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Ramsey–Turán theory

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Dedicated to the memory of Paul Erdős

Abstract

Ramsey- and Turán-type problems were always strongly related to each other. Motivated by an observation of Paul Erdős, it was Turán who started the systematic investigation of the applications of extremal graph theory in geometry and analysis. This led the second author to some results and problems which, in turn, led to the birth of Ramsey–Turán-type theorems. Today this is a wide field of research with many interesting results and many unsolved problems. Below we give a short survey of the most important parts of this field: starting with a historical sketch we continue by describing the

- Ramsey–Turán-type problems and results.
- Related problems in Ramsey theory.
- Some applications.

Keywords: Graphs; Hypergraphs; Ramsey theory; Turán-type extremal graph theory; Ramsey–Turán theorems; Geometry

Notation. For a set Q, $|Q|$ will denote its cardinality. We shall primarily consider graphs without loops and multiple edges. However, (as tools) we shall also use colored graphs with weighted edges and vertices. Given a graph G, $e(G)$ will denote the number of its edges, $v(G)$ the number of its vertices, $\chi(G)$ its chromatic number, $\alpha(G)$ the maximum size of an independent set in it. Given a graph, the (first) subscript will denote the number of vertices: $G_n, S_n, \ldots$ will always denote graphs on n vertices. [3] $R(k_1, \ldots, k_r)$ will denote the usual *lower* Ramsey number, that is the *maximum* t such

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[3] The only case when the (first) subscript in the notation of a graph is not the number of vertices is when we speak of the excluded graphs $L_1, \ldots, L_r$. Of course, in case of sets, etc., e.g. in case of sets A_i, V_i, the subscript is not necessarily the cardinality.

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that there exists an edge-coloring of K_t in r colors where K_t contains no K_{k_i} in the ith color (sometimes denoted by χ_i) for $1 \leqslant i \leqslant r$. For given graphs $L_1, \dots, L_r$, $R(L_1, \dots, L_r)$ will denote the corresponding Ramsey number, that is, the maximum t for which K_t has an edge-coloring in r colors where K_t contains no (not necessarily induced) L_i in the ith color for $1 \leqslant i \leqslant r$. If we use two colors χ_1 and χ_2, we shall call the first color RED, the second one BLUE. Occasionally, when we need to indicate the number of colors used, like in $R(a, \dots, a)$, — to avoid ambiguity — we shall use the more precise notation $R_{[r]}(a, \dots, a)$. $K_k^{(r)}$ denotes the complete r-uniform hypergraph on k vertices.

Given a graph G and a set U of vertices of G, $G[U]$ will denote the subgraph of G induced (spanned) by U. The number of edges in a subgraph spanned by a set U of vertices of G will be denoted by $e(U)$. We shall say that X is *completely joined* to Y if every vertex of X is joined to every vertex of Y.

Given two points x, y in the Euclidean space $\mathbb{E}^h$, (or in any given metric space) $\rho(x, y)$ will denote their distance.

1. Introduction

1.1. Ramsey theorem, Turán theorem and generalizations

Ramsey theorem [115] and Turán extremal graph theorem [147,148,154], are both among the basic theorems of graph theory. Both served as starting points of whole branches in graph theory and both are applied in many fields of mathematics. [4] In the late 1960s a whole new theory emerged, connecting these fields.

In 1930 Ramsey proved the famous

Theorem 1 (Ramsey theorem for 2 colors, complete graphs [115]). *Given a positive integer k there exists a threshold integer $R = R(k)$ such that if $n > R(k)$ and the edges of K_n are colored in two colors arbitrarily, then it contains a monochromatic K_k.*

Motivated by this theorem, Turán posed the following question in 1940 [147]:

> What is the maximum number of edges a graph G_n can have without containing a complete graph K_k?

Obviously, if we partition n vertices into $k-1$ classes as equally as possible and join two vertices iff they belong to different classes, then we obtain a $k-1$-chromatic graph, not containing K_k. This graph will be denoted by $T_{n,k-1}$, and called the *Turán graph* on n vertices and $k-1$ classes.

P. Turán proved:

[4] On Ramsey theory see the book of Graham et al. [85], and on extremal graph theory see the book of Bollobás [12] or the survey of Simonovits [135].

Theorem 2 (Turán theorem for complete graph [147,148]). *Given n and $k, (1 < k \leqslant n)$, every graph G_n on n vertices not containing a K_k has at most $t(n, k-1) := e(T_{n,k-1})$ edges, and this maximum is attained only by $T_{n,k-1}$.*

Note that this completely solves the extremal problem:
If $n = \ell(k-1) + d$, $0 \leqslant d < k-1$, then

$$t(n, k-1) = \frac{1}{2}\left(1 - \frac{1}{k-1}\right)(n^2 - d^2) + \binom{d}{2} = \left(1 - \frac{1}{k-1}\right)\binom{n}{2} + O(1).$$

Below we formulate another version of Ramsey theorem which may seem to be more general but is equivalent to the previous one. Then we formulate a generalization of Turán's theorem. Here, writing that 'G contains an L' we do not necessarily assume that L is an induced subgraph of G.

Theorem 3 (Ramsey theorem for many colors and arbitrary graphs). *Let $L_1, \ldots, L_r$ be fixed graphs. There exists a threshold integer $R = R(L_1, \ldots, L_r)$ such that if $n > R(L_1, \ldots, L_r)$ and the edges of K_n are colored in r colors arbitrarily, then for some $i \leqslant r$ it contains an L_i in the ith color.*

Theorem 4 (Erdős–Stone–Simonovits [64]). *Let $\mathscr{L}$ be a family of graphs and let $\mathbf{ext}(n, \mathscr{L})$ denote the maximum number of edges a graph G_n can have without containing any subgraph $L \in \mathscr{L}$. Put*

$$p := p(\mathscr{L}) = \min_{L \in \mathscr{L}} \chi(L) - 1.$$

Then

$$\mathbf{ext}(n, \mathscr{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

As we see, in case of Turán type extremal problems the chromatic number determines the answer asymptotically. (For $p = 1$ this gives only $\mathbf{ext}(n, \mathscr{L}) = o(n^2)$ and to find finer estimates is mostly a very difficult, open problem. We shall return to this problem in Section 2.5, more precisely, to the problem of finding lower bounds.)

As Erdős [40] and Simonovits [132] proved, not only $\mathbf{ext}(n, L)$ but the extremal or almost extremal graphs are also near (in structure) to a Turán graph.

As to the Ramsey functions, even in the simplest case, for $R(k)$ (two colors, symmetric case) no asymptotics is known.

1.2. The Ramsey–Turán problem

Observe, that the extremal graph in Turán's theorem has a very strict structure. It is very regular, and the chromatic number is 'small', the vertex set is the disjoint union of a few 'large' independent sets. Its structure is as far as possible from what we would call randomlike. The Ramsey problems are also extremal problems but there everybody thinks that the good Ramsey structures are randomlike. One question of this field could be: how does the maximum number of edges of a graph G_n not containing some fixed subgraph L changes if we add some extra conditions which move the structure of G_n away from the regular, simple structures towards the randomlike ones. In other words, how 'stable' the extremal graph is.

Problem A. For a given m let G_n be a graph not containing K_k and having independence number $\alpha(G_n) < m$. What is the maximum number of edges such a graph can have?

This simple question is motivated by Ramsey and Turán theorems and also by some applications discussed later, in Section 8. Most probably it was Andrásfai who — answering some questions of Erdős — first started the investigation of this problem systematically, see Remark 63. Considering the general formulation of Ramsey's theorem, it is also natural to ask the analogous Turán-type question.

Problem B (Turán-type extremal problem for colored graphs). Let $L_1, \dots, L_r$ be fixed graphs. What is the maximum number of edges an r-edge-colored G_n can have under the condition that it does not contain an L_i in the ith color, for any $1 \leqslant i \leqslant r$.

The maximum will be denoted by $T(n, L_1, \dots, L_r)$.

The more general problem — a common generalization of the above problems is:

Problem C. Let $L_1, \dots, L_r$ be fixed graphs. Let G_n be a graph such that

(a) $\alpha(G_n) < m$ and
(b) the edges of G_n are colored by r colors so that the subgraph $G_n^{(i)}$ defined by the edges of the ith color contains no L_i for any $i = 1, \dots, r$.
What is the maximum number of edges such a graph can have?

The maximum will be denoted by

$$\mathbf{RT}(n; L_1, \dots, L_r, m)$$

or, when $L_i = K_{k_i}$, by

$$\mathbf{RT}(n; k_1, \dots, k_r, m).$$

Of course, for fixed m and large n — by Ramsey theorem — there are no graphs with the above properties: the maximum is taken over the empty set. However, we are interested mainly in the case $m \to \infty$, $m = o(n)$, but $m/n \to 0$ very slowly. We will always assume that the set of graphs is nonempty, which is equivalent with $n \leqslant R(L_1, \dots, L_r, K_m)$. Later we shall generalize Problem C into two directions:

- to hypergraphs and
- by generalizing the notion of independence number to $\alpha_p(G)$, see Section 7.

1.3. Applications of Ramsey's and Turán's theorem

It is an interesting piece of history that Erdős and Szekeres rediscovered the Ramsey Theorem to apply it in the solution of a problem of Eszter Klein (Mrs Szekeres) in geometry. A detailed description of the 'story' of the Erdős–Szekeres theorem can be found in the 'Preface'-paper of Szekeres included in the Art of Counting [141].

Theorem 5 (Erdős–Szekeres [72]). *For every k there is a threshold $F(k)$ such that if at least $F(k)$ points are given in the plane, no three on a line, then there are always k of them forming a convex k-gon.*

Proof (*Sketch*). The basic idea of one of the standard proofs is the following:

Take n points in the plane and consider the corresponding complete 4-uniform hypergraph $K_n^{(4)}$. Color its hyperedges by RED and BLUE as follows: if for four points their convex hull is a 4-gon then the 4-tuple be RED, otherwise it is BLUE.

Claim 6. *This coloring has no RED complete 4-uniform graph $K_5^{(4)}$.*

Since Ramsey theorem holds also for hypergraphs, by Claim 6 the RED–BLUE colored $K_n^{(4)}$ contains a **BLUE** complete k-graph $K_k^{(4)}$, assumed that n is sufficiently large. The corresponding k points form a convex k-gon. □

Remark 7. Erdős and Szekeres have shown that

$$F(k) \leqslant \binom{2k-4}{k-2}$$

and they conjectured that

$$F(k) = 2^{k-2} + 1.$$

One can easily show that $F(4)=5$, $F(5)=9$ is a difficult result of Makai and Turán [107]. The general case is still unsolved. [5]

Remark 8. (a) The Erdős–Szekeres paper contains another proof as well.

(b) There are alternative proofs of the corresponding fact using similar Ramsey arguments but for triplets instead of 4-tuples.

The first application of Turán's theorem to geometry was given by Erdős.

Theorem 9 (Erdős [34]). *If $\{P_1,\ldots,P_n\}$ is a set of n points in the plane of diameter (of maximum distance) 1, then at least $3\binom{[n/3]}{2} \sim n^2/6$ pairs P_i,P_j have distance $\leqslant \frac{1}{\sqrt{2}}$.*

[5] Recently G. Szekeres has obtained some new results in connection with this problem: he reformulated a more general problem so that it became much more algebraic and therefore the small cases of the original problem can be handled even by a computer.

The proof is based on the following simple geometrical fact:

If $Q_i, 1 \leqslant i \leqslant 4$ are points in the plane, of maximum distance 1, then the smallest distance among them is at most $\frac{1}{\sqrt{2}}$.

Therefore, if G_n is the graph the vertices of which are $1, \dots, n$ and the edges of which are the pairs (i, j) for which the distance $\varrho(P_i, P_j) > \frac{1}{\sqrt{2}}$, then $K_4 \not\subseteq G_n$. Hence Turán's theorem implies the result.

Remark 10. This result is sharp in two different ways.

(a) The proportion $\frac{1}{3}$ cannot be increased: taking the vertices A, B, C of an equilateral triangle in $\mathbb{R}^2$ and replacing each vertex by $\approx n/3$ different vertices near to the corresponding point we have a situation where $\approx \frac{2}{3}\binom{n}{2}$ distances are near 1 and all the others are very small.

(b) The distance $\frac{1}{\sqrt{2}}$ cannot be decreased without decreasing the proportion: taking the vertices of a unit square and replacing each by roughly $n/4$ different points from their ε-neighborhood, inside the unit square, we get a set of n points where the diameter is 1 and only $4\binom{n/4}{2} \approx n^2/8$ of the distances are smaller than $\frac{1}{\sqrt{2}} - \delta$.

It is somewhat annoying that these two constructions are completely different.

In the late 1960s Turán started applying the main idea of the above proof of Theorem 9 in several different situations. As Turán observed, both Ramsey theorem and his theorem are in some sense generalizations of the Pigeon Hole Principle and therefore it is not so surprising that they are applicable in so many areas of graph theory and other branches of mathematics. [6] We shall not consider here any applications to combinatorics (which are perhaps not so surprising), but only applications to geometry, analysis, potential theory, and probability theory. Work in the first two areas was initiated by Turán [149], continued first by him in [150–152],... and then continued by Erdős, Meir, T. Sós and Turán [54–57] and others. The application to probability theory is due to Katona [88–91,94], Katona–Stechkin, [95], and Sidorenko [124–130].

- One of the new ideas in Turán's papers was that the constant $1/\sqrt{2}$ is a so-called 'packing' constant of the plane and the above method works for the other packing constants as well. Moreover, one can apply the method with many different packing constants simultaneously.
- Another important new feature was that this method works for arbitrary (let us say, 'reasonable') metric spaces. Therefore, there is a wide spectrum of cases where extremal graph theory can be applied.
- The same approach can be observed in the works of Katona and then of Sidorenko: Take a problem in probability theory, where the distribution of sums of (scalar or vector valued) random variables should be estimated. Find the appropriate geometric graph, find out, which subgraphs are excluded and apply the appropriate extremal

[6] This meta-mathematical remark is due to Turán.

graph theoretical result to this geometric graph. In many cases there are no such results at hand, and this way we encounter new problems in graph theory.

