q}(xt) \sin_{p,q}(yt) = \sum_{k=0}^{\infty} S_{k,p,q}(x, y) \frac{t^k}{[k]_{p,q}!}, \tag{3.6}$$

Some particular cases are

$$C_{2n,p,q}(0, y) = (-1)^n p^{\binom{2n}{2}} y^{2n}, \quad C_{2n+1,p,q}(0, y) = 0$$

and

$$S_{2n,p,q}(0, y) = 0, \quad S_{2n+1,p,q}(0, y) = (-1)^n p^{\binom{2n+1}{2}} y^{2n+1}.$$

The following Lemma will be useful in the derivation of several results.

**Lemma 1** ([9]). *The following elementary series manipulations hold*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} A(k, n - 2k), \tag{3.7}$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B(k, n - 1 - 2k). \tag{3.8}$$

**Theorem 2.** *The following representations hold*

$$C_{n,p,q}(x, y) = p^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[ \begin{matrix} n \\ 2k \end{matrix} \right]_{p,q} p^{2k(k-n)} x^{n-2k} y^{2k} \tag{3.9}$$

and

$$S_{n,p,q}(x, y) = p^{\binom{n-1}{2}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \left[ \begin{matrix} n \\ 2k+1 \end{matrix} \right]_{p,q} p^{4k^2-2kn} x^{n-2k-1} y^{2k+1} \tag{3.10}$$

*Proof.* By series manipulation procedure (3.7), we have

$$\begin{aligned} e_{p,q}(xt) \cos_{p,q}(yt) &= \left( \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} (xt)^n \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} (yt)^{2n} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{p^{\binom{n-2k}{2}}}{[n-2k]_{p,q}!} (xt)^{n-2k} \frac{(-1)^k p^{\binom{2k}{2}}}{[2k]_{p,q}!} (yt)^{2k} \\ &= \sum_{n=0}^{\infty} \left( p^{\binom{n}{2}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[ \begin{matrix} n \\ 2k \end{matrix} \right]_{p,q} p^{2k(k-n)} x^{n-2k} y^{2k} \right) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which proves (3.5). The proof of (3.6) is similar by means of the series manipulation method (3.8).

□

**Theorem 3.** *The following derivative rules are valid*

$$D_{p,q,x} C_{k,p,q}(x, y) = [k]_{p,q} C_{k-1,p,q}(px, y), \quad (3.11)$$

$$D_{p,q,y} C_{k,p,q}(x, y) = -[k]_{p,q} S_{k-1,p,q}(x, py), \quad (3.12)$$

$$D_{p,q,x} S_{k,p,q}(x, y) = [k]_{p,q} S_{k-1,p,q}(px, y), \quad (3.13)$$

$$D_{p,q,y} S_{k,p,q}(x, y) = [k]_{p,q} C_{k-1,p,q}(x, py). \quad (3.14)$$

*Proof.* Relation (3.5) yields

$$\begin{aligned} \sum_{n=1}^{\infty} D_{p,q,x} C_{n,p,q}(x, y) \frac{t^n}{[n]_{p,q}!} &= t e_{p,q}(pxt) \cos_{p,q}(yt) = \sum_{n=0}^{\infty} C_{n,p,q}(px, y) \frac{t^{n+1}}{[n]_{p,q}!} \\ &= \sum_{n=1}^{\infty} C_{n-1,p,q}(px, y) \frac{t^n}{[n-1]_{p,q}!} \\ &= \sum_{n=0}^{\infty} [n]_{p,q} C_{n-1,p,q}(px, y) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

proving (3.11). The other equations (3.12), (3.13) and (3.14) can be similarly proved.  $\square$

**Theorem 4.** *The following relations are valid*

$$C_{n,p,q}(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} C_{k,p,q}(0, y) x^{n-k}, \quad (3.15)$$

$$S_{n,p,q}(x, y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} S_{k,p,q}(0, y) x^{n-k}. \quad (3.16)$$

**Theorem 5.** *The following power representations hold*

$$(-1)^n p^{\binom{2n}{2}} y^{2n} = \sum_{k=0}^{2n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_{p,q} C_{2n-k,p,q}(x, y) x^k, \quad (3.17)$$

and

$$(-1)^n p^{\binom{2n+1}{2}} y^{2n+1} = \sum_{k=0}^{2n+1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{p,q} S_{2n+1-k,p,q}(x, y) x^k. \quad (3.18)$$

