THE UNLABELLED SPEED OF A HEREDITARY GRAPH PROPERTY

JÓZSEF BALOGH, BÉLA BOLLOBÁS, MICHAEL SAKS, AND VERA T. SÓS

Abstract. A *property* of graphs is a collection $\mathcal{P}$ of graphs closed under isomorphism; we call $\mathcal{P}$ *hereditary* if it is closed under taking induced subgraphs. Given a property $\mathcal{P}$, we write $\mathcal{P}^n$ for the set of graphs in $\mathcal{P}$ with vertex set $[n] = \{1, \dots, n\}$, and $\mathcal{P}_n$ for the isomorphism classes of graphs of order n that are in $\mathcal{P}$. The cardinality $|\mathcal{P}^n|$ is the *labelled speed* of $\mathcal{P}$ and $|\mathcal{P}_n|$ is the *unlabelled speed*. In the last decade numerous results have been proved about the labelled speeds of hereditary properties, with emphasis on the striking phenomenon that only certain speeds are possible: there are various pairs of functions $(f(n), F(n))$, with $F(n)$ *much* larger than $f(n)$, such that if the labelled speed is infinitely often larger than $f(n)$ then it is also larger than $F(n)$ for all sufficiently large values of n. Putting it concisely: the speed *jumps* from $f(n)$ to $F(n)$. Recent work on hereditary graph properties has shown that "large" and "small" labelled speeds of hereditary graph properties do jump.

The aim of this paper is to study the unlabelled speed of a hereditary property, with emphasis on jumps. Among other results, we shall show that the unlabelled speed of a hereditary graph property is either of polynomial order or at least $S(n)$, the number of ways of partitioning a set with n indistinguishable elements.

1. Introduction

For a graph property $\mathcal{P}$, the n^{th} *labelled slice* of $\mathcal{P}$ is the set $\mathcal{P}^n$ of graphs in $\mathcal{P}$ with vertex set $[n] = \{1, \dots, n\}$. The *labelled speed* of a

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1991 *Mathematics Subject Classification.* 05C.

Key words and phrases. graph properties, monotone, hereditary, speed, size.

Research of the first author was supported in part by NSF grants DMS 0302804, DMS 0603769 and DMS 0600303, UIUC Campus Research Board 06139 and by OTKA grant 049398.

Research of the second author was supported in part by NSF grant CCR-0225610, DMS-0505550 and W911NF-06-1-0076.

Research of the third author was supported in part by NSF grants CCR-9988526 and CCR-0515201.

Research of the fourth author was supported in part by OTKA grants T032236, T038210 and T042750.

property $\mathcal{P}$ is the function $n \mapsto |\mathcal{P}^n|$. Similarly, the n^{th} *unlabelled slice* of $\mathcal{P}$ is the set $\mathcal{P}_n$ of isomorphism classes of graphs of order n that are in $\mathcal{P}$, and the *unlabelled speed* of $\mathcal{P}$ is the function $n \mapsto |\mathcal{P}_n|$. Trivially, $|\mathcal{P}_n| \le |\mathcal{P}^n| \le n!|\mathcal{P}_n|$ for every n.

In what follows, by a "subgraph" we always mean an *induced subgraph*, so that a graph property is *hereditary* if it is closed under taking subgraphs. Also, two graphs are considered to be the "same" if they are isomorphic. Otherwise, the notation and terminology in this note are standard. Thus, K_n is a complete graph on n vertices, and E_n the "empty graph" of order n, i.e., the graph on n vertices with no edges. Also, G_n denotes a graph on n vertices. The *neighborhood* of a vertex $x \in V(G)$ is $\Gamma(x) = \{y : \ xy \in E(G)\}$, and the *degree* of x is $d(x) = |\Gamma(x)|$. For a $U \subset V(G)$ write $G[U]$ for that graph spanned by G on U.