- There are many applications of Turán and Ramsey theorems in number theory and in computer science. We shall skip discussing the computer science applications and return very briefly to number theory in Section 8.9. Yet we should emphasize here that the connection to number theory is among the most important ones.
  Erdős has written several papers on applications of Ramsey theory and of extremal graph theory in number theory but their descriptions is beyond the scope of this paper. As a 'random selection' we mention [45,27,5,46,59,15].

These applications provided the motivation for the Ramsey–Turán-type problems, (Problem C) [138], however, Ramsey–Turán problems are interesting on their own and — we feel — they should have been posed and investigated even without these applications.

In Sections 2–7 we shall primarily discuss the results related to Problems A–C, and in Section 8 we return to the applications in geometry, analysis and probability theory.

(1) Problem (B) is much simpler than Problems A, C and it shows the clear relation between Ramsey and Turán Theorems. In Problems A, C asymptotic results are known only in some special cases. The general results indicate that probably the chromatic number — playing an important role in extremal graph problems — should be replaced by some version of the arboricity number $\mathbf{arb}(L)$, where $\mathbf{arb}(L)$ is the minimum number of classes into which $V(L)$ can be partitioned so that each class spans a tree or forest in L. Some version $\mathbf{ARB}(L)$ of $\mathbf{arb}(L)$ will be defined in Section 2.6. Yet, no sharp asymptotics on $\mathbf{RT}(n, L, m)$, are known for general $m = o(n)$.

It is easy to see that

Theorem 11 (Sós [138]). *Let $R := R(k_1, \ldots, k_r)$ be the Ramsey number and $T(n, k_1, \ldots, k_r)$ be the maximum number of edges a G_n can have if G_n can be r-colored without a K_{k_i} in the ith color, $i = 1, \ldots, r$. Then*

$$T(n, k_1, \ldots, k_r) = \mathbf{ext}(n, K_{R+1}) = \left(1 - \frac{1}{R}\right)\binom{n}{2} + O(1).$$

Here $e(G_n) \leqslant \mathbf{ext}(n, K_{R+1})$ is obvious: assuming the contrary we would have an r-colored $K_{R+1} \subseteq G_n$ and for some i we would have in it a K_{k_i} of the ith color. The lower bound follows from the following.

Construction 12 (Sós [138]). *Let $V(K_R) = \{x_1, \ldots, x_R\}$. Fix an r-*edge-coloring*: $\varphi : E(K_R) \to [1, r]$, where the ith color contains no K_{k_i}, $(i = 1, \ldots, r)$. Consider $T_{n,R}$ and color all the edges between the classes C_h and C_k by color $\varphi(x_h, x_k)$ $(1 \leqslant h < k \leqslant R)$.*

Clearly, $T_{n,R}$ colored this way contains no K_{k_i} of color i $(i = 1, \ldots, r)$.

The problem of $\mathbf{T}(n, L_1, \ldots, L_r)$ is still 'easy'. Burr, Erdős and Lovász, [28] introduced the following Ramsey function: Let $t = t(L_1, \ldots, L_r)$ be the smallest integer for which, if $v > 0$ is sufficiently large, then *for any* r-coloring of $K_t(v,\dots,v)$ there exists an i for which there is a monochromatic L_i in the ith color. (Here $K_t(v_1,\dots,v_t)$ denotes the complete t-partite graph with v_i vertices in its ith class.)

Theorem 13.[7] *Given r sample graphs*, $L_1,\dots,L_r$, *then* (*for some constant* $c > 0$)

$$\mathbf{T}(n, L_1, \dots, L_r) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + O(n^{2-c}).$$

Proof. Indeed, if G_n can be colored in r colors so that the ith color contains no L_i for $i = 1,\dots,r$, then (by the definition above) $G_n \not\supseteq K_t(v,\dots,v)$. Applying the Erdős–Stone Theorem [71] we immediately obtain the slightly weaker

$$e(G_n) \leqslant \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2).$$

To get the stronger error term (i.e. Theorem 13) one should use the (stronger) Erdős–Simonovits Theorem [40,132]. □

2. Ramsey–Turán theorems for complete graphs

We distinguish three ranges: the 'no-restriction' case settled by Theorems 11 and 13, the intermediate, i.e. where $\alpha(G_n) \leqslant cn$ for some fixed $c \in (0,1)$ and the $\alpha(G_n) = o(n)$ ranges. The most interesting case is the last one, discussed here the most.

2.1. *The definition of* $\mathbf{RT}(n, L_1, \dots, L_r, o(n))$

$\mathbf{RT}(n, L_1, \dots, L_r, f(n))$ is well defined for any function $f(n)$. Yet, the notation $\mathbf{RT}(n, L_1, \dots, L_r, o(n))$, needs some clarification. Put

$$\vartheta(L_1,\dots,L_r) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{\mathbf{RT}(n, L_1, \dots, L_r, \varepsilon n)}{n^2}.$$

Then $\mathbf{RT}(n, L_1, \dots, L_r, o(n))$ is the family of functions[8] $\vartheta(L_1,\dots,L_r)n^2 + o(n^2)$. An easy application of Cantor diagonalization shows that there exist best $f(n)$:

Claim 14. *For every* $L_1,\dots,L_r$ *there exists a function* $f(n) = o(n)$ *for which*

$$\mathbf{RT}(n, L_1, \dots, L_r, f(n)) = \vartheta(L_1,\dots,L_r)n^2 + o(n^2).$$

See also Definition 52 and Problem 9 on threshold functions.

[7] This theorem is explicitly formulated in [50] where we refer to it as if it were from [28], but there it is (at least) difficult to find this otherwise easy statement.

[8] More loosely, any of these functions.

2.2. The o(n) range, complete graphs, odd case

First we consider the case $r=1$, $L=K_{2k+1}$, and $m\to\infty$, $m=f(n)=o(n)$. Trivially,

$$\mathbf{RT}(n,K_3,o(n))=o(n^2),$$

since the condition implies

$$d_{\max}(G_n)=o(n).$$

Theorem 15 (Erdős–Sós [66]).

$$\mathbf{RT}(n,K_{2k+1},o(n))=\left(1-\frac{1}{k}\right)\binom{n}{2}+o(n^2).$$

Construction 17 below provides the lower bound of this result:

Claim 16 (Erdős graph, F_m). *For any fixed integer $\ell>2$ there exists a constant $c>0$ such that there exist graphs F_m with girth $g(F_m)>\ell$, and $\alpha(F_m)<m^{1-c}$, for any $m>m_0(\ell)$.*

Claim 16 has several different proofs. The first one is a 'random construction' from [37]. (Now this is enough for our purposes and we shall return to a slightly more detailed analysis of this question in Section 2.5.)

Construction 17 ($U_{n,k}$). *Take a Turán graph $T_{n,k}$ with classes $C_1,\ldots,C_k$ and put a graph F_m ($m=\lfloor n/k\rfloor$) defined in Claim 16 into each of its classes.*

Clearly, the resulting graph $U_{n,k}$ contains no K_{2k+1}, since that would imply that one of its classes (i.e. the graph F_m) contained a K_3. Further, $\alpha(U_{n,k})=O(n^{1-c})$. This provides the lower bound in Theorem 15.

Intuitively, the theorem asserts that for large n, if we add (in Turán's theorem on K_{2k+1}) the extra condition that $\alpha(G_n)=o(n)$, that will have roughly the same effect on the maximum number of edges as excluding a complete graph K_{k+1}.

Construction 17 (used for one color) can be generalized:

Construction 18 (Erdős–Sós [66]). *Put $R=R(k_1,\ldots,k_r)$. Consider $T_{n,R}$ and add to each class $C_1,\ldots,C_R$ an 'Erdős' graph F_m, $m=n/R$. Color the edges between different classes according to the appropriate Ramsey coloring of K_R, and color the edges in C_i ($i=1,\ldots,R$) arbitrarily.*

This coloring gives

$$\mathbf{RT}(n,2k_1+1,\ldots,2k_r+1,n^{1-c})\geqslant\left(1-\frac{1}{R}\right)\binom{n}{2}+o(n^2).$$

The lower bound obtained this way can be improved. We shall return to the case of $\mathbf{RT}(3,\dots,3,o(n))$, in Section 3.1.

2.3. The Bollobás–Erdős graph

The even case $q=2k$ is much more difficult. Even the simplest case $q=4$ is a deep theorem. We start with the corresponding construction: with the Bollobás–Erdős graph, which is one of the most important constructions in this area. So it deserves some explanation.

The basic geometric idea is that if we take on the h-dimensional unit sphere $\mathbb{S}^h$ four points x,y,x^*,y^* so that x,x^* and y,y^* are almost antipodal, then these four points are almost in a plane and they form an almost-rectangle and therefore at least one of the sides of this 4-gon xyx^*y^* is longer than $\sqrt{2}-\eta$, where η is a small error-term.

Construction 19 (Bollobás–Erdős graph [14]). *For a given $\varepsilon>0$ and a large integer h we fix a sufficiently large $n_0(\varepsilon,h)$ and assume that $n>n_0(\varepsilon,h)$ is even. Put $\mu=\varepsilon/\sqrt{h}$. Fix a high-dimensional sphere $\mathbb{S}^h$ and partition it into $n/2$ domains $D_1,\dots,D_{n/2}$, of equal measure and diameter $<\frac{1}{2}\mu$. (This is possible!) Choose a vertex $x_i\in D_i$ and an $y_i\in D_i$ (for $i=1,\dots,n/2$) and put $X=\{x_i,\dots,x_{n/2}\}$ and $Y=\{y_1,\dots,y_{n/2}\}$. Let $X\cup Y$ be the vertex-set of our graph to be defined.*

(a) *Join an $x\in X$ to a $y\in Y$ if $\rho(x,y)<\sqrt{2}-\mu$;*
(b) *join an $x\in X$ to an $x'\in X$ if $\rho(x,x')>2-\mu$;*
(c) *join a $y\in Y$ to a $y'\in Y$ if $\rho(y,y')>2-\mu$.*

Denote the resulting graph by $\mathbb{BE}_n$ or $\mathbb{BE}(n,h,\varepsilon)$.

Claim 20. $\alpha(\mathbb{BE}_n)=o(n)$.

The idea behind this is that if we choose $\frac{1}{2}cn$ vertices from among $x_1,\dots,x_{n/2}$, then the union U of the corresponding domains give a subset or relative measure $>c$ and therefore (by a corresponding isoperimetric theorem) U contains two points A,B with $\rho(A,B)\approx 2$. Since the diameter of the domains is small, there are two vertices of the graph, x near to A and y near to B with $\rho(x,y)\approx 2$, so they are *joined.*

Claim 21. *$\mathbb{BE}_n$ contains no K_4.*

The idea behind this claim was explained above.

Claim 22. *Each vertex of $\mathbb{BE}_n$ has degree $n/4+o(n)$, as $\varepsilon\to 0$ and therefore $n\to\infty$.*

Indeed, each x_i is joined to the y_j's on the 'opposite approximate halfsphere'.

2.4. *The* o(n) *range, complete graphs, even case*

Theorem 23 (Szemerédi [142] and Bollobás–Erdős [14]).

$$\mathbf{RT}(n, K_4, \mathrm{o}(n)) = \frac{n^2}{8} + \mathrm{o}(n^2).$$

The upper bound was proved by Szemerédi, the lower bound by Bollobás and Erdős, (by the above construction), and even after having this result it took years to determine $\mathbf{RT}(n, K_{2k}, \mathrm{o}(n))$.

Theorem 24 (Erdős–Hajnal–Sós–Szemerédi [52]). *For* $q = 2k$,

$$\mathbf{RT}(n, K_q, \mathrm{o}(n)) = \frac{1}{2}\frac{3q-10}{3q-4}n^2 + \mathrm{o}(n^2).$$

For large q this again means that the effect of condition $\alpha(G_n) = \mathrm{o}(n)$ is roughly the same as excluding a $K_{[q/2]}$ without any restriction on $\alpha(G_n)$. Though the formula above may seem mysterious, it becomes transparent if we rephrase the above theorem as follows. We need the following.

For a given property $\mathscr{A}$ of graphs the corresponding extremal problem is to

$$\text{maximize } e(G_n) \text{ for } G_n \in \mathscr{A}.$$

Definition 25. A sequence of graphs, (S_n) will be called *asymptotically extremal* if $S_n \in \mathscr{A}$ and

$$e(S_n) \geqslant (1 - \mathrm{o}(1)) \max_{G_n \in \mathscr{A}} e(G_n).$$

Theorem 26. *Put* $m := \lceil 4n/(3k-2) \rceil$. *Take a Bollobás–Erdős graph* $\mathbb{BE}(m, h, \varepsilon_m)$ *and a Turán graph* $T_{n-m,k-1}$. *Join each vertex of* $\mathbb{BE}(m, h, \varepsilon_m)$ *to each vertex of* $T_{n-m,k-1}$. *Put an Erdős graph* F_{m^*} *into each class of* $T_{n-m,k-1}$, (*to spoil the large independent sets*). *Choosing* ε_m *and* h *appropriately*, *the resulting graph* H_n *is approximately regular*:

$$d_{\max}(H_n) - d_{\min}(H_n) = \mathrm{o}(n),$$

and H_n *is an* '*asymptotically extremal sequence*' *for the problem of* $\mathbf{RT}(n, K_{2k}, \mathrm{o}(n))$:

(a) $K_{2k} \not\subseteq H_n$,
(b) $\alpha(H_n) = \mathrm{o}(n)$,
(c) $e(H_n) \geqslant \mathbf{RT}(n, K_{2k}, \mathrm{o}(n)) - \mathrm{o}(n^2)$.