*Proof.* Multiplying both sides of (3.5) by  $E_{p,q}(-xt)$  and using (2.4), it follows that

$$\sum_{n=0}^{\infty} (-1)^n p^{\binom{2n}{2}} y^{2n} \frac{t^{2n}}{[n]_{p,q}!} = \left( \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-x)^n t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} C_{n,p,q}(x, y) \frac{t^n}{[n]_{p,q}!} \right)$$



$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} C_{n-k,p,q}(x, y) x^k \right) \frac{t^n}{[n]_{p,q}!},$$

which proves (3.17). The proof of (3.18) is similar. □

**Theorem 6.** *The following connection formulas hold*

$$C_{2n+1,p,q}(x, y) = \sum_{k=0}^{2n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n+1 \\ k+1 \end{bmatrix}_{p,q} C_{2n-k,p,q}(x, y) x^{k+1}, \quad (3.19)$$

and

$$S_{2n,p,q}(x, y) = \sum_{k=0}^{2n-1} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} 2n \\ k+1 \end{bmatrix}_{p,q} S_{2n-k-1,p,q}(x, y) x^{k+1}. \quad (3.20)$$

*Proof.* From the relation

$$\sum_{n=0}^{\infty} (-1)^n p^{\binom{2n}{2}} y^{2n} \frac{t^{2n}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} C_{n-k,p,q}(x, y) x^k \right) \frac{t^n}{[n]_{p,q}!},$$

it follows that

$$\sum_{k=0}^{2n+1} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{p,q} C_{2n+1-k,p,q}(x, y) x^k = 0.$$

Hence (3.19) is proved. We prove (3.20) in the same way. □

We can now introduce two kinds of bivariate  $q$ -Bernoulli polynomials as

$$\frac{te_{p,q}(xt)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) = \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!}, \quad (3.21)$$

and

$$\frac{te_{p,q}(xt)}{e_{p,q}(t) - 1} \sin_{p,q}(yt) = \sum_{n=0}^{\infty} B_{n,p,q}^{(s)}(x, y) \frac{t^n}{[n]_{p,q}!}. \quad (3.22)$$

Upon setting  $x = y = 0$  for both polynomials in (3.21) and (3.22), we have  $B_{n,p,q}^{(c)}(0, 0) = B_{n,p,q}^{(s)}(0, 0) := B_{n,p,q}$  which are called  $(p, q)$ -Bernoulli polynomials defined in [3]. When  $y = 0$  in (3.21) and (3.22), we get the usual  $(p, q)$ -Bernoulli polynomials, denoted by  $B_{n,p,q}(x)$ , see [3, 11].

Next, we give some basic properties of these polynomials.

**Theorem 7.**  $B_{n,p,q}^{(c)}(x, y)$  and  $B_{n,p,q}^{(s)}(x, y)$  can be represented in terms of  $(p, q)$ -Bernoulli numbers as follows

$$B_{n,p,q}^{(c)}(x, y) = \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right]_{p,q} B_{k,p,q} C_{n-k,p,q}(x, y), \quad (3.23)$$

and

$$B_{n,p,q}^{(s)}(x, y) = \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right]_{p,q} B_{k,p,q} S_{n-k,p,q}(x, y), \quad (3.24)$$

*Proof.* Using the Cauchy product rule, we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!} &= \frac{t}{e_{p,q}(t) - 1} e_{p,q}(xt) \cos_{p,q}(yt) \\ &= \left( \sum_{n=0}^{\infty} B_n \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} C_{n,p,q}(x, y) \frac{t^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right]_{p,q} B_{k,p,q} C_{n-k,p,q}(x, y) \right) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which proves (3.23). The proof of (3.24) is similar.  $\square$

We now state the following theorem.

**Theorem 8.** The following connection formulas are valid

$$B_{n,p,q}^{(c)}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[ \begin{matrix} n \\ 2k \end{matrix} \right]_{p,q} B_{n-2k,p,q}(x) p^{\binom{2k}{2}} y^{2k}, \quad (3.25)$$

and

$$B_{n,p,q}^{(s)}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[ \begin{matrix} n \\ 2k+1 \end{matrix} \right]_{p,q} B_{n-1-2k,p,q}(x) p^{\binom{2k+1}{2}} y^{2k+1}. \quad (3.26)$$

*Proof.* The formula (3.25) follows from (3.7) since

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_q} &= \frac{t e_{p,q}(xt)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\ &= \left( \sum_{n=0}^{\infty} B_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} (yt)^{2n} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[ \begin{matrix} n \\ 2k \end{matrix} \right]_{p,q} B_{n-2k,p,q}(x) p^{\binom{2k}{2}} y^{2k} \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