Turning to less standard concepts, given a graph G, we define a relation $\sim$ on $V(G)$: for two vertices $x, y \in V(G)$ we call x and y *twins* and write $x \sim y$ if $\Gamma(x) \cup \{x, y\} = \Gamma(y) \cup \{x, y\}$. This relation $\sim$ is an equivalence relation; we call its equivalence classes the *homogeneous* classes of the graph. A *homogeneous k-part graph* is a graph with k-partition $(V_1, \dots, V_k)$ such that each pair of vertices in the same set V_i are twins. (Note that every graph of order n is a homogeneous n-part graph.) Let $S(n)$ be the number of partitions of a set with n indistinguishable elements into nonempty subsets. Thus $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 5$, $S(5) = 7$ and $S(n) = \exp(\Theta(\sqrt{n}))$. Also, denote by $B(n)$ the number of partitions of a set with n distinguishable elements, so that $B(1) = 1$, $B(2) = 2$, $B(3) = 5$, $B(4) = 15$ and $B(n) \approx (n/\log n)^n$. It is clear that $B(n)$ and $S(n)$ have different order of growths.

Let $\mathcal{S}$ denote the property that consists of all graphs whose components are cliques, and set $\bar{\mathcal{S}} = \{G : \bar{G} \in \mathcal{S}\}$, i.e., let $\bar{\mathcal{S}}$ be the class of complete k-partite graphs for $k \ge 1$.

Clearly $\mathcal{S}$ has unlabelled speed $S(n)$ and labelled speed $B(n)$, and so does $\bar{\mathcal{S}}$. Let $\mathcal{T}$ denote the property consisting of all star forests, i.e., graphs whose components are stars, and put $\bar{\mathcal{T}} = \{G : \bar{G} \in \mathcal{T}\}$. Also, denote by $\mathcal{F}$ the property consisting of all the path forests, i.e., graphs whose components are paths, and set $\overline{\mathcal{F}} = \{G : \bar{G} \in \mathcal{F}\}$. Clearly, each of $\mathcal{T}$, $\bar{\mathcal{T}}$, $\mathcal{F}$ and $\overline{\mathcal{F}}$ has unlabelled speed $S(n)$, and labelled speed greater than $B(n)$.

Let us start by recalling some results concerning labelled speeds of hereditary properties. Parts (i) and (ii) are from [3], part (iii) is from [4] and [5], and (iv) is from [9].

Theorem A. *Let $\mathcal{P}$ be a hereditary property of graphs. Then one of the following assertions holds.*
(i) There exist $N, k \in \mathbb{N}$ and a collection $\{p_i(n)\}_{i=0}^k$ of polynomials such that, for all $n > N$, $|\mathcal{P}^n| = \sum_{i=0}^k p_i(n) i^n$.
(ii) For some $t \in \mathbb{N},\ t > 1$, we have $|\mathcal{P}^n| = n^{(1-1/t+o(1))n}$.
(iii) For n large enough, $n^{(1+o(1))n} = B(n) \leq |\mathcal{P}^n| \leq 2^{o(n^2)}$.
(iv) There exists $k \in \mathbb{N},\ k > 1$, such that $|\mathcal{P}^n| = 2^{(1-1/k+o(1))n^2/2}$. □

Our aim in this paper is to prove an analogue of cases (i) and (ii) of Theorem A for unlabelled speeds. Note that these are the cases when the number of labellings, $n!$, is larger than the labelled speed of the property, so the crude bounds $|\mathcal{P}_n| \leq |\mathcal{P}^n| \leq n!|\mathcal{P}_n|$ hardly tell us anything about $|\mathcal{P}_n|$.

Theorem 1. *For every hereditary graph property $\mathcal{P}$ one of the following assertions holds.*
(i) There are integers ℓ and t such that if n is large enough then every graph $G \in \mathcal{P}^n$ is the symmetric difference of a homogeneous ℓ-part graph and a graph in which every component has at most t vertices. The unlabelled speed $\mathcal{P}_n$ is polynomially bounded; even more, there is a positive integer k and a rational number c such that

$$|\mathcal{P}_n| = c \cdot n^k + O(n^{k-1}). \tag{1}$$

(ii) If n is large enough, $|\mathcal{P}_n| \geq S(n)$. Furthermore, equality holds for n large enough if and only if $\mathcal{P}$ is one of the six hereditary properties $\mathcal{S}, \bar{\mathcal{S}}, \mathcal{T}, \overline{\mathcal{T}}, \mathcal{F}$ and $\overline{\mathcal{F}}$.