Replacing $\mathrm{o}(n)$ in some problems $\mathbf{RT}(n, L, \mathrm{o}(n))$ by slightly smaller functions, say by $f(n) = n/\log n$ perhaps one could get smaller upper bounds.

Problem 1. *Is it true that for some* $c > 0$,

$$\mathbf{RT}\left(n, K_4, \frac{n}{\log n}\right) < \left(\frac{1}{8} - c\right) n^2?$$

Similarly, we could ask, what happens if $o(n)$ is replaced by $O(n^{1-c})$ for some fixed but small constant $c > 0$:

Problem 2. *Under which conditions on L can one state that there exist two positive constants* $c, c_1 > 0$ *for which*

$$\mathbf{RT}(n, L, o(n)) - \mathbf{RT}(n, L, f(n)) > c_1 n^2 \quad \textit{for every} \quad f(n) = O(n^{1-c}) \ ?$$

2.5. Geometric constructions, isoperimetric problems

The interaction between graph theory and other parts of mathematics, e.g., geometry, number theory, etc. became more and more evident and intensive in the last two decades.

We have already mentioned that there is a connection between some geometric problems and Ramsey theory, see Section 1.2. As to the connection of *extremal graph theory* and *geometry*, one could say that this connection is perhaps even stronger and more many sided. Indeed, right at the beginning, Erdős applied Turán's theorem and other extremal graph results in geometry. Among others, he applied these methods to give the first, fairly simple estimates on the number of unit distances in his famous problem:

Problem 3 (Unit distances). *Given n points* $x_1, \dots, x_n$ *in* $\mathbb{R}^h$, *what is the maximum number of pairs* (x_i, x_j) *for which* $\rho(x_i, x_j) = 1$ (*or any constant*)?

Let us consider the graph whose vertices are the points x_i and the edges are the pairs with $\rho(x_i, x_j) = 1$. Using the observation that *in the plane* this graph does not contain $K_{2,3}$ — since two circles intersect in at most two points — Erdős concluded that in the plane the number of unit distances is $O(n^{3/2})$. Similarly, in $\mathbb{R}^3$ the graph does not contain $K_{3,3}$, therefore the number of unit distances is $O(n^{5/3})$. (Unfortunately these estimates are far from the conjectured $O(n^{o(1)})$, see [35].)

Later, in some sense these observations were (implicitly) used in the opposite direction: Erdős, Rényi and T. Sós [61] and Brown [17] constructed finite geometric graphs which showed that

$$\mathbf{ext}(n, K_{2,2}) = \tfrac{1}{2} n^{3/2} + o(n^{3/2})$$

and

$$\mathbf{ext}(n, K_{3,3}) \geqslant c_{3,3} n^{5/3} + o(n^{5/3}).$$

All these [9] and many other cases [116,11,134,79] show that geometric graphs can often be transformed into finite geometric graphs to get interesting constructions

[9] Above we were interested in $K_2(2,3)$, not in $K_2(2,2)$ but the result for $K_2(2,2)$ is the transparent one which we wanted to emphasize here, to show the interaction between geometry and extremal graph theory, For our reasons the Eszter Klein from [33] construction would be equally good.

in graph problems. For a detailed discussion of such interactions see the survey of Sós [139].

Remark 27. More recently algebraic geometric methods also provided beautiful constructions, a breakthrough in the area of extremal problems with bipartite excluded subgraphs, see [102], and then [6].

The reason why geometric observations can be used to get lower bounds in *ordinary extremal graph theory* is that if in Euclidean or affine or projective geometry some configuration is excluded, that often can be translated into graph-theoretical language. This provides an infinite graph without some subgraph L. To get a finite graph construction, first we should describe geometry in terms of analytic geometry, then replace the field of real numbers by a finite field. Often L will be excluded in the resulting finite graph. [10]

It is perhaps much less known, that in Ramsey–Turán problems *High Dimensional Isoperimetric Theorems* play important role.

In our simplest case we would need graph sequences (G_n) for which

$$\alpha(G_n) = o(n) \quad \text{and} \quad K_3 \nsubseteq G_n. \tag{1}$$

Clearly, (1) implies that

$$\chi(G_n) \to \infty. \tag{2}$$

The random graph construction of Erdős [37] has both properties (1) and (2) and therefore it can be used in many Ramsey–Turán problems (see e.g. Construction 18). However, to solve the problem of $\mathbf{RT}(n, K_4, o(n))$ we are interested in more explicit graphs, because, following the construction of Bollobás and Erdős, we want to take two copies of such graphs and join them by many edges, (i.e., by positive edge density) without getting K_4. However this breaks down in case of random graphs. There are (at least) three famous graphs which could replace the random construction in such cases: the Borsuk graph, the Kneser graph, [104] and the Margulis–Lubotzky–Phillips–Sarnak graphs [108–110,105,106].

Bollobás and Erdős used a discretized version of the Borsuk graph to provide a lower bound for $\mathbf{RT}(n, K_4, o(n))$, in [14]. The fact that for the graph $\mathbb{BE}_n$ constructed by them $\alpha(\mathbb{BE}_n) = o(n)$ was proved by applying an isoperimetric theorem.

The Borsuk graph is defined as follows: [11]

Construction 28 (Borsuk graph). *The vertices of $\mathscr{B}(h, \varepsilon)$ are the points of an h-dimension sphere $\mathbb{S}^h$ and we join two points x, y by an edge if $\rho(x, y) > 2 - \varepsilon$.*

---

[10] But not always, e.g. if we choose the parameters in the Brown construction carelessly, the three-dimensional spheres will contain straight lines and the proof will not work.

[11] Here $\mathbb{S}^h$ denotes $(h-1)$-dimensional unit sphere in $\mathbb{R}^h$.

One of its important features is that it contains no short odd cycles, since each edge is joined only to 'almost antipodal' vertices. The other important feature is that its chromatic number is $h+1$. It has also a third important feature, connected to the high chromatic number: its independent sets are of small measure. If we wish to find a large independent subset of $\mathbb{S}^h$, that means that we are looking for a large subset without distances $\geqslant 2-\varepsilon$. A corresponding 'isoperimetric' theorem asserts that

Theorem 29 (Schmidt [120]).[12] *If $A \subseteq \mathbb{S}^h$ is an arbitrary measurable set not containing two points of distance $\geqslant 2-\varepsilon$ and $\mathbf{B}$ is a spherical cap in $\mathbb{S}^h$ of diameter $2-\varepsilon$, then $\lambda(A) \leqslant \lambda(\mathbf{B})$ (where λ is the Lebesgue measure).*

Corollary 30. *If $A \subseteq \mathbb{S}^h$ is an arbitrary measurable set not containing two points of distance $\geqslant 2-\varepsilon$[13] then*

$$\lambda(A) \leqslant 2\mathrm{e}^{-(1/2)\varepsilon h}.$$

Construction 31. *The Kneser graph $\mathbf{KN}(m,\ell)$ is defined as follows: An m-element set S is fixed and the vertices of the graph are the $n := \binom{m}{\ell}$ ℓ-subsets of this S; two such 'vertices', i.e. ℓ-tuples are joined iff their intersection is empty.*

It is interesting to note that, though the Kneser graph is similar in many respects to the Borsuk graph, it is useless for our purposes since it may have too large independent sets, e.g., put $m = 3\ell - 1$. Here $\mathbf{KN}(3\ell-1,\ell) \not\supseteq K_3$ and $\chi(\mathbf{KN}(3\ell-1,\ell)) = \ell+1$. However, the independence number is

$$\alpha(\mathbf{KN}(3\ell-1,\ell)) \approx \tfrac{1}{3}v(\mathbf{KN}(3\ell-1,\ell)).$$

To close this short section we remark that in connection with the K_p-independence number Erdős conjectured and Bollobás proved the corresponding generalization of Schmidt's result, see Section 7.

2.6. Missing asymptotics and the arboricity number

One of the basic problems in Ramsey–Turán theory (see [52]) is:

Given a graph L, which graph-theoretic properties of L influence $\vartheta(L)$ defined in 2.1?

As we have mentioned in the Introduction, for ordinary Turán-type extremal problems the Erdős–Stone–Simonovits theorem [64] immediately provides the asymptotical behavior of the solution if $p > 1$, and this asymptotics depends only on the chromatic numbers of the forbidden graphs. No analog results are known for Ramsey–Turán problems. Perhaps the chromatic number of Theorem 4 should be replaced by a modification of $\mathbf{arb}(L)$. In this section we shall discuss: when does the '$\alpha(G_n) = o(n)$'

[12] Here we could formulate the results also in a more general form, see [13], yet we restrict ourselves to Lebesgue-measurable sets.

[13] In the previous statement it did not matter, which way do we measure the distances: in the space or on the surface, along geodesics, here we rather fix that in the space.

condition change the extremal number significantly and when it does not. Further, we shall discuss also: why is the arboricity important here?

Claim 32. *If for a graph L the arboricity* $\mathbf{arb}(L) = \chi(L)$, *then*

$$\mathbf{ext}(n, L) - \mathbf{RT}(n, L, o(n)) = o(n^2).$$

The heuristic explanation of this claim is that if coloring the vertices of L in $\chi(L)$ colors we have many edges between the color classes in the sense that any two classes must span a cycle, then the extra condition on the independence number does not decrease the maximum but by $o(n^2)$. On the other hand, many examples suggest that if there are color classes weakly joined to each other, then the extremal number noticeably drops.

To prove Claim 32, put $p := \chi(L) - 1$. Take Construction 17 with an F_m having large girth, say $g(F_m) > v(L)$. Clearly, the resulting graph $U_{n,p}$ contains no L and $\alpha(U_{n,p}) = o(n)$. Hence (by Theorem 4)

$$\mathbf{ext}(n, L) \geqslant \mathbf{RT}(n, L, o(n)) \geqslant e(U_{n,p}) > e(T_{n,p}) = \mathbf{ext}(n, L) - o(n^2).$$

$K(3,3,3)$ is a good example here. Indeed, it is trivial that $\mathbf{arb}(K(3,3,3)) = \chi(K(3,3,3)) = 3$. Therefore

Theorem 33.

$$\mathbf{RT}(n, K(3,3,3), o(n)) = \mathbf{ext}(n, K(3,3,3)) + o(n^2) = \left(1 - \frac{1}{2}\right)\binom{n}{2} + o(n^2).$$

More generally, if $p < s \leqslant t$, *then*

$$\mathbf{RT}(n, K_{p+1}(s, t, \ldots, t), o(n)) = \mathbf{ext}(n, K_{p+1}(s, t, \ldots, t)) + o(n^2)$$
$$= \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

One can also see that

- $\mathbf{arb}(L) \geqslant 2$ except if L is a tree or a forest.
- If L can be colored in h colors so that (the coloring is a 'good' vertex-coloring and) the first color is used only $s < h$ times then $\mathbf{arb}(L) < \chi(L)$.

Below we need a modified version of the arboricity.

Definition 34 (*Modified arboricity*). The 'modified' arboricity $\mathbf{ARB}(L)$ of a graph L is the minimum ℓ for which the following holds:

- either ℓ is even and $\mathbf{arb}(L) \leqslant \ell/2$,
- or ℓ is odd and we can delete a set V^* of independent vertices so that $\mathbf{arb}(L - V^*) \leqslant \frac{1}{2}(\ell - 1)$.

To compare the two notions, observe that $\mathbf{arb}(K_\ell) = \lceil \ell/2 \rceil$, $\mathbf{ARB}(K_\ell) = \ell$. One of the main results of [52] is that for given $\mathbf{ARB}(L)$ the complete graph is 'the worst':

Theorem 35 (Arboricity, one color, Erdős, Hajnal, Sós and Szemerédi [52]). *If* $\mathbf{ARB}(L) \leqslant \ell$ *then*

$$\mathbf{RT}(n, L, o(n)) \leqslant \mathbf{RT}(n, K_\ell, o(n)) + o(n^2).$$

Right now the case of $K_3(2,2,2)$ seems to be the first real difficulty. Since $\mathbf{ARB}(K_3(2,2,2)) = 4$, therefore

$$\vartheta(K_3(2,2,2)) \leqslant \vartheta(K_4) = \tfrac{1}{8}.$$

No improvement of this bound is known. One way to settle this question would be to show that the Bollobás–Erdős graph (or some slight modification of it) contains no $K_3(2,2,2)$. We cannot decide even this (seemingly simple) question.

Problem 4. (a) *Decide if* $\mathbf{RT}(n, K_3(2,2,2), o(n)) = o(n^2)$ *or not*.

(b) *Decide if* $\mathbf{RT}(n, K_3(2,3,3), o(n)) = o(n^2)$ *or not*.

(c) *Can one prove that* (*some version of*) *the Bollobás–Erdős graph contains no* $K_3(2,2,2)$?

2.7. Ramsey–Turán problems, Szemerédi lemma, weighted extremal problems, multigraph problems

There are only a few cases where we can solve satisfactorily the Ramsey–Turán problems. In some other cases we do not know the extremal densities or the (asymptotically) extremal structures, yet we can prove that there exist relatively simple asymptotically extremal graph sequences. One such case is when we consider many colors and complete graphs.

For two disjoint sets of vertices, $X, Y \subseteq V(G)$, we denote by $e(X,Y)$ the number of edges joining them and define the *density*

$$d(X,Y) := \frac{e(X,Y)}{|X| \cdot |Y|}.$$

First we formulate one of our results in a rather simplified form.