The proof of (3.26) is similar via (3.8). □

**Theorem 9.** *The following connection formulas are valid*

$$C_{n,p,q}(x, y) = \sum_{k=0}^n \frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{n,p,q}^{(c)}(x, y), \tag{3.27}$$

$$S_{n,p,q}(x, y) = \sum_{k=0}^n \frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{n,p,q}^{(s)}(x, y). \tag{3.28}$$

*Proof.* From (3.21), we have

$$\sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!} = \frac{te_{p,q}(xt)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) = \frac{t}{e_{p,q}(t) - 1} \sum_{n=0}^{\infty} C_{n,p,q}(x, y) \frac{t^n}{[n]_{p,q}!}.$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} C_{n,p,q}(x, y) \frac{t^n}{[n]_{p,q}!} &= \frac{e_{p,q}(t) - 1}{t} \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!} \\ &= \left( \sum_{n=0}^{\infty} \frac{p^{\binom{n+1}{2}}}{[n+1]_{p,q}} \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \frac{p^{\binom{k+1}{2}}}{[k+1]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{n,p,q}^{(c)}(x, y) \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned}$$

Thus (3.27) follows. (3.28) is proved in a similar way. □

**Proposition 4.** *For every  $n \in \mathbb{N}$ , the following identities hold*

$$B_{n,p,q}^{(c)}((1 \oplus_{p,q} x), y) - B_{n,p,q}^{(c)}(x, y) = [n]_{p,q} C_{n-1,p,q}(x, y), \tag{3.29}$$

$$B_{n,p,q}^{(s)}((1 \oplus_{p,q} x), y) - B_{n,p,q}^{(s)}(x, y) = [n]_{p,q} S_{n-1,p,q}(x, y). \tag{3.30}$$

*Proof.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}((1 \oplus_{p,q} x), y) \frac{t^n}{[n]_{p,q}!} &= \frac{te_{p,q}[(1 \oplus_{p,q} x)t]}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\ &= \frac{te_{p,q}(xt)[e_{p,q}(t) - 1 + 1]}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\ &= te_{p,q}(xt) \cos_{p,q}(yt) + \frac{te_{p,q}(xt)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\ &= \sum_{n=0}^{\infty} C_{n,p,q}(x, y) \frac{t^{n+1}}{[n]_{p,q}!} + \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which proves (3.29). The identity (3.30) is proved similarly. □

**Corollary 2.** *The following relations hold*

$$\begin{aligned} B_{2n+1,p,q}^{(c)}(1,y) - B_{2n+1,p,q}^{(c)}(0,y) &= [2n+1]_{p,q}(-1)^n p^{\binom{2n}{2}} y^{2n}, \\ B_{2n,p,q}^{(s)}(1,y) - B_{2n,p,q}^{(s)}(0,y) &= [2n]_{p,q}(-1)^{n+1} p^{\binom{2n-1}{2}} y^{2n-1}. \end{aligned}$$

*Proof.* If we replace  $n$  by  $2n+1$  in (3.29), and  $x$  by 0, we obtain

$$B_{2n+1,p,q}^{(c)}(1,y) - B_{2n+1,p,q}^{(c)}(0,y) = [2n+1]_{p,q} C_{2n,p,q}(0,y).$$

The first relation is proved since from (3.9) we have  $C_{2n,q}(0,y) = (-1)^n p^{\binom{2n}{2}} y^{2n}$ . The second relation is proved similarly.  $\square$

**Proposition 5.** *For every  $n \in \mathbb{N}$ , the following identities hold*

$$B_{n,p,q}^{(c)}((x \oplus_{p,q} z), y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}(x) C_{n-k,p,q}(y, z), \quad (3.31)$$

and

$$B_{n,p,q}^{(s)}((x \oplus_{p,q} z), y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}(x) S_{n-k,p,q}(y, z), \quad (3.32)$$

*Proof.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}((x \oplus_{p,q} z), y) \frac{t^n}{[n]_{p,q}!} &= \frac{t e_{p,q}((x \oplus_{p,q} z)t)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\ &= \frac{t e_{p,q}(xt)}{e_{p,q}(t) - 1} \times e_{p,q}(zt) \cos_{p,q}(yt) \\ &= \left( \sum_{n=0}^{\infty} B_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} C_{n,p,q}(y, z) \frac{t^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} B_{k,p,q}(x) C_{n-k}(y, z) \right) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which proves (3.31). The proof of (3.32) is similar.  $\square$