The structure of the paper is as follows. In Section 2 we show that any hereditary property satisfying condition (i) or condition (ii) of Theorem A, satisfies condition (i) of Theorem 1. In Section 3 we show that for any hereditary property for which neither conditions (i) or (ii) of Theorem A hold, condition (ii) of Theorem 1 holds. In the final section we make some remarks about properties with higher speeds.

2. Case (i) of Theorem 1

In this section we show that any hereditary property satisfying condition (i) or condition (ii) of Theorem A also satisfies condition (i) of Theorem 1.

To do this, we will need results from [3] that provide more detailed information about hereditary graph properties that satisfy conditions (i) or (ii) of Theorem A. We start with some more terminology and notation. We write $G(A, B)$ for a *template*, a graph whose vertex set is partitioned into two classes, A and B: the *anchor* and the *body* of the graph. A template is allowed to contain loops, but only at the vertices of B. (Our notation also suggests the fact that the vertices in B will be 'blown up' into many vertices, while those in A will just 'anchor' the new structure.)

Let $x_1, \dots, x_b$ be an enumeration of the vertices in B, so that $|B| = b$. Given non-negative integers $m_1, \dots, m_b$, let $G(A; B\cdot(m_i)_1^b)$ be the graph obtained from the template $G(A, B)$ by replacing each x_i by m_i vertices, and joining two vertices if the original vertices were joined by an edge or loop. Thus, $H = G(A; B\cdot(m_i)_1^b)$ has $|A|+\sum_{i=1}^b m_i$ vertices and, e.g., two of the m_i vertices replacing x_i are joined by an edge if and only if $G(A, B)$ has a loop at x_i. We say that H is obtained from $G(A, B)$ by blowing up the vertices of B, or multiplying each vertex x_i by m_i.

For a template $G(A, B)$, let $\mathcal{P}(G(A, B))$ be the following set of graphs:

$$\mathcal{P}(G(A, B)) = \mathcal{P}(G(A, B)) = \{G : \ G \cong G(A; B\cdot(m_i)_1^b),\ m_i \geq 1 \text{ for every } i\},$$

and write $\mathcal{P}^n(G(A, B))$ for the set of graphs in $\mathcal{P}(G(A, B))$ with n vertices, and $\mathcal{P}_n(G(A, B))$ for the set of their isomorphism classes.

We shall examine the cases (i) and (ii) in Theorem 1. Considerably more is known about these cases than what we stated in Theorem A; in the arguments below we shall make use of this additional information as well; much of what we shall need will be given in Theorem B.

(i) We have here three subcases. The first was described by Scheinerman and Zito [11]: for large n, the property contains only some of E_n and K_n, and so the speed is $0, 1$ or 2. It is trivial that in this case the unlabelled speed equals the labelled speed.

In the second subcase, the labelled speed is polynomial (see [3], Theorem 10). As shown there, the structures of the graphs in these properties are as follows. There is a finite set of templates $G(A_i, B_i)$, $i = 1, \dots, \ell$, with each B_i a single vertex or a vertex with a loop, such that for n large enough, $\mathcal{P}^n = \cup_{i=1}^{\ell}\mathcal{P}^n(A_i, B_i)$. It is easy to see that, for n sufficiently large, the unlabelled speed is constant.

The remaining type of hereditary graph property to be studied here is the *exponential*. As the terminology indicates, these types are distinguished by the labelled speeds of the properties: a hereditary property $\mathcal{P}$ is exponential if $|\mathcal{P}^n|/n^k \to \infty$ for every k and $|\mathcal{P}^n| = n^{o(n)}$; in this case by Theorem A (i) we have $|\mathcal{P}^n| = (c + o(1))^n$ for some constant $c > 1$.

Recall that two vertices in a graph are said to be *twins* if their neighbourhoods coincide. Call a template *irreducible* if no vertex of its body has a twin (in either the body or anchor). Note that if $b \in B$ has a twin in $a \in A$ then for $G' = G - a$ and $A' = A \setminus \{a\}$ we have

$\mathcal{P}(G(A,B)) \subset \mathcal{P}(G'(A',B))$. Similarly, if $b \in B$ has a twin in B then $\mathcal{P}(G(A,B)) \subset \mathcal{P}(G'(A,B'))$ for $G' = G - b$ and $B' = B \setminus \{b\}$. Thus if $\mathcal{P} = \cup_{i=1}^{\ell} \mathcal{P}(G(A_i,B_i))$ is a hereditary property then we may assume that $G(A_i,B_i)$ is irreducible.