Theorem 36 (Erdős, Hajnal, Simonovits, Sós and Szemerédi [50]). *Given the integers* $k_1, \ldots, k_r \geqslant 3$, *for* $\mathbf{RT}(n, K_{k_1}, \ldots, K_{k_r}, o(n))$ *there exists a fixed* t *and a sequence of asymptotically extremal graphs* (S_n) *such that the vertices of* S_n *can be partitioned into* t *classes* $V_{1,n}, \ldots, V_{t,n}$ *where*

(a) $e(V_{i,n}) = o(n^2)$ *for* $i = 1, 2, \ldots, t$, *and*
(b) *either* $d(V_{i,n}, V_{j,n}) = \frac{1}{2} + o(1)$ *or* $d(V_{i,n}, V_{j,n}) = 1 + o(1)$ *for* $1 \leqslant i < j \leqslant t$.

As a matter of fact, in [50] we formulate a more general form of Theorem 36, asserting that in the above cases there are always asymptotically extremal graph sequences which are generalized Bollobás–Erdős graphs. There the extra information is that in the above theorem, in case when two classes are connected by density $\frac{1}{2}$, then the corresponding two classes span a Bollobás–Erdős graph.

One of the key tools used in this area is the Szemerédi Regularity lemma [143] generalized to many colors [50]. (For these and other applications of the Regularity Lemma and for some generalizations see [100].)

Regularity condition: Given a graph G_n and two disjoint vertex sets in it, X and Y, we shall call the pair (X, Y) ε-regular if for every subset $X^* \subset X$ and $Y^* \subset Y$ satisfying $|X^*| > \varepsilon|X|$ and $|Y^*| > \varepsilon|Y|$,

$$|d(X^*, Y^*) - d(X, Y)| < \varepsilon.$$

The regularity condition means that the edges behave (in some weak sense) as if they were random. If the graph G is edge-colored in r colors, let $d_v(X, Y)$ denote the density in color χ_v.

Generalized Regularity Lemma. *For every* $\varepsilon > 0$, *and integers* r *and* κ_0 *there exists a* $\Lambda_0(\varepsilon, r, \kappa_0)$ *such that for every* r*-edge-colored* G_n $V(G_n)$ *can be partitioned into sets* $V_0, V_1, \dots, V_\lambda$ — *for some* $\kappa_0 < \lambda < \Lambda_0(\varepsilon, r, \kappa_0)$ — *so that* $|V_0| < \varepsilon n$, $|V_i| = m$ (*the same*) *for every* $i > 0$, *and for all but at most* $\varepsilon\binom{\lambda}{2}$ pairs (i, j), *for every* $X \subseteq V_i$ *and* $Y \subseteq V_j$, *satisfying* $|X|, |Y| > \varepsilon m$, *we have*

$$|d_v(X, Y) - d_v(V_i, V_j)| < \varepsilon$$

for every $1 \leqslant v \leqslant r$.

The above theorem does not explicitly deal with the edges inside the classes V_i. This is why we need to put a lower bound κ_0 on the number of classes. If we choose κ_0 large then the number of edges inside the classes will be negligible compared to the total number of edges, so in most problems we may forget about them. On the other hand, Λ_0 is an upper bound on the number of classes which enables us to treat the whole graph as if it were the union of just a few randomlike bipartite subgraphs $G(V_i, V_j)$.

2.7.1. Multigraphs, weighted graphs

To solve the Turán–Ramsey problem $\mathbf{RT}(n, K_{2\ell}, o(n))$, in [52] weighted extremal graph problems were used: having applied the Regularity lemma, one obtained a 'reduced' graph the edges of which had weight 1 and $\frac{1}{2}$, depending on (b) of Theorem 36. In these weighted extremal graph problems a weight function $w : E(G_n) \to [0, 1]$ is given on the edges of G_n and a family $\mathscr{L}$ of weighted subgraphs is also given. Here $L \subset G_n$ means that each edge of L has $\leqslant$ weight in L than in G_n. Of course, the weighted graph extremal problems are strongly connected to multigraph extremal problems: in some sense they are equivalent.

Harary and Brown [26] and then Brown, Erdős and Simonovits [19–25] considered multigraph extremal problems where the multiplicities were 1 and 2. Above the multiplicities (or weights) are 1/2 and 1 and that is not much difference. So it turned out that these Ramsey–Turán problems are strongly connected with a particular kind of multigraph extremal problems, which can algorithmically be solved. In those cases where we want to solve a Ramsey–Turán problem, one can define weighted complete graphs and weighted Ramsey theorems and reduce the solution of Ramsey–Turán problems (for many colors and complete graphs) to the solution of weighted Ramsey problems. For details see [50]. It may happen that *all* the Ramsey–Turán problems for ordinary graph — assuming that we look for a solution up to $o(n^2)$ edges — can be reduced to such multigraph extremal problems. However, that we cannot prove, not even for one color.

2.8. Permissible densities

Turán- or Ramsey–Turán-type problems may be asked, investigated in various settings. By Theorem 4, for ordinary graphs — in the Turán problem — the densities

$$\lim_{n\to\infty} \frac{\mathbf{ext}(n,L)}{\binom{n}{2}}$$

have very special forms: $1 - 1/p$. One can ask in other settings the same: which are the possible densities? It turned out, that — though in the simplest case of multigraph extremal problems these densities form a well ordered set — practically we do not know too much about the set of densities in the other cases. The following consequence of Theorem 35 shows that (at least) these densities for Ramsey–Turán problems cannot be arbitrary:

Theorem 37 (Erdős, Hajnal, T. Sós and Szemerédi [52]). *Let L be an arbitrary fixed graph. Let*

$$a_\ell = \frac{1}{2}\frac{3\ell-9}{3\ell-3} \quad \textit{if } \ell \textit{ is odd and} \quad a_\ell = \frac{1}{2}\frac{3\ell-10}{3\ell-4} \quad \textit{if } \ell \textit{ is even.}$$

Then for some odd ℓ

$$\vartheta(L) = \lim_{n\to\infty} \frac{\mathbf{RT}(n,L,o(n))}{\binom{n}{2}} \in [a_\ell, a_{\ell+1}].$$

The sequence $(a_3, a_4, a_5, a_6, \ldots) = (0, \frac{1}{8}, \frac{1}{4}, \frac{2}{7}, \ldots)$ is strictly increasing. As a result, e.g., there is no density in $(\frac{1}{8}, \frac{1}{4})$.

3. Some results on many colors

3.1. Triangles

Theorem 38 (Erdős and Sós [68]).

$$\mathbf{RT}(n;3,3,o(n)) = \frac{n^2}{4}(1+o(1)).$$

Erdős, and Sós [69] conjectured what was proved only later:

Theorem 39 (Erdős, Hajnal, T. Sós and Szemerédi [52]).

$$\mathbf{RT}(n;3,3,3,\mathrm{o}(n)) = \left(\frac{2}{5}+\mathrm{o}(1)\right)n^2.$$

To explain the general case we need a definition.

Definition 40. $R^*_{[r]}(L_1,\ldots,L_r)$ is the maximum R for which one can r-color K_R so that

(a) no monochromatic L_i of color χ_i is in K_R, for $1\leqslant i\leqslant r$,
(b) each vertex is incident only to at most $r-1$ colors.

Now, Theorem 39 immediately follows from

Theorem 41 (Erdős–Hajnal–Sós–Szemerédi [52]).

$$\mathbf{RT}(n;3,3,\ldots,3,\mathrm{o}(n)) = \left(1-\frac{1}{R^*_r(3,3,\ldots,3)}\right)\binom{n}{2}+\mathrm{o}(n^2).$$

The reason why we use here $R^*_{[r]}(3,3,\ldots,3)$ is that if we color a T_{n,R^*} according to Definition 40, so that some color $\chi_{\pi(i)}$ is missing from the colors used for the edges between C_i and the other classes, then we can put into each class C_i of this T_{n,R^*} an Erdős graph $F_{[n/R^*]}$ and color it with $\chi_{\pi(i)}$. Thus we get an r-coloring of T_{n,R^*} without monochromatic triangles.

Remark 42 (*Local Ramsey numbers*). We have mentioned that one of the beautiful aspects of Ramsey–Turán theory is that this area is intrinsically connected to many other areas. One of these areas is the 'Theory of Local Ramsey Numbers', where for given $L_1,\ldots,L_r$ and $\ell<r$ we consider r-colorings of K_n where each vertex is incident to at most ℓ colors and (a) of Definition 40 is satisfied. Here we see the connection to this field, for $\ell=r-1$.

Actually, [66] was the first place where the notion of *Local Ramsey Coloring* arose. Later (1987) Truszczynski and Tuza [146] and Gyárfás, Lehel, Schelp and Tuza [81] started more systematic investigation of Local Ramsey Coloring, they and Nešetřil and Rödl extended these investigations to hypergraphs, [82] and somewhat later Galluccio, Simonovits and Simongi rediscovered it, again in connection with a Ramsey theoretical problem [80].

Erdős conjectured that $R^*_{[r]}(3,\ldots,3)=R_{[r-1]}(3,\ldots,3)$, but this was disproved by Fan Chung [29], (oral communication).

Remark 43. Denote by $m_3(r)$ the largest integer for which one can edge-color $K_{m_3(r)}$ by r colors so that none of the colors contains a monochromatic triangle. $m_3(2)=5$, $m_3(3)=16$ are well-known, but Folkman proved $m_3(4)<64$. (A trivial induction gives $m_3(r+1)<(r+1)m_3(r)+1$ and Folkman's result shows that equality does not hold for $r=4$.) The exact determination or even to find good bounds on $m_3(r)$ seems very difficult. It is not even known if $m_3(r)^{1/r}\to\infty$ is true.

3.2. The other end?

Above we excluded one or more sample graphs L and considered (in the 'inverse formulation' the problem: if a graph G_n has e edges, how large must $\alpha(G_n)$ be. Mostly we were interested in the case when $e(G_n)$ is relatively large and we wish to find the minimum of $\alpha(G_n)$. In some applications Ajtai, Komlós and Szemerédi needed the other end: the case when $e(G_n)$ is small.

Theorem 44 (Ajtai, Komlós and Szemerédi [3]). *There exists a constant $c>0$ such that if $e(G_n)=tn$ and $K_3 \nsubseteq G_n$ then*

$$\alpha(G_n) > c\frac{n}{t}\log t.$$

The theorem was generalized to arbitrary excluded K_p by Ajtai, Erdös, Komlós and Szemerédi [1], and to hypergraphs by Ajtai, Komlós, Pintz, Spencer and Szemerédi [2], see below.

Remark 45. One of the applications was the estimate on the Ramsey number $R(3,k)$. Both the upper bound, using Theorem 44, [3] and the matching lower bound obtained by Kim [97] are among the most important results in Ramsey Theory for ordinary graphs:

$$\frac{c_1n^2}{\log n} < R(3,n) < \frac{c_2n^2}{\log n}.$$

Another important application was to give an existence proof for an infinite Sidon sequence a_k for which the number of a_k's below n for any n is greater than $c(n\log n)^{1/3}$ for some $c>0$, [4]. (A sequence of integers is called a Sidon sequence if all the pairwise sums are distinct.)

That time this was a breakthrough, now it is strongly superseded by Ruzsa [119].

We close this part with the following

Problem 5 (Minimum independence number). *For given n and e, put*

$$a(n,e) := \min\{\alpha(G_n) : e(G_n)=e \text{ and } K_3 \nsubseteq G_n\}.$$

Determine (or estimate?) the minimum of $a(n,e)$ as e varies from 1 to $[n^2/4]$.

4. Hypergraph results and problems

For an r-uniform hypergraph $G^{(r)}$ we denote by $\alpha(G^{(r)})$ the largest subset of $V(G_n)$ not containing any hyperedge of $G^{(r)}$.

The basic problems to be solved here are of the following types:

1. *Turán hypergraph problems*: Given a forbidden $L^{(r)}$, determine or estimate the maximum number of hyperedges $G_n^{(r)}$ can have without containing $L^{(r)}$.

2. *Ramsey problems*: As we know, Ramsey theorem holds for hypergraphs as well. The problems on the corresponding Ramsey functions are even more difficult.
3. *Ramsey–Turán hypergraph problems*: Given a forbidden $L^{(r)}$, and the integers n, and $m \leqslant n$ determine or estimate the maximum number of triples $G_n^{(r)}$ can have without containing $L^{(r)}$ and having independence number $\alpha(G_n^{(r)}) < m$.
4. Given a sequence $m_n = o(n)$, under which condition on $L^{(r)}$ is there an essential difference between the answers for the first and third problems above.

Here for complete graphs there is a sharp difference between ordinary graphs ($r=2$) and hypergraphs ($r > 2$). We may define two corresponding constants

$$\tau(L^{(r)}) = \lim_{n\to\infty} \frac{\mathbf{ext}(n, L^{(r)})}{n^r}$$

and

$$\gamma(L^{(r)}) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\mathbf{RT}(n, L^{(r)}, \varepsilon n)}{n^r}. \tag{3}$$

The existence of the limit τ follows from a simple averaging argument [96], while the existence of the limit of $\lim_{n\to\infty} \mathbf{RT}(n, L^{(r)}, \varepsilon n)/n^r$ in (3) follows relatively easily from vertex-multiplication. In general, there are some results showing that in some cases these constants are equal and in some others they differ. Below we shall discuss these new phenomena for $r > 2$.