**Proposition 6.** *For every  $n \in \mathbb{N}$ , the following identities hold*

$$B_{n,p,q}^{(c)}((x \oplus_{p,q} z), y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} B_{k,p,q}^{(c)}(x, y) z^{n-k}, \quad (3.33)$$

and

$$B_{n,p,q}^{(s)}((x \oplus_{p,q} z), y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} B_{k,p,q}^{(s)}(x, y) z^{n-k}. \quad (3.34)$$

*Proof.* We have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}((x \oplus_{p,q} z), y) \frac{t^n}{[n]_{p,q}!} &= \frac{t e_{p,q}((x \oplus_{p,q} z)t)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \\ &= \frac{t e_{p,q}(xt)}{e_{p,q}(t) - 1} \cos_{p,q}(yt) \times e_{p,q}(zt) \\ &= \left( \sum_{n=0}^{\infty} B_{n,p,q}^{(c)}(x, y) \frac{t^n}{[n]_{p,q}!} \right) \left( \sum_{n=0}^{\infty} p^{\binom{n}{2}} z^n \frac{t^n}{[n]_{p,q}!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} B_{k,p,q}^{(c)}(x, y) z^{n-k} \right) \frac{t^n}{[n]_{p,q}!}, \end{aligned}$$

which proves (3.33). The proof of (3.34) is similar.  $\square$

**Proposition 7.** *The following equations can be concluded*

$$\sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B_{k,p,q}^{(c)}(x, y) = [n+1]_{p,q} C_{n,p,q}(x, y), \tag{3.35}$$

$$\sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B_{k,p,q}^{(s)}(x, y) = [n+1]_{p,q} S_{n,p,q}(x, y). \tag{3.36}$$

*Proof.* From (3.33), we have

$$B_{n+1,p,q}^{(c)}((x \oplus_{p,q} 1), y) - B_{n+1,p,q}^{(c)}(x, y) = \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B_{k,p,q}^{(c)}(x, y).$$

Hence, by using (3.29), relation (3.35) is derived. The proof of (3.36) is concluded in a similar way.  $\square$

**Corollary 3.** *Relations (3.35) and (3.36) imply that*

$$\begin{aligned} \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B_{k,p,q}^{(c)}(0, y) \\ = \begin{cases} (-1)^m [2m+1]_{p,q} p^{\binom{2m}{2}} y^{2m} & \text{if } n = 2m \text{ is odd,} \\ 0 & \text{if } n = 2m+1 \text{ is even,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^n \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n+1-k}{2}} B_{k,p,q}^{(s)}(0, y) \\ = \begin{cases} 0 & \text{if } n = 2m \text{ is odd,} \\ (-1)^m [2m+2]_{p,q} p^{\binom{2m+1}{2}} y^{2m+1} & \text{if } n = 2m+1 \text{ is even.} \end{cases} \end{aligned}$$

**Corollary 4.** For every  $n \in \mathbb{N}$ , the following partial  $(p, q)$ -differential equations hold

$$\begin{aligned} D_{p,q,x} B_{n,p,q}^{(c)}(x, y) &= [n]_{p,q} B_{n-1,p,q}^{(c)}(px, y), \\ D_{p,q,y} B_{n,p,q}^{(c)}(x, y) &= -[n]_{p,q} B_{n-1,p,q}^{(c)}(x, py), \\ D_{p,q,x} B_{n,p,q}^{(s)}(x, y) &= [n]_{p,q} B_{n-1,p,q}^{(s)}(px, y), \end{aligned}$$

and

$$D_{p,q,y} B_{n,p,q}^{(c)}(x, y) = [n]_{p,q} B_{n-1,p,q}^{(s)}(x, py). \quad (3.37)$$

**Corollary 5.** The following equations are valid

$$\begin{aligned} \int_0^1 B_{2n,p,q}^{(c)}(px, y) d_{p,q}x &= (-1)^n p^{\binom{2n}{2}} y^{2n}, \\ \int_0^1 B_{2n+1,p,q}^{(s)}(px, y) d_{p,q}x &= (-1)^{n+1} p^{\binom{2n+1}{2}} y^{2n+1}, \end{aligned}$$

which are proved by combining Proposition 4 and Corollary 2 using the definition of the  $(p, q)$ -integral.

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