Given a natural number D, let us define the following subset of $\mathcal{P}(G(A,B))$:

$$\mathcal{P}_D(G(A,B)) = \{G(A; B\cdot(m_i)_1^b) : \ m_i \geq D \text{ and } |m_i - m_j| \geq D \text{ for } i \neq j\}.$$

Thus $G \in \mathcal{P}(G(A,B))$ belongs to $\mathcal{P}_D(G(A,B))$ if all the multipliers of G differ from 0 and each other by at least D. We shall use the self-explanatory notation $\mathcal{P}_D^n(G(A,B)) = \mathcal{P}_D(G(A,B))^n$ and $\mathcal{P}_{D,n}(G(A,B)) = \mathcal{P}_D(G(A,B))_n$.

Let us recall the following structural theorem, Theorem 18 of [3].

Theorem B. *Let $\mathcal{P}$ be an exponential hereditary graph property. Then there are finitely many (non-isomorphic) templates, $G_1(A_1,B_1),\ldots,$ $G_s(A_s,B_s)$, each irreducible, such that if n is large enough then*

$$\mathcal{P}^n = \bigcup_{i=1}^{s} \mathcal{P}^n(G_i(A_i,B_i)).$$

For exponential properties, Theorem B implies the structural part of Theorem A (i).

Needless to say, in the union above the sets $\mathcal{P}^n\,(G_i(A_i,B_i))$ need not be disjoint; however as the next result shows, we can make these sets disjoint if we make them slightly smaller.

Lemma 2. *Let $G_1(A_1,B_1),\ldots,G_s(A_s,B_s)$ be non-isomorphic templates, each irreducible. Then for $D = \max_i\,(|A_i|+|B_i|+1)$ the sets $\mathcal{P}_{D,n}(G_1(A_1,B_1))$, $\ldots$, $\mathcal{P}_{D,n}(G_s(A_s,B_s))$ are pairwise disjoint.*

Proof. Let $G \in \mathcal{P}_{D,n}(G_i(A_i,B_i))$. Then, since $G_i(A_i,B_i)$ is irreducible each vertex of B_i gives rise to a distinct homogenous class. Thus G has $|B_i|$ large homogeneous classes, i.e., classes with at least D vertices each. These are precisely the classes obtained from blowing up the vertices of B_i. In addition to these large homogeneous classes, G has $|A_i|$ more vertices. Hence, if G is also in $\mathcal{P}_{D,n}(G_j(A_j,B_j))$, then since the homogeneous classes have different sizes we get one-to-one maps between the homogeneous classes of B_i and B_j, and between A_i and A_j. These maps induce an isomorphism between $G_i(A_i,B_i)$ and $G_j(A_j,B_j)$, mapping A_i into A_j and B_i into B_j, showing that $G_i(A_i,B_i)$ and $G_j(A_j,B_j)$ are isomorphic templates. $\square$

Now we are ready to prove that there are constants k and c such that $|\mathcal{P}_n|$ satisfies (1).

By Theorem B and Lemma 2, we can find a constant D and templates $G_1(A_1, B_1), \dots, G_s(A_s, B_s)$, such that

$$\bigcup_{i=1}^{s} \mathcal{P}_{D,n}\ (G_i(A_i, B_i)) \subset \mathcal{P}_n \ = \ \bigcup_{i=1}^{s} \mathcal{P}_n\ (G_i(A_i, B_i)),$$

and the classes $\mathcal{P}_{D,n}(G_i(A_i, B_i))$ are pairwise disjoint. Now, for every i,

$$|\mathcal{P}_n\ (G_i(A_i, B_i)) - \mathcal{P}_{D,n}\ (G_i(A_i, B_i))| \ = \ O(n^{k_i-1}), \tag{2}$$

and

$$|\mathcal{P}_{D,n}\ (G_i(A_i, B_i))| \ = \ c_i n^{k_i} \ + \ O(n^{k_i-1}) \tag{3}$$

for some natural numbers k_i and strictly positive rationals c_i. Indeed, as we shall see shortly, $k_i = b_i - 1 = |B_i| - 1$. Clearly, relations (2) and (3) imply (1) with $k = \max k_i$ and $c = \sum\{c_i : k_i = k\}$.