An easy consequence of a theorem of Erdős is

Proposition 46 (Erdős [38]).

$$\mathbf{ext}(n, L^{(r)}) = o(n^r)$$

iff $L^{(r)}$ *has a vertex-coloring in r colors where each hyperedge has r distinct colors.*

This characterizes the cases when $\tau(L^{(r)}) = 0$ and consequently, $\gamma(L^{(r)}) = 0$ as well. Erdős and Sós proved that

Theorem 47 (Erdős and Sós [69]). *If an r-uniform hypergraph* $L^{(r)}$ *is such that for each hyperedge e of* $L^{(r)}$ *there exists another hyperedge, f intersecting e in at least* 2 *vertices, then*

$$\tau(L^{(r)}) = \gamma(L^{(r)}).$$

Obviously, the complete hypergraphs $K_t^{(r)}$ satisfy the condition of Theorem 47 ($r \geqslant 3, t > r$). This implies e.g., that when $L^{(r)} = K_t^{(r)}$ is the complete r-uniform hypergraph, the limits coincide. It is easy to see [96] that

$$\lim \frac{\mathbf{ext}(n; K_t^{(r)})}{\binom{n}{t}} = \beta_{r,t} > 0 \quad \text{as } n \to \infty$$

exists, but the value of $\beta_{r,t}$ — even in the simplest case of $K_4^{(3)}$ is not known. Yet, this yields that $\tau(K_t^{(r)}) = \gamma(K_t^{(r)})$. (The famous conjecture of Turán asserts that this $\beta_{3,4} = \frac{5}{9}$ and one of the extremal structures is obtained as follows: n points are divided into three classes C_1, C_2, C_3 and the triples are all the ones having two points in a C_i and one in C_{i+1} ($i = 1, 2, 3$, where $C_4 := C_1$) and all the transversal triples, i.e. where the three vertices belong to three different classes.)

The same way, Theorem 47 implies that if $L^{(3)}(4;3)$ is the 3-uniform hypergraph of four vertices and three triples, then the two limits coincide for this excluded subhypergraph as well. There is a third important consequence of Theorem 47.

Let $L^{(r)}$ be an arbitrary r-uniform hypergraph and $L^{(r)}[t]$ be the hypergraph obtained from $L^{(r)}$ by 'blowing up': by replacing each vertex v by a set C_v of t new vertices and joining r new vertices $z_1, \dots, z_r$ belonging to r distinct classes $C_{v_1}, \dots, C_{v_r}$, respectively, by a hyperedge if $(v_1, \dots, v_r)$ formed a hyperedge in $L^{(r)}$.

Theorem 48 (Erdős [43]). *For any fixed integer* t,

$$\mathbf{ext}(n, L^{(r)}[t]) - \mathbf{ext}(n, L^{(r)}) = \mathrm{o}(n^r).$$

(This can be regarded as a generalization of the Erdős–Stone theorem.) Now, if we take any $\mathbf{ext}(n, L^{(r)}[t])$ for $t \geqslant 2$, that will satisfy the conditions of Theorem 47. As a matter of fact, if we fix a representing set S of the hyperedges in $L^{(r)}$ and double only the vertices in S, the resulting $L_S^{(r)}$ will also satisfy the condition. So for every hypergraph L there is a slightly larger L', obtained by blowing up some vertices of L, for which $\gamma(L') = \tau(L')$.

These results seem to show that for $r > 2$ the extra condition: 'the largest independent set has size $\mathrm{o}(n)$' has no significant effect here. This might be surprising, at least for complete graphs, knowing that for ordinary complete graphs the opposite is true (see Section 2.2) and that the conjectured extremal hypergraphs for $K_4^{(3)}$ have independent sets of size $n/3$. [14]

On the other hand, there are cases when the two constants differ.

Denote by $L^{(3)}(5;4)$ the hypergraph having the vertices x, y, z_1, z_2, z_3 and the edges (x, y, z_i), $i = 1, 2, 3$ and (z_1, z_2, z_3). Clearly

$$\mathbf{ext}(n; L^{(3)}(5;4)) > cn^3$$

and Erdős and Sós proved [69] that

$$\mathbf{RT}(n : L^{(3)}(5,4), \mathrm{o}(n)) = \mathrm{o}(n^3).$$

A more general case where the two constants differ is

[14] Originally there was one conjectured extremal graph, described above, but then Brown gave a 1-parameter family of extremal graph structures [18], Kostochka extended it to a many-parameter family [103] and van der Flaass simplified Kostochka's construction [74]. Yet all these conjectured extremal graphs have large independent sets.

Theorem 49 (Erdős and Sós [69]). *Assume that L is an r-uniform hypergraph with the following property: the vertices of L can be r-colored by $1,\dots,r$ and the edges 2-colored*[15] *in RED and BLUE so that*

(a) *All RED edges contain one vertex from each vertex-color-class.*
(b) *All BLUE edges are contained in the rth vertex-color-class.*
(c) *The BLUE edges can be enumerated so that each BLUE edge intersects the union of the previous BLUE edges in at most one vertex. Then*

$$\mathbf{RT}(n, L, o(n)) = o(n^r).$$

If L satisfies the conditions of this theorem but cannot be colored in r colors so that each hyperedge has r distinct colors then the two constants differ: $\gamma = 0$, $\tau > 0$.

Condition (c) may seem to be somewhat artificial but it comes from the fact that in the (indirect) proof the L is built up recursively: the edges are found in this order.

The following problem refers to the simplest case not covered by Theorem 49.

Problem 6 (Erdős and Sós [69]). *Let $L^{(3)}(7;11)$ be the hypergraph having the vertices $x; y_1, y_2, y_3, z_1, z_2, z_3$ and the 11 triples (x, y_i, z_j), (y_1, y_2, y_3), (z_1, z_2, z_3). Is it true that*

$$\mathbf{RT}(n; L^{(3)}(7;11), o(n)) = o(n^3)?$$

Until now we have seen cases where the two limits were positive and equal, and where τ was positive and γ was 0. The following problem was posed in [69];

Problem 7. *Does there exist a hypergraph $L^{(r)}$ $(r \geqslant 3)$ for which*

$$0 < \gamma(L^{(r)}) < \tau(L^{(r)})?$$

Frankl and Rődl [76] proved the existence of graphs for arbitrary $r \geqslant 3$, using random graph methods. Sidorenko — using many ideas of Frankl and Rődl — replaced their existence proof by a simple construction, for $r = 3$.

Construction 50 (Sidorenko [131]). *Let $L^{(3)}_{2m+1}$ be the 3-uniform hypergraph whose vertices are $a_0, \dots, a_m$ and $b_1, \dots, b_m$ with the triples*

$$\{a_i b_i a_j\} \quad \text{and} \quad \{a_i b_i b_j\} \quad \text{for } 1 \leqslant j < i \leqslant m.$$

Theorem 51 (Sidorenko [131]).

$$0 < \gamma(L^{(3)}_7) < \tau(L^{(3)}_7).$$

If one lists the hypergraph extremal *results*, one must realize that it is very seldom that $\tau(L^{(r)}) > 0$ and its value is known as well. So one question is whether we know at all results on hypergraph Ramsey–Turán problems where we know both constants

[15] Here coloring is not a proper coloring, just a partition.

$\tau(L^{(r)}) > 0$ and $\gamma(L^{(r)})$ and $\tau(L^{(r)}) > \gamma(L^{(r)})$. The field is full with difficult questions. We close this part with the following

Problem 8. *Find a function $f(n) \to \infty$, 'not too small', for which*

$$\mathbf{RT}(n, K_4^{(3)}, f(n)) = o(n^3).$$

More generally, the same question may be asked for any graph or hypergraph L instead of $K_4^{(3)}$ (for which $\gamma(L) > 0$).

Definition 52 (*Threshold*). Call $f(n)$ a 'threshold function' for $L^{(r)}$ if $g(n) = o(f(n))$ implies

$$\mathbf{RT}(n, L^{(r)}, g(n)) = o(n^3),$$

but if $g(n)/f(n) \to \infty$ then

$$\mathbf{RT}(n, L^{(r)}, g(n)) = \gamma n^3$$

for some positive constant γ.

Problem 9. *Does there exist such a threshold function for every $L^{(r)}$? If not, give conditions when it does?*

Remark 53. Obviously, if $\mathbf{RT}(n, L^{(r)}, o(n)) = o(n^r)$ but $\mathbf{ext}(n, L^{(r)}) > cn^r$, then n is a threshold function.

4.1. Applications

One important application of Hypergraph Ramsey–Turán problem was where Komlós, Pintz and Szemerédi [99] improved the lower bound in Heilbronn's problem (for a more detailed account see also Beck [10]), thus disproving Heilbronn's conjecture:

> For n points in a unit disk in the plane, no three on a line, take the minimum of the areas of the corresponding $\binom{n}{3}$ triangles. Let $\Delta(n)$ be the maximum of this minimum, taken over all the positions of the n points. What is the order of magnitude of this $\Delta(n)$?

Conjecture 10 (Heilbronn). *$\Delta(n) \leqslant c/n^2$, for some constant $c > 0$.*

If we have n points in the unit disk, $P_1, \dots, P_n$, then the half-lines $P_1 \to P_j$ cut the disk into $n-1$ parts. So the area of the smallest part is at most $\pi/(n-1)$. Erdős constructed n points so that the minimum area is larger than c/n^2, by taking an $n \times n$ square grid and n points from it, no three on a line. [16] It took roughly 30 years to show that there are cases where the minimum area is much larger than $1/n^2$.

[16] The existence of n such points is nontrivial but not too difficult.

Theorem 54 (Komlós, Pintz and Szemerédi [99]).

$$\Delta(n) > c\frac{\log n}{n^2}.$$

Remark 55. As to improving the upper bound, Roth proved [117] that $\Delta(n) < 1/n^{1+\mu}$ if $\mu < \mu_0 = 1.117\ldots$ and n is large enough. Komlós, Pintz and Szemerédi improved Roth's result, showing that $\mu < 8/7 = 1.142857$ would also do above [98].

The basic tool to prove the lower bound was an extension of Theorem 44 to hypergraphs, due by Ajtai et al. [2].

Theorem 56. *Let G_n be a $(k+1)$-uniform hypergraph with n vertices and average degree t. It is proved that if $k \ll t \ll n$ and if G_n contains no cycle of length 2, 3 or 4, then the stability number $\alpha(G_n) \geqslant c_k(n/t)(\log t)^{1/k}$.*

5. Positive edge densities in graphs and hypergraphs

Here we shall discuss edge-density conditions and their connection to quasi-random graphs. Quasi-random graphs and hypergraphs form a more and more important area in random graph theory.

Thomason [144,145] and Chung, Graham and Wilson [32] gave some characterization of randomlike graph sequences, Chung and Graham [31] extended this to hypergraph sequences, Frankl, Rödl and Wilson [77] gave some characterizations of 'randomlike' matrix sequences, etc. Some results of Erdős and Sós [69] on hypergraphs is also one of the roots of the theory of quasi-random combinatorial structures. For some results related to this topic see [77,30–32,136,137]. There are — among others — two main 'themes' in this field: that the edges (hyperedges) are uniformly distributed and that all small graphs occur in these graphs.

One of the weakest 'edge-density' conditions is that $\alpha(G_n^{(r)}) = o(n)$. Let $(G_n^{(r)})$ be a sequence of r-uniform hypergraphs. Below we shall define several different uniformity conditions, which form kind of a hierarchy.

Condition A. For every $\varepsilon > 0$ there exist an $\eta = \eta(\varepsilon) > 0$ and $n_0(\varepsilon)$ so that if $n > n_0(\varepsilon)$, then for every induced subgraph $H_m^{(r)} \subseteq G_n^{(r)}$, with $m > \varepsilon n$, we have

$$e(H_m^{(r)}) > \eta\binom{m}{r}.$$

Condition B(c). For a fixed $c > 0$, for every $\varepsilon > 0$ and $n_0(\varepsilon)$, if $n > n_0(\varepsilon)$, and $m > \varepsilon n$ then for every induced subgraph $H_m^{(r)} \subseteq G_n^{(r)}$, we have

$$e(H_m^{(r)}) > c\binom{m}{r}.$$

Condition C (c,δ,ε). For fixed $c>0$, $\delta,\varepsilon>0$ (where $\delta<\min(c,1-c)$), there exists an n_0 such that if $n>n_0$ and $m>\varepsilon n$, then for every induced subgraph $H_m^{(r)}$ of $G_n^{(r)}$,

$$(c-\delta)\binom{m}{r}<e(H_m^{(r)})<(c+\delta)\binom{m}{r}.$$

Condition D(c). For fixed $c>0$, for every δ where $\delta<\min(c,1-c)$, there exist an $\varepsilon=\varepsilon(\delta)\in(0,\frac{1}{2})$, with $\varepsilon(\delta)\to 0$ as $\delta\to 0$, and there is an n_0 such that if $n>n_0$ and $m>\varepsilon n$ then for every induced subgraph $H_m^{(r)}\subseteq G_n^{(r)}$, we have

$$(c-\delta)\binom{m}{r}<e(H_m^{(r)})<(c+\delta)\binom{m}{r}.$$

Erdős and Sós showed that

Theorem 57 (Erdős and Sós [69]). *For ordinary graphs* ($r=2$), *and for every fixed* $k>2$, *if* (G_n) *is any graph sequence satisfying Condition A*, *then* $K_k\subseteq G_n$ *for n large enough.*

Rödl proved that Condition C for $r=2$ has an even stronger consequence:

Theorem 58 (Rödl [118]). *For ordinary graphs*, *for every positive integer k and every* $c>0$ *and* $\delta>0$ *such that* $\delta<\min(c,1-c)$ *there exists an* $\varepsilon>0$ *and a positive integer* n_0 *such that if* $n\geqslant n_0$ *and* G_n *is a graph for which every induced subgraph* H_m *with* $m>\varepsilon n$ *vertices satisfies*

$$(c-\delta)\binom{m}{2}<e(H_m)<(c+\delta)\binom{m}{2},$$

then G_n *contains all graphs with k vertices as induced subgraphs.*

Remark 59. This is (again) strongly connected to the theory of quasi-random graphs. Condition D is already a ‘quasi-random graph’ property. Restricting ourselves to ordinary graphs ($r=2$) condition D(c) implies that G_n contains each H as an induced subgraph asymptotically as many times as in the random graph of edge-probability c.

However, this is not true for hypergraphs. As we mentioned, in [69] Erdős and Sós constructed a 3-uniform hypergraph $G^{(3)}$ satisfying Condition A but not containing $K_4^{(3)}$, not even a $L^{(3)}(4,3)$.