It remains to prove (2) and (3). Note that the left-hand side of (2) is at most the number of multipliers $(m_1, \dots, m_{b_i})$ such that $\sum_{j=1}^{b_i}\ m_j = n - a_i$ and either

$$\min\{m_j :\ 1 \le j \le i\} < D$$

or

$$\min\{|m_{j_1} - m_{j_2}| :\ j_1 \ne j_2\} < D.$$

This is clearly no more than

$$(b_i - 1)\binom{n - a_i}{b_i - 2} D,$$

proving (2).

Also, if $\sum_{j=1}^{b_i} m_j = n - a_i$,

$$\min\{m_j : 1 \le j \le i\} \ge D$$

and

$$\min\{|m_{j_1} - m_{j_2}| :\ j_1 \ne j_2\} \ge D$$

then the number of ways to choose the multipliers is $(1+O(n^{-1}))\binom{n}{b_i-1}$. Two different multipliers may give isomorphic graphs. For example, if Λ is the set (group) of permutations of B that can be extended to an automorphism of G, then for any vector of multipliers, the $|\Lambda|$ vectors obtained by permuting m by the permutation of Λ all generate isomorphic copies of the same graph. Using the irreducibility of the template $G_i(A_i, B_i)$ and an argument similar to that used to prove Lemma 2, one can show that this is the only way to generate isomorphic graphs by different multipliers. Thus the number of isomorphism types of graphs generated from these multipliers is $\frac{1}{|\Lambda|}(1 + O(n^{-1}))\binom{n}{b_i-1}$. This proves

(3) and so completes the proof of Theorem 1 for the exponential case.

Note that the smallest unbounded speed, $\lceil (n+1)/2 \rceil$, can be achieved by two properties of exponential type: the homogeneous bipartite graphs and their complements.

Now we turn to the proof of Theorem 1 (i) in the factorial range. This range is studied in both [3] and [6]; in particular the proof of Theorem A (ii) is in [3]. The results of [3] and [6] imply the structural part of Theorem 1 (i), so our task is only to justify relation (1). To this end, we shall imitate the proof in the exponential range; however, in order to count as there, we shall introduce a rather technical system of notation. It is in this form that we shall recall the results we shall need (see Theorem C).

For the rest of this section we shall assume that $\mathcal{P}$ is a hereditary property with $|\mathcal{P}^n| = n^{(1-1/t+o(1))n}$ for some $t > 1$.

Let us define a *unit* to be a graph in which every vertex has a label i, $0 \le i \le r$, and the vertices with label 0 are linearly ordered. Note that the same label may be used for many vertices. The vertices with label 0 span the *server*, and the remaining vertices the *terminal.* For a unit U_i, we write S_i for its server and T_i for its terminal. Also, we write u_i, s_i and t_i for the orders of these graphs, so that $u_i = s_i + t_i$.

Two units are *compatible* if their servers are isomorphic, and there is an order-preserving isomorphism. Loosely speaking, two units are *compatible* if their servers coincide.

Let us fix r, s and t; in what follows, we suppress the dependence of our objects on these parameters. Let $U_1, \dots, U_\ell$ be all the units with $s_i = |S_i| \le s$ and $t_i = |T_i| \le t$, and labels $0, 1, \dots, r$. Let $J \subset L = \{1, \dots, \ell\} = [\ell]$ be such that the units $U_j, j \in J$ are compatible. Given natural numbers $m_j, j \in J$, let $H = H((m_j)_{j\in J})$ be the graph obtained as follows. For $j \in J$, take m_j copies of unit U_j such that all $\sum_{j\in J} m_j$ of these units are pairwise disjoint. We call $\mathbf{m} = (m_j)_{j\in J}$ the sequence of *multipliers* of H. The graph H is obtained from these units by identifying their (isomorphic) servers. Note that

$$|H| = s + \sum_{j\in J} m_j t_j,$$

where $s = s_j$ for every $j \in J$. Clearly, $V(H) = \cup_{i=0}^{r} V_i$, where $V_i = V_i(H)$ is the set of vertices of the constituent units that are labelled i. Thus $|V_0| = s$ and, for $i \ge 1$, $|V_i| = \sum_{j\in J} m_j t_{i,j}$, where $t_{i,j}$ is the number of vertices of U_j that are labelled i.