A surprisingly simple construction of Füredi (see [75]) gives the even stronger result: even the stronger Condition D does not imply the existence of a $L^{(3)}(4,3)$. Füredi took a random tournament T_n and defined $G_n^{(3)}$ on its vertex set as the family of triples which spanned a directed 3-cycle in T_n. One can easily see that this hypergraph does not contain $L^{(3)}(4,3)$: on any four points it has 0 or 2 triangles and satisfies D.

This shows that for hypergraphs even Condition D is not enough to imply the existence of $L^{(3)}(4,3)$. It is somewhat surprising that — as Frankl and Rődl proved in

[76] — there is an infinite (recursively given) sequence $(H_i^{(r)})$ of r-uniform hypergraphs such that if a sequence $(G_n^{(r)})$ satisfies Condition A, then $H_i^{(r)} \subseteq G_n^{(r)}$ if n is sufficiently large. As a special case, $L^{(3)}(7,11)$ of Problem 6 is such a graph.

The case of $L^{(3)}(4;3)$ is fairly important to make a short detour. First we formulate an old conjecture on its extremal number, which was disproved by Frankl and Füredi. Then we explain the motivation of the original construction and the basic idea of its disproof. Finally we clarify, how these assertions are connected to our hypergraph Ramsey–Turán problems.

Below we mostly (but not entirely) restrict ourselves to 3-uniform hypergraphs and to the case of one excluded 3-uniform hypergraph $L^{(3)}$.

Construction 60. *Take n vertices and partition them into three (roughly) equal classes. Take all the* $\approx \frac{1}{27}n^3$ *triplets joining each of the three classes. Then subdivide each of the three classes into three classes of size* $\approx n/9$ *vertices and take all the*

$$\approx 3\ \frac{1}{27}\left(\frac{n}{3}\right)^3$$

triplets which are completely in an original class and intersect all the three subclasses of it. Iterate this k times, for $k \to \infty$, *getting* $\frac{1}{24}n^3 + o(n^3)$ *triplets.*

Remark 61. This construction can also be described in a less transparent but more compact way: if the vertices are the integers $1,\dots,n$, then we take those triplets (i,j,k) which — written in ternary form — for some $t = t(i,j,k) \in [0,\log_3 n]$ have the same digits in the positions $1,\dots,t-1$ and three different digits in the tth position.

Erdős conjectured that this is the extremal configuration for $L^{(3)}(4;3)$ but this was disproved by Frankl and Füredi [75]. They noticed that the above iteration method (i.e., taking a hypergraph, replacing each of its vertices by groups of vertices and putting into the new groups smaller hypergraphs not containing $L^{(3)}(4;3)$) works in general as well. They also noticed that this method provides a better construction if one takes the $L^{(3)}(4;3)$-extremal hypergraph $Q_6^{(3)}$ on six vertices and applies the iteration to the blown up version: $S_n^{(3)} := Q_6^{(3)}[n/6]$. Here 'blown up' means that each vertex of $Q_6^{(3)}$ is replaced by $[n/6]$ independent vertices. The $Q_6^{(3)}$ can more explicitly be described by fixing that its vertices are $1,\dots,6$ and the triplets are

$(1,2,3)$, $(1,2,4)$, $(3,4,5)$, $(3,4,6)$ $(5,6,1)$, $(5,6,2)$, $(1,2,5)$, $(1,4,6)$, $(2,3,6)$, $(2,4,5)$.

Frankl and Füredi obtained that

$$\lim_{n\to\infty} \frac{\mathbf{ext}_3(n, L^3(4,3))}{\binom{n}{3}} \geqslant \frac{2}{7},$$

or, in another form,

$$\mathbf{ext}_3(n, L^3(4,3)) \geqslant \frac{1}{21}n^3 + o(n^3).$$

As a matter of fact, we could say that it is not that surprising that for $L^{(3)}(4;3)$, γ and τ are equal (by Theorem 47) since it is easy to check that the Construction 60 contains only $o(n)$ independent vertices. (As a matter of fact, only $O(n^{1-c})$.) The same holds for the improved construction. So, if the extremal graph is such an 'iterated' construction, then the two constants are trivially equal. (Watch out: we know that $\gamma=\tau$ in this case but we do not know the extremal graphs, neither that they are 'iterated' graphs.)

Problem 11. *Is there a hypergraph $L^{(3)}$ for which there exists an (asymptotically) extremal graph sequence which satisfies Condition* A?

Problem 12. *Is there a hypergraph $L^{(3)}$ for which there exists an (asymptotically) extremal graph sequence which satisfies Condition* B?

Problem 13. *Is there a hypergraph $L^{(3)}$ for which there exists an (asymptotically) extremal graph sequence which satisfies Condition* C? (*or Condition* D?)

Erdős and Sós [69] conjecture that such extremal graphs do not exist. Strengthening the uniformity condition we would get a weaker version:

Problem 14. *Assume that k is fixed and that a sequence of* 3*-uniform hypergraphs $\{G_n^{(3)}: n \in \mathbb{N}\}$ is such that there exists a constant $c>0$ so that for every sufficiently large $n \in \mathbb{N}$ every induced subgraph $H_m \subseteq G_n$ with $m > n/\log n$ vertices has at least $(c+o(1))\binom{m}{3}$ edges. Is it true that*

$$K_k^{(3)} \subseteq G_n^{(3)}?$$

The problem is unsolved even for $k=4$. Moreover, even for $L^{(3)}(4,3) \subseteq G_n^{(3)}$ is unknown, see [69].

6. Ordinary graphs, the intermediate range

The intermediate range is when we assume that $\alpha(G_n) \leqslant cn$ and $c>0$ is small but fixed. Ramsey Theorem and Construction 12 give

Corollary 62. *For*

$$m > \frac{n}{R(k_1,\ldots,k_r)}, \tag{4}$$

$$\mathbf{RT}(n,k_1,\ldots,k_r,m) = \mathbf{ext}(n,K_{R+1}) = \left(1-\frac{1}{R}\right)\binom{n}{2} + O(1).$$

Hence we will assume that m is large but not too large: $n < R(k_1,\ldots,k_r,m)$ but (4) does not hold. Mostly we are interested in the case when $\alpha(G_n)=o(n)$. One could think that the case $\alpha(G_n) \leqslant cn$ for small but fixed $c>0$ is perhaps also tractable.

Below we ask only the simplest questions.

If G_n contains no K_3 and $\alpha(G_n) \leqslant cn$, then replacing each vertex of G_n by t independent vertices and joining them as the original vertices were joined we get a graph G_{nt} without triangles and with $\alpha(G_{nt}) \leqslant cnt$. This implies the existence of $H(c)$ in the next problem.

Problem 15. *Determine* $H(c)$ *where*

$$\mathbf{RT}(n; 3, cn) = (H(c) + o(1))n^2.$$

Remark 63. Clearly, $H(c) \leqslant c$, since the regarded graphs have maximum degree $\leqslant cn$. The solution of this problem, at least for some range of c, is 'hidden' in some papers of Andrásfai [7,8], and treated in more details in the Habilitation Thesis of Stephan Brandt [16].

Of course, it is enough to regard here graphs G_n which are triangle-free, with $\alpha(G_n) \leqslant cn$ and which are maximal with respect to being triangle free: adding any edge to it produces a triangle. For some structural information on such graphs see also Pach [113].

Problem 16. *Determine* $h(c)$ *where*

$$\mathbf{RT}(n; 3, 3, cn) = (h(c) + o(1))n^2.$$

At first, Problem 16 may seem to be easy, but it is not. Using the coloring of the edges of K_{16} by three colors, say, RED, BLUE and YELLOW, none of which contains a triangle, one may consider the graph $T_{n,16}$ colored according to the above Ramsey coloring and then delete the YELLOW edges. In the resulting graph G_n we have $\alpha(G_n) \leqslant \lceil n/8 \rceil$ and the degrees are around

$$n - \frac{n}{16} - \frac{5n}{16} = \frac{5n}{8}.$$

So Erdős and Sós conjectured that

$$\mathbf{RT}\left(n; 3, 3, \frac{n}{8}\right) = \left(\frac{5}{16} + o(1)\right) n^2,$$

suggesting that to determine $h(c)$ may not be so easy [67].

7. K_p-independence results

Let the K_p-independence number $\alpha_p(G)$ of a graph G be the maximum order of an induced subgraph in G which contains no K_p. (So K_2-independence number is just the maximum size of an independent set.)

Definition 64. For given integers $r, p, m > 0$ and graphs $L_1, \dots, L_r$, we define the corresponding Ramsey–Turán function $\mathbf{RT}_p(n, L_1, \dots, L_r, m)$ to be the maximum number of edges in a graph G_n of order n such that $\alpha_p(G_n) \leqslant m$ and there is an edge-coloring of G_n with r colors such that the jth color class contains no copy of L_j, for $j = 1, \dots, r$.

The concept of $\alpha_p(G)$ was introduced long ago by Hajnal, and also investigated by Erdős and Rogers, see [62]. (A similar 'independence notion' is investigated for random graphs in a paper of Eli Shamir [123], where he generalizes some results on the chromatic number of random graphs.)

We start with a result and an open problem, stated in 'elementary' terms, related to the Szemerédi theorem and Bollobás–Erdős construction on $\mathbf{RT}(n, K_4, o(n))$.

Theorem 65 (Erdős, Hajnal, Simonovits, Sós and Szemerédi [51]). *Assume that* (G_n) *is a graph sequence,* $\alpha_3(G_n) = o(n)$.

(a) *If* $K_5 \not\subseteq G_n$ *then*

$$e(G_n) \leqslant \tfrac{1}{12} n^2 + o(n^2).$$

(b) *If* $K_6 \not\subseteq G_n$ *then*

$$e(G_n) \leqslant \tfrac{1}{6} n^2 + o(n^2).$$

On the other hand, for every $p \geqslant 2$,

(c) *There is a sequence of graphs* (G_n) *not containing* K_{2p}, *with* $\alpha_p(G_n) = o(n)$, *for which*

$$e(G_n) \geqslant \tfrac{1}{8} n^2 + o(n^2). \tag{5}$$

Problem 17. *Is it true that* $\alpha_3(G_n) = o(n)$ *and* $K_5 \not\subseteq G_n$ *imply* $e(G_n) = o(n^2)$?

Conjecture 18. *The asymptotically extremal graphs for* $\mathbf{RT}_p(n, K_k, o(n))$ *have the following structure*:

Let $k = pq + \ell$, $(\ell = 1, 2, \dots, p)$. *Then n vertices are partitioned into* $q + 1$ *classes* $V_{0,n}, \dots, V_{q,n}$. *For each pair* $i \neq j$, $\{i, j\} \neq \{0, 1\}$ $V_{i,n}$ *is almost completely joined to* $V_{j,n}$ *in the sense that every* $x \in V_{i,n}$ *is joined to every* $y \in V_{j,n}$ *with a possible exception of* $o(n^2)$ *pairs xy. Further,* $d(V_0, V_1) = (\ell - 1)/p + o(1)$ (*as* $n \to \infty$), *and* V_0, V_1 *are joined* $o(1)$*-regularly. Finally,* $e(V_i) = o(n^2)$, $i = 1, \dots, p$.

Remark 66. For graphs of this kind the optimal sizes of the classes V_i can easily be computed. The optimal class-sizes are

$$|V_i| = \frac{1}{2 + (q-1)(2 - (\ell - 1)/p)} n + o(n) \quad \text{for } i = 0, 1$$

and

$$|V_i| = \frac{(2 - (\ell - 1)/p)}{2 + (q-1)(2 - (\ell - 1)/p)} n + o(n) \quad \text{for } 2 \leqslant i \leqslant q.$$

From this $e(S_n)$ can easily be calculated: $e[V_i]=o(n^2)$ can be neglected. If S_n is the graph described in the conjecture, it is almost regular, the degrees in V_2 are $n-|V_2|$. Hence

$$e(S_n)\approx\frac{1}{2}(n-|V_2|)n\approx\left(1-\frac{(2p-\ell+1)}{q(2p-\ell+1)-\ell+1}\right)\binom{n}{2}.$$

Problem 19. *We see from Theorem* 65 *that* $\vartheta_3(K_6)\in[\frac{1}{8},\frac{1}{6}]$. *Determine its exact value.*

Until now we were mostly interested in the $o(n)$-range. However, for an arbitrary fixed f, like $f(n)=n^c$ or $f(n)=n/(\log n)^t$, etc. we may ask analogous questions: give estimates on $\mathbf{RT}_p(n,L_1,\ldots,L_r,o(f(n)))$. We shall define (similarly to $\mathbf{RT}(n,L_1,\ldots,L_r,o(n))$ and $\vartheta(L_1,\ldots,L_r)$ the much more general) $\mathbf{RT}_p(n,L_1,\ldots,L_r,o(f(n)))$ and $\vartheta_{p,f}(L_1,\ldots,L_r)$:

Definition 67.

$$\vartheta_{p,f}(L_1,\ldots,L_r)=\lim_{\varepsilon\to0}\limsup_{n\to\infty}\frac{\mathbf{RT}_p(n,L_1,\ldots,L_r,\varepsilon f(n))}{\binom{n}{2}}.$$

The case of general f is investigated in [50] but here we shall restrict ourselves to the simplest case $f(n)=n$. The meaning of the next theorem is that (a) ϑ is an upper bound for any $\varepsilon_n\to0$ (b) but it can be achieved by an appropriate $\varepsilon_n^*\to0$ and (c) one can choose this ε_n^* to be 'maximal' in some sense.