The sequences $\mathbf{m} = (m_j)_{j\in J}$ we shall be interested in are such that the non-empty classes V_i of $H(\mathbf{m})$ form an initial segment $0, 1, 2, \ldots, h$; we call such sequences *permissible*.
Given a permissible sequence $\mathbf{m}$, let $V_0, V_1, \ldots, V_h$ be the classes of $H = H(\mathbf{m})$. Call a graph K *compatible* with $H = H(\mathbf{m})$ if its vertex sets is $[h]$. Given H and a graph K compatible with H, define a graph G on $V = \cup_{i=0}^h V_i$ as follows. Let $x \in V_i$ and $y \in V_{i'}$, $x \neq y$. If ii' is not an edge (or loop) of K, then $xy \in E(G)$ iff $xy \in E(H)$; otherwise, $xy \in E(G)$ iff $xy \notin E(H)$. We write $H\Delta K$ for this graph G. Thus $H\Delta K$ is obtained from H by toggling the edges according to the instructions coded by K.

Let $J \subset L$ be such that if $\mathbf{m} = (m_j)_{j\in J}$, with $m_j \geq 1$ for every j, then $H(\mathbf{m})$ has classes $V_0, V_1, \ldots, V_h$, and let K be compatible with H. Let $\mathcal{P}(J, K)$ be the set of isomorphism classes of graphs $G(H, K) = H\Delta K$ with $H = H(\mathbf{m})$, $\mathbf{m} = (m_j)_{j\in J}$, and K compatible with H. Note that $\mathcal{P}(J, K)$ need not be a hereditary property of graphs.
Finally, for $D \geq 0$, let $\mathcal{P}_D(J, K)$ be the set of members of $\mathcal{P}(J, K)$ in which each multiplier m_j is at least D, and $|m_j - m_{j'}| \geq D$ whenever $j, j' \in J$, $j \neq j'$.

After all this preparation, we are ready to state a (structural) result about factorial hereditary properties that will enable us to deduce relation (1) in Theorem 1 (i). This result is hardly more than a reformulation of Theorem 28 [3], including a statement analogous to Lemma 2.

Theorem C. *Let $\mathcal{P}$ be a hereditary property of graphs with factorial labelled speed. Then we have*

- *integers ℓ, s, t, N and D,*
- *subsets J_i of $L = [\ell]$, $i = 1, \ldots, N$,*
- *sets of units $\{U_1, \ldots, U_N\}$, with $|S_i| \leq s$ and $|T_i| \leq t$ for each i,*
- *a graph K_i on $[h_i]$ for each i,*

such that

$$\bigcup_{i=1}^{N} \mathcal{P}_D^n(J_i, K_i) \subseteq \mathcal{P}^n \subseteq \bigcup_{i=1}^{N} \mathcal{P}^n(J_i, K_i), \tag{4}$$

and $\mathcal{P}_D^n(J_i, K_i) \cap \mathcal{P}_D^n(J_{i'}, K_{i'}) = \emptyset$ whenever $i \neq i'$.

□

The rest of the proof of (1) in the factorial range goes as in the exponential range. All we have to do is to justify that $\mathcal{P}_{D,n}(J_i, K_i)$ approximates $\mathcal{P}_n(J_i, K_i)$ in the sense that, for some positive integer k_i and a rational c_i, we have

$$|\mathcal{P}_{D,n}(J_i, K_i) - \mathcal{P}_{D,n}(J_i, K_i)| = O(n^{k_i - 1}) \tag{5}$$

and

$$|\mathcal{P}_{D,n}(J_i, K_i)| = c_i n^{k_i} + O(n^{k_i-1}). \tag{6}$$

These relations can be proved as (2) and (3); we leave the details to the reader.

Note that the structural part of Theorem 1 was mainly proved in [3]. It is easy to see that in the polynomial case we have $\ell = 1$, and in the exponential case $t = 1$. The factorial case is more technical, relation (4) implies that there are integers ℓ, t and C such that if n is large enough then every graph $G \in \mathcal{P}^n$ is such that for some set V_0 of at most C vertices, the graph $G - V_0$ is the symmetric difference of a homogeneous ℓ-part graph and a graph in which every component has at most t vertices. Observing that each vertex of V_0 can be a class in a homogeneous partition of a graph yields the result.