Theorem 68 (Erdős et al. [51]). *For any* $k_1,\ldots,k_r$ *for* $f(n)=n$, *for any* $\varepsilon_n\to0$,

(a) *Let* (S_n) *be an external graph sequence for* $\mathbf{RT}_p(n,K_{k_1},\ldots,K_{k_r},\varepsilon_n n)$. *Then*

$$\limsup_{n\to\infty}\frac{e(S_n)}{\binom{n}{2}}\leqslant\vartheta_{p,f}(K_{k_1},\ldots,K_{k_r}).\tag{6}$$

(b) *There exists an* $\varepsilon_n^*\to0$ *for which on the left-hand side of* (6) *the limit exists and*

$$\lim_{n\to\infty}\frac{e(S_n)}{\binom{n}{2}}=\vartheta_{p,f}(K_{k_1},\ldots,K_{k_r}).\tag{7}$$

(c) *For every* $\varepsilon_n\to0$ *with* $\varepsilon_n\geqslant\varepsilon_n^*$ *the same — namely,* (7) *— holds.*

Here $f(n)=n$ means that we consider the case $\alpha_p(G_n)=o(n)$. We restrict ourselves to complete graphs, and assert the existence of the limit which we do not know in the general case!

Some further results of [51] assert that in the general case there are asymptotically extremal graph sequences of fairly simple structure, where 'simple' means that the structure depends on n weakly. This is a weak generalization of the Erdős–Stone–

Simonovits theorem (from ordinary extremal graph theory) [64,71]. Formally, the results of [51] assert that in many cases 'there exists a matrix A for which the *optimal matrix graph sequence* $(A(n))$ is asymptotically extremal' for the Ramsey–Turán problem considered by us. Here the optimal matrix graph sequences — in some sense — generalize the Turán graphs, while the so called matrix graphs generalize the complete t-partite graphs (see also [40,132]). We refer the reader to [51], since the explanation of the notion of optimal matrix graph sequences would require some further important but too technical definitions.

The isoperimetric inequality behind Theorem 65(c): As we explained, proving the lower bound on $\mathbf{RT}(n, K_4, o(n))$ Bollobás and Erdős used an 'isoperimetric' theorem. The lower bound (5) is a generalization of the Bollobás–Erdős result. So it is natural to use a generalization of the original isoperimetric inequality. This generalization was conjectured by Erdős and proved by Bollobás.

We need the following definition.

Definition 69. For $k \geqslant 2$ define the kth *packing constant* [17] of a set A in a metric space by

$$d_k(A) = \sup_{x_1,\dots,x_k \in A} \min_{i<j} \rho(x_i, x_j).$$

A spherical cap is the intersection of an h-dimensional unit sphere $\mathbb{S}^h$ and a halfspace Π.

Theorem 70 (Bollobás [13]). *Let A be a nonempty subset of the h-dimensional unit sphere $\mathbb{S}^h$ of outer measure $\mu^*(A)$* [18] *and let C be a spherical cap of the same measure. Then $d_k(A) \geqslant d_k(C)$ for every $k \geqslant 2$.*

Below, whenever we speak of 'measure', we shall always consider relative measure which is the measure of the set on the sphere $\mathbb{S}^h$ divided by the measure of the whole sphere.

Denote by $\delta = \delta_p$ the diameter of a p-simplex. ($\delta_2 = 2$, $\delta_3 = \sqrt{3}, \dots$)

Corollary 71 (Erdős et al. [51]). *Let the integer p and two small constants ε and $\eta > 0$ be fixed. Then for $h > h_0(p, \varepsilon, \eta)$, if A is a measurable subset of $\mathbb{S}^h$ of relative measure $> \varepsilon$, then there exist p points $x_1, \dots, x_p \in A$ so that all $d(x_i, x_j) > \delta_p - \eta$.*

This is what we needed to get the lower bound (5).

[17] k-diameter in [13].

[18] In applications we use only 'nice sets' but Bollobás formulated his result in this generality. The reader can replace 'outer measure' by 'measure'.

8. Application of Turán's theorem

8.1. Application to geometry and to metric spaces

We have already described in the Introduction, how Turán [149], setting out from an observation of Erdős [34] initiated the systematic application of Turán-type extremal results in geometry, analysis, general metric space [150–152], etc.

Turán's basic observation was as follows:

Given n points in the space (or in any bounded metric space), for every $c > 0$ we can define a graph $G(c)$ by joining the points P and Q iff $\rho(P,Q) > c$. By establishing some appropriate geometric facts, we may ensure that $G(c)$ contains no complete $p = p(c)$-graph. Hence we know (by Turán's theorem) that the number of pairs (P,Q) with $\rho(P,Q) > c$ cannot be too large. Assume that we apply this method with many constant $c_1 > c_2 > \cdots > c_k > 0$. If $f(x)$ is a monotone decreasing function and we are interested in

$$\sum f(\rho(P_i,P_j))$$

then we may obtain lower bounds on this expression by replacing all the distances between c_i and c_{i+1} by c_i. The 'only' problem to be solved is:

How to choose the constants $c_1 > c_2 > \cdots > c_k > \cdots > 0$ to get the best results?

This was the point, where the *packing constants* came in:

Let $\mathbb{M}$ be a metric space, and let $\mathscr{F}$ be a family of finite subsets of $\mathbb{M}$ each of which has diameter at most c, for some fixed constant c. Typical examples are

(i) the family of all finite subsets with diameter at most c of a closed set $D \subseteq \mathbb{M}$;
(ii) the family of all subsets of a bounded set $D \subseteq \mathbb{M}$.

We are interested in the distribution of distances $\rho(P_j,P_j)$ for an n-element set $\{P_1,\dots,P_n\} \in \mathscr{F}$. In characterizing these distributions, we find that the 'packing constants' (Definition 69) are very useful. The kth packing constant is

$$d_k = \sup_{\{Q_1,\dots,Q_k\}\in\mathscr{F}} \min_{i\neq j} \rho(Q_i,Q_j).$$

Clearly, if there are at least $k+1$ different points in $\mathbb{M}$, then for the kth packing constant $d_{k+1} \leqslant d_k$. If M is a bounded subset of the m-dimensional Euclidean space $\mathbb{R}^m$, then $d_k \to 0$. Such constants (depending largely on the geometric situation) are called packing constants. Their investigation goes back at least to a dispute between Newton and Gregory [73,151].

Observe that, by the definition of d_k, if $\{P_1,\dots,P_n\} \in \mathscr{F}$ and if G_n is the graph defined on the vertex set $V = \{P_1,\dots,P_n\}$ by joining P_i and P_j by an edge if and only if $\rho(P_i,P_j) > d_{k+1}$, then G_n contains no complete subgraph K_{k+1}. Applying Turán's theorem to this G_n we obtain a slightly simplified version of Turán's distance distribution theorem [149].

Theorem 72. *For any* $\{P_1,\dots,P_n\} \in \mathscr{F}$, *the number of distances* $\rho(P_i,P_j)\leqslant d_{k+1}$ *is at least*

$$\frac{1}{2k}n(n-k).$$

Under some quite natural additional conditions, Theorem 72 becomes sharp.

It is not worth giving a detailed description of the results obtained this way, since the Introduction of [54] does it.

In the next part we shall regard the applications of Turán's graph theorem to the distribution of distances in metric spaces. Using the distance-distribution results Turán, Sós [138], and later Erdős et al. [54–56], could give estimates on certain integrals.

8.2. The dual problem

Definition 73 (*kth covering constant*). Given a metric space $\mathbb{M}$, the kth covering constant $c_k(\mathbb{M})$ is defined as the infimum of those r for which there exist k points $P_1,\dots,P_k$ and r-balls $\mathscr{B}(P_i,r)$ around them so that

$$\mathbb{M} = \bigcup_{i=1}^{k} \mathscr{B}(P_i,r).$$

An equivalent formulation is

$$c_k := \inf_{(P_1,\dots,P_k)} \sup_{Q} \min_{i=1,\dots,k} \rho(Q,P_i),$$

where $Q\in\mathbb{M}$, $P_i\in\mathbb{M}$.

Theorem 74 (Sos [138]). *If* $A=\{P_1,\dots,P_n\}$ *is a point set in the plane, having kth covering constant* c_k, *then at least*

$$e(n,k):=(k-1)(n-1)+\left[\frac{n-k+2}{2}\right]$$

of the distances $\rho(P_i,P_j)$, $(1\leqslant i\leqslant j\leqslant n)$ *satisfy* $\rho(P_i,P_j)\geqslant c_k$.

The result is sharp, the proof follows from a theorem of Erdős and Leo Moser [58] on k-universal graphs:

Theorem 75 (Erdős–Moser). *If* G_n *is a graph of order n with the property that to every k-vertex subset* $X\subseteq V(G_n)$ *there is an* $x\in V(G_n)$ *joined to all the vertices of X then*

$$e(G_n)\geqslant(k-1)(n-1)+\left[\frac{n-k+2}{2}\right].$$

8.3. Embeddability

Assume that we are given n points in a metric space with their distances and we wish to decide if they can be embedded into a low-dimensional Euclidean space.

Proposition 76 (Erdős, Meir, Sós and Turán [54]). *If $P_1,\dots,P_n$ are n points in a metric space $(\mathbb{M},\rho)$ and $\max\rho(P_i,P_j)\leqslant 1$ and if*

$$\rho(P_i,P_j) > \sqrt{\frac{k}{k+2}}$$

for more than $(k/(2k+2))n^2$ pairs (i,j), then this point-set cannot be embedded into $\mathbb{R}^k$.

It is known (see below) that in $\mathbb{R}^k$ for the case when $\mathscr{F}$ is the family of sets of diameter at most 1, then

$$d_1 = d_2 = \cdots = d_{k+1} = 1 \tag{8}$$

and

$$d_{k+2} = \sqrt{\frac{k}{k+2}} \tag{9}$$

if k is even and

$$d_{k+2} = \sqrt{\frac{k^2+2k-1}{k^2+4k+3}}$$

if k is odd. The second expression is the smaller! Clearly, (8) and the above estimates of d_k are obtained by putting $\lceil k/2\rceil+1$ points into the vertices of a $\lceil k/2\rceil$-dimensional simplex and the remaining $\lfloor k/2\rfloor+1$ points into the vertices of another simplex in an orthogonal plane. (The related results come from Schönberg [121] Schütte, Seidel [122], etc. and one can find a simple and elementary proof of this geometric fact in a note of Bárány with a related more general conjecture [9].)

The above proposition can be stated as follows: if we have too many distances larger than the above d_{k+2} then — by Turán's theorem — we have a set of $k+2$ points having pairwise distances $\geqslant\sqrt{k/(2k+2)}\geqslant d_{k+2}$, and this shows that we cannot embed even these $k+2$ points into $\mathbb{R}^k$. The positive feature of applying Turán's theorem is that to check this distance-distribution takes only $\binom{n}{2}$ trivial steps while checking the existence of those $k+2$ points with pairwise large distances takes more steps.

8.4. Chromatic number of geometric graphs

There is a large area of combinatorial geometry, where extremal graph results can well be applied. Instead of going into details we mention just one subfield and refer the reader to the Handbood of Combinatorics [60] and to the book of Pach and Agarwal [114].

(i) The topic we wish to mention here is embedding graphs into low-dimensional Euclidean spaces so that the adjacent vertices be at unit distances in the space. One of the papers to be mentioned in this field was that of Erdős, Meir, Sós, Turán [53]. In [53] the following (not too difficult) assertion is proved:

Theorem 77. *If G is a d-chromatic graph then it can be embedded into $\mathbb{R}^{2d}$ so that if two vertices are adjacent then the representing points have distance* 1.

The proof idea is that regard in $\mathbb{R}^{2d}$ d circles of radii $1/\sqrt{2}$ around the origin, in pairwise orthogonal planes, say $C_1,\dots,C_d$, and put the vertices of the ith color class of G onto the ith circle C_i, for $i=1,\dots,d$.

(ii) The minimum dimension d for which G can be embedded into $\mathbb{R}^d$ so that the edges join vertices of distance 1 is the *dimension* of the graph.[19]

(iii) For each graph G we can also ask its *faithful dimension*. This is the minimum dimension d for which G can be embedded into $\mathbb{R}^d$ so that x and y are joined in the graph if and only if their distance is 1 in the space. (The faithful dimension can be much larger than the ordinary dimension, e.g., the dimension of bipartite graphs is at most 4 and their faithful dimension can be arbitrary large.)

(iv) It would be interesting to know the chromatic number of $\mathbb{R}^d$, i.e. of the (infinite) graph the vertices of which are the points of $\mathbb{R}^d$ and two vertices (points) are joined if their distance is 1. This problem can easily be transformed into a question on finite graphs, using the de Bruijn–Erdős theorem. Larman and Rogers [101] proved that this chromatic number is smaller than $(3+o(1))^d$ and much later Frankl and Wilson [78] proved that it is at least $(1-o(1))\cdot 1.2^d$.

(v) Another related paper on some geometric dimension of a graph G where the extremal graph theoretical approach is used is an Erdős–Simonovits paper [65]. Here the essential chromatic number of $\mathbb{R}^d$ is defined. This equals to t if in any graph (G_n) embedded into $\mathbb{R}^d$ one can delete $o(n^2)$ edges (as $n\to\infty$) so that the resulting graph has chromatic number $\leqslant t$ (in sense of (ii)).

There are many problems and results on application of extremal graph theory in geometry also in the paper of Erdős [41].

8.5. Applications in analysis

There are many quantities in analysis depending on distance-distributions in a set. Energy integrals are among them. Some other quantities occur in connection with the theory of analytic functions, conformal mappings and so on. Such quantities are the ‘capacity’ of a plane set, and the conformal radius, among others. In this section — for the sake of brevity — we shall skip the definitions, and refer the reader to Chapter 7 of the book of Goluzin, available both in Russian and English [83,84].

[19] Here two vertices can have distance 1 even if they are not joined!

8.5.1. Transfinite diameter, capacity

In the typical applications of Turán's theorem Turán, and later Erdős, Meir, Sós and Turán in [54–56] used to assume some regularity conditions which sometimes are not really needed (only for the sharpness) but are natural in the applications. Here are the conditions assumed in [54]:

Let (X,ρ) be a complete metric space and $\mathscr{F}$ be a family of point sets in it, satisfying

1. *There exists an R such that all the sets of $\mathscr{F}$ are in $\mathbb{B}(0,R)$.*
2. *If $S\in\mathscr{F}$ and $S_1\subset S$ in finite, then $S_1\in\mathscr{F}$.*
3. *If $S\in\mathscr{F}$ is finite and $P\in F$, then for any $\varepsilon>0$ there is a $P_1\in X$ so that $P_1\neq P$, $\rho(P,P_1)<\varepsilon$ and $S\cup\{P_1\}\in\mathscr{F}$.*

Theorem 78. *Suppose that B is a bounded closed set in the plane and that ∂B belongs to an $\mathscr{F}$-family of sets satisfying Conditions 1–3 and having packing constants d_k. If*

$$\sum_k \frac{1}{k^2}\log\frac{1}{d_k}$$

diverges, then the capacity of B is 0.

Perhaps a heuristic explanation of this theorem is that in some sense a set is small if its capacity is 0, in some other sense it is small if its packing constants tend to 0 fast and this theorem connects the two quantities.

8.5.2. Outer conformal radius

Theorem 79. *If B is a bounded closed continuum whose complement is simply connected and ∂B belongs to an $\mathscr{F}$-family of sets satisfying 1, 2, 3 and having packing constants d_k, then the outer conformal radius $r=r(B)$ satisfies*

$$r(B)\leqslant\prod_{k=2}^{\infty}(d_k)^{1/k(k-1)}.$$

8.5.3. Potential theory

Let $f(r)$ be a decreasing function, and let $\rho(x,y)$ be the distance between x and y in $\mathbb{R}^m$. If D is a closed subset of $\mathbb{R}^m$ and μ is a mass distribution (or measure) on D, then the generalized potential is defined by

$$I(f)=\int_{D\times D} f(\rho(x,y))\,\mathrm{d}\mu_x\,\mathrm{d}\mu_y.$$

(In classical physics, $f(r)=-\log r$ for $m=2$, and $f(r)=r^{2-m}$ for $m=3,4,\ldots$.)

Theorem 72 immediately implies the following result (see [149]):

Theorem 80 (Turán's Potential Theorem). *If* $D \subset \mathbb{R}^m$ *is compact, if* d_k *is its kth packing constant, and if* $f(r) \geqslant d_0$ *for* $r \in (0, d_2)$, *then*

$$I(f) \geqslant \mu(D) \sum_{k \geqslant 2} \frac{f(d_k)}{k^2 - k}.$$

8.6. Lipschitz functions

Theorem 81 (Erdős, Meir, T. Sós and Turán [56]). *Let* $\mathscr{F}$ *denote the set of functions in* $C[0,1]$ *satisfying* $f(0) = 0$ *and* $|f(x_1) - f(x_2)| \leqslant |x_1 - x_2|$ *whenever* $0 \leqslant x_1 \leqslant x_2 \leqslant 1$. $||f - g||$ *is the usual maximum difference norm. For* $k = 1, 2, \ldots,$ *if* $n > 2^k$ *an* $f_1, \ldots, f_n \in \mathscr{F}$ *then the number of pairs* (f_i, f_j) *with* $||f_i - f_j|| \leqslant 2/k$ *is at least*

$$\frac{n^2}{2^k} - \frac{n}{2}.$$

This estimate is sharp.

A corollary of this theorem is that the probability that randomly chosen $f, g \in \mathscr{F}$ satisfy $||f - g|| \leqslant 2/k$ is at least $1/2^{k-1}$, for $k = 2, \ldots$.

8.7. Triangle functionals

Up to now we have considered only binary functionals. Of course, the same methods can be applied to value-distributions of geometric data depending on more than two points. Thus e.g., we may ask

Problem 20. *Given a set of points*, $x_1, \ldots, x_n$ *in a metric space* $\mathbb{M}$,

- *If for any triple* $(x_i x_j x_k)$ *the perimeter of the triangle is at most* 1, *how many triples* $(x_i x_j x_k)$ *may have perimeter larger than* t?
- *If for any triple* $(x_i x_j x_k)$ *the area of the triangle is at most* 1, *how many triangles* $(x_i x_j x_k)$ *may have area larger than* t?

Obviously, these are related to extremal hypergraph problems (on 3-uniform hypergraphs). For related results see [55].

8.8. Probability theory

Katona started applying extremal graph theory to probability theory [88–91,94]. The germ of these application was the observation that if we have two random variables ζ and η, then — knowing their distribution and applying Turán-type extremal graph theory — we can derive estimates on the probabilities $P(\zeta + \eta < x)$, i.e. on the distribution of the sum. One typical result in this field was

Theorem 82 (Katona's Inequality [88]). *If ξ and η are vector-valued independent random variables with the same distribution, then*

$$P(|\xi+\eta|>x)\geqslant \tfrac{1}{2}P(|\xi|>x)^2.$$

Katona and later Sidorenko have published several results on this topic [125–130].

We do not give a detailed description of the topic here, primarily since Katona gave an excellent survey of the field in [94].

One could say that the probability applications are mostly application of geometry, with one new feature: instead of describing the distribution of pairwise distances between geometric points one has to describe length distribution of certain vector sums. If e.g., we want to prove that Theorem 82 holds in an arbitrary Hilbert space, then we have to use

Lemma 83. *For any three vectors $a_1, a_2, a_3 \in \mathbb{R}^3$ of length at least 1 we may choose two appropriate ones, a_i and a_j for which $|a_i + a_j| > 1$.*

This implies that for an arbitrary Hilbert space $\mathscr{H}$,

Lemma 84. *Let $a_1, \dots, a_n \in \mathscr{H}$ with $||a_i|| > x$ and let the vertices of a graph G_n be these vectors and the edges be the pairs (a_i, a_j) for which $||a_i + a_j|| > x$. Then $K_3 \not\subset G_n$.*

Then — using the last lemma and Turán's theorem — one can show relatively easily that

Theorem 85 (Katona). *Let ξ, η be independent random variables with values from a Hilbert space $\mathscr{H}$ which have m different values of equal probabilities. Then*

$$P(|\xi+\eta|>x) \quad \geqslant \quad P(||\xi+\eta||\geqslant x, ||\xi||\geqslant x, ||\eta||\geqslant x) \quad \geqslant \quad \tfrac{1}{2}P(|\xi|>x)^2.$$

If ξ, η have a continuous range, they can have arbitrary many values but if the probability space is 'atom'-free [20] then the previous argument can be modified to give the same result.

The general case, when atoms are also allowed, can also be handled. But we skip here the discussion of this case.

8.8.1. Related extremal graph results

Some new types of extremal problems occur in these applications. To be precise, the results motivated by these applications often have occurred earlier but without being

[20] i.e. each set A of positive probability can be partitioned into two measurable subsets of strictly smaller measure.

applied to other fields.

(a) One has to extend Turán type extremal problems to continuous versions.
(b) Sidorenko proved that to prove certain kind of probability distribution results is equivalent with solving extremal graph problems with colored vertices. Here we fix a set of colors, say, $\chi_1,\dots,\chi_\ell$ and the graph G_n will be replaced by a vertex-colored graph: $G_n \subseteq K_\ell(n_1,\dots,n_\ell)$ where these class-sizes are treated as arbitrary but known, fixed parameters. The forbidden graphs also $L \in \mathscr{L}$ are vertex-colored (where vertices of the same color are allowed to be joined). We allow $L_i \subseteq G_n$ in *most positions*, but we exclude $L_i \subseteq G_n$ so that the vertices of the jth color of L_i are in the jth color-class of G_n, $(j=1,\dots,\ell)$. Such problems were earlier investigated in [133], in connection with hypergraph extremal problems.
(c) In some other applications weighted extremal graph problems have to be solved. One such case was discussed in Section 2.7.

8.9. Applications in number theory

Extremal graph theory has several roots, and the two most important ones among them are Turán's paper [147] and the 'Tomsk' paper of Erdős [33].

Erdős often arrived at graph problems from the applications in other fields. A detailed description of this 'story' can be found in the 'preface'-paper of Szekeres included in the Art of Counting [141]. Here we are interested in the birth of extremal graph theory.

In 1938 Erdős published a paper [33] which has several interesting features.

- It contains the first extremal graph problem Erdős dealt with, namely, the problem of excluding the C_4 and (not too surprisingly) it contains the first application of 'finite geometrical' methods to provide lower bounds for extremal graph problems.
- This paper seems to be the first example where Erdős used extremal graph theory in other fields, namely, in number theory.

A few years ago Erdős, A. Sárkőzy and T. Sós [63] returned to this field, solved some further problems from combinatorial number theory, using extremal graph theory, then G. Sárkőzy and Győri proved some related extremal graph results (Győri's results [87] completely settled the related graph theoretical question) and earlier Faudree, Simonovits, de Caen and Székely had some related results.

A somewhat more detailed description of the problem can be found in [63,87].

This connection of number theory and combinatorics was the topic of Sós' lecture at the SIAM meeting, Toronto, 1998 [140].

9. Further open problems

There are various intriguing open problems in connection with the above theorems. Some of them are mentioned in the corresponding sections, some further ones are listed below.

9.1. Ramsey–Turán problems

The problem below is motivated by the Bollobás–Erdős construction. Assume that g is fixed. Here we are looking for two graphs, F_m and H_m, with girth $\geqslant g$ and $\alpha(G_m)=\alpha(H_m)=o(m)$ and try to join them by $\frac{1}{2}m^2-o(m^2)$ edges so that the resulting G_{2m} contains no K_4. Observe that in the Bollobás–Erdős construction we have large odd girth but our graphs are full with large complete bipartite graphs $K_2(p,p)$. (It would be nice also to have $F_m=H_m$.)

Of course, we have constructions for graphs with large girth H_m but we do not know if they can be joined densely without getting a K_4.

Problem 21 (Generalized $\mathbb{BE}$-graphs). *Can one construct a graph similar to the Bollobás–Erdős graph, where the girth of the components is arbitrary large: Find a graph G_{2m} with two vertex-disjoint subgraphs F_m and H_m, so that $g(F_m), g(H_m) \geqslant g$ and $\alpha(F_m)=o(n)$ and $\alpha(H_m)=o(n)$ and $e(G_{2m})=\frac{1}{8}(2m)^2$ and $K_4 \not\subseteq G_{2m}$.*

A solution to this problem would (almost) imply that it is the Arboricity which determines the asymptotical value of $\mathbf{RT}(n, L, o(n))$ (see Section 2.6).

The following problem seems to be very difficult and may be raised in several different settings but we cannot solve it even in the simplest case. We formulate the problem here for graphs and many colors.

Problem 22 (The spectrum). *Let $\mathscr{RT}(n, L_1, \ldots, L_r, m)$ be the* set of integers *e for which there exists a graph G_n with $\alpha(G_n) < m$, and $e(G_n)=e$ which can be edge-colored in r colors so that the ith color contains no L_i, for $i=1,\ldots,r$. Is this set an interval?*

As it is remarked e.g. in [52], this problem may be relevant in several different cases, e.g. in investigating size-Ramsey numbers.

Problem 23. *Determine $\vartheta(K_q, K_3)$.*

Perhaps the following is true.

Construction 86. *Let $t=R(q,s)$. Color $T_{n,t}$ by RED and BLUE canonically (with respect to the classes of $T_{n,t}$) — i.e. the coloring of an edge depends only on the classes it joins — so that the colored graph should contain neither RED K_q, nor BLUE K_s. Put into each class of this graph a RED Erdős graph F_m* of *Construction* 16. *The resulting graph $U_n=U(n,q,s)$ will contain neither a RED K_{2q-1}, nor BLUE K_s. Clearly,*

$$e(U_n) \geqslant e(T_{n,t})$$

and

$$\alpha(U_n)=o(n).$$

Hence

$$RT(n, 2q-1, s, o(n)) \geqslant e(T_{n,t}).$$

Conjecture 24. *$U(n,q,3)$ of Construction 86 is extremal for* $\mathbf{RT}(n, 2q-1, 3, o(n))$.

9.2. K_p-independence problems

Various open problems are stated in [51]. Here we list some of them. The first two of these are the simplest special cases of Conjecture 18 where we got stuck.

Problem 25. *Determine $\vartheta_3(K_{11})$ and $\vartheta_3(K_{14})$.*

According to Conjecture 18, $\vartheta_3(K_{11}) = \frac{11}{32}$ and $\vartheta_3(K_{14}) = \frac{8}{21}$.

Problem 26. *Is there a finite algorithm to find the limit*

$$\vartheta_p(L) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\mathbf{RT}_p(n, L, \varepsilon n)}{\binom{n}{2}}?$$

We have proved in [50] that there is a finite algorithm to find $\vartheta_2(L_1, \ldots, L_r)$ if the sample graphs L_i are complete graphs. A paper of Brown, Erdős and Simonovits [24] shows that for the digraph extremal problems without parallel arcs (which seems to be very near to the Turán–Ramsey problems) there is an algorithmic solution, though far from being trivial. What is the situation in case of $\vartheta_p(L_1, \ldots, L_r)$?

9.3. Related Ramsey problems

Ramsey–Turán problems often lead to interesting and difficult questions in Ramsey theory. Erdős and Sós formulated several conjectures on the Ramsey function [67]. We mention here two of them.

Conjecture 27.

$$\lim_{n \to \infty} \frac{R(3,3,n)}{R(3,n)} \to \infty \tag{10}$$

and

$$\lim_{n \to \infty} (R(3, n+1) - (R(3,n)) \to \infty. \tag{11}$$

It is very surprising that (10) and (11), which seem trivial at first sight, cause serious difficulties.

Conjecture 28.

$$\lim_{n \to \infty} \frac{R(3,3,n)}{n^2} \to \infty$$

and perhaps $R(3,3,n) > n^{3-\varepsilon}$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$.

10. Uncited references

[36,39,42,44,47–49,70,86,92,93,111,112,153,155]

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