3. Case (ii) of Theorem 1

In this section we show that any hereditary property satisfying neither condition (i) nor condition (ii) of Theorem A satisfies condition (ii) of Theorem 1.

Here we have the following strategy: by making use of a result in [3] we shall show that, given n, there is an $N = N(n)$ such that $\mathcal{P}^N$ contains a graph with at least $S(n)$ many different subgraphs of order n.

Let us recall from [6] a detailed description of the properties which are not in classes (i) and (ii) of Theorem A. In this description we have two main cases (A and B), and several subcases. Let then $\mathcal{P}$ be a hereditary property not in classes (i) an (ii) of Theorem A. In [6] it was proved that $\mathcal{P}$ satisfies one of the cases below.

Case A: The property $\mathcal{P}$ contains at least one of the 24 types of graph from a list (see [6], Section 5 and Theorem 20). The graphs are as follows.

Fix positive integers m and t. Let $V(G) = U \cup V_1 \cup \ldots \cup V_t$ be a vertex partition of G with $U = \{u_1, \ldots, u_t\}$ and $|V_i| = m$ for $i = 1, \ldots, t$. Furthermore, we have the following structural restrictions:

- the set U is either independent or spans a complete graph (2 possibilities),
- either every V_i is an independent set, or every V_i spans a complete graph (2 possibilities),
- either every vertex of every V_i is joined to every vertex of $V_j \neq V_i$, or there is no edge vw with $v \in V_i$ and $w \in V_j \neq V_i$ (2 possibilities),

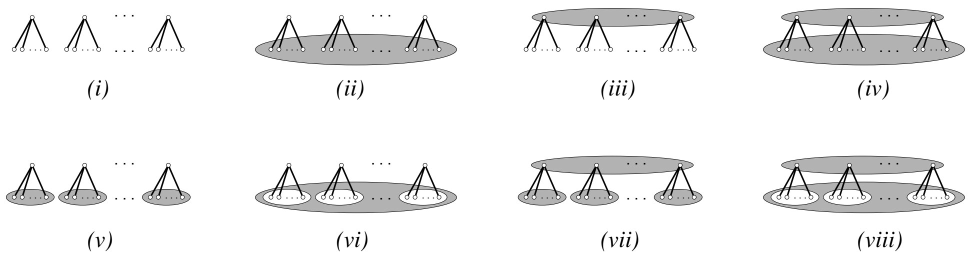

FIGURE 1. The eight possibilities for case A.1. The grey ovals indicate sets which induce a clique, while an empty oval within a grey oval represents an induced independent set within an otherwise fully connected group of vertices (i.e., a Turán graph). In each figure, the top vertices form $U = \{u_1, \ldots, u_t\}$ and the bottom vertices make up $V_1 \cup V_2 \cup \ldots \cup V_t$.

- there are 3 possibilities for the edges between a vertex $u_i \in U$ and a set V_j, $(1 \leq i, j \leq t)$, namely
  A.1. u_i is adjacent to v_j iff $i = j$,
  A.2. u_i is adjacent to v_j iff $i \neq j$,
  A.3. u_i is adjacent to v_j iff $i \leq j$.

We can greatly reduce the number of cases to be analyzed. First, observe that if we have a property $\mathcal{P}$ then the "complementary" property $\underline{\mathcal{P}} = \{G : \underline{G} \in \mathcal{P}\}$ has the same (labelled or unlabelled) speed.

Since a graph occurring in case A.2. is the complement of a graph in case A.1, there is no need to consider the graphs occurring in case A.2. (See Figure 1 for the eight types of case A.1.) We can reduce the number of graphs to be studied in case A.3. if we delete u_1 and then pair four-four graphs with each other (see Figure 2): simply relabel $u_2, \ldots, u_t$ as $u_t, \ldots, u_2$. (In the proof we use only the graphs obtained after deletion of u_1.) We can easily take care of additional six cases, since if a subgraph of the graph spanned by $V_1 \cup \ldots \cup V_t$ is not a clique or an independent set, then the property contains all members of $\mathcal{S}$ or $\underline{\mathcal{S}}$, and has unlabelled speed at least $S(n)$.

Thus, altogether there are six types of graph to consider, namely those shown in (i)-(iv) of Figure 1, and in Figure 3 (i) and (ii). First, let us consider the case A.1.

Before going into the details we sketch our general strategy. Consider a graph in $\mathcal{P}_N$ whose existence is guaranteed, and the aim is to find many different subgraphs of it with n vertices. To prove that there are

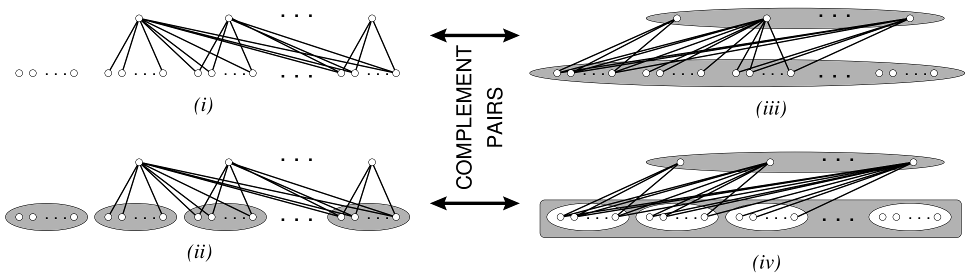

FIGURE 2. These are four possibilities (out of eight) from case A.3. The horizontal pairings indicate complementary pairs of graphs. The grey ovals indicate sets which induce a clique, while an empty oval within a grey oval represents an induced independent set within an otherwise fully connected group of vertices (i.e., a Turán graph). In each figure, the top vertices are $U = \{u_2, \dots, u_t\}$ and the bottom vertices are $V_1 \cup V_2 \cup \ldots \cup V_t$.

at least $S(n) = |\mathcal{S}_n|$ subgraphs, we would like to find a surjective map from $\mathcal{P}_n$ to $\mathcal{S}_n$.

In our proof we shall always assume that $n > 100$ and N is large (to be specified later) compared to n. Consider a graph $G_N \in \mathcal{P}_N$, where G_N is any one of the 6 graphs. The goal is to find at least $S(n)$ different n-subgraphs of G_N. From now on, by a *vector* $\mathbf{a}$ we mean a vector $\mathbf{a} = (a_i)_1^r = \langle a_1, \dots, a_r \rangle$ with positive integer coordinates whose sum is n. Let $\mathbf{a} = (a_i)_1^r$ be a decreasing vector, i.e., such that $a_1 \geq a_2 \geq \cdots \geq a_r$; the number of these vectors is $S(n)$. We shall map these vectors into different n-subgraphs of G_N. Given G_N and an r-vector $\mathbf{a} = \{a_1, \dots, a_r\}$, let $A(\mathbf{a})$ denote the graph spanned by the vertices $\{u_1, \dots, u_r\} \cup V_1' \cup \ldots \cup V_r'$ in G_N, where $V_i' \subset V_i$ and $|V_i'| = a_i - 1$ for every $1 \leq i \leq r$. Our case analysis consists of two steps: first we prove that in this way we generate at least $S(n) - O(n)$ different n-graphs, and then we find some additional graphs. Unfortunately, no global argument seems to be available, so we have to check the six cases separately.

(i) This case is shown in Figure 1 (i). It is clear that in this case $\mathcal{P}$ contains $\mathcal{T}$.

(ii) This case is shown in Figure 1 (ii). Let $\mathbf{a}$ be an r-vector. We claim that every vector $\mathbf{a}$ can be reconstructed from the graph $A = A(\mathbf{a})$. As the largest independent set of A contains r vertices, A determines r. Furthermore, there is a unique clique cover of A consisting of r cliques,

with order sequence $\mathbf{a}$, which determines uniquely $\mathbf{a}$.
To obtain one more n-subgraph of G_N in order to show that $|\mathcal{P}^n| > S(n)$, consider the graph spanned by $\{u_1\} \cup V_1' \cup V_2'$, where $V_1' \subset V_1, V_2' \subset V_2$ with $|V_1'| = 10$, $|V_2'| = n - 11$.

(iii) This case is shown in Figure 1 (iii). Now we claim that for $r \geq 2$ every vector $\mathbf{a} = (a_i)_1^r$ is determined by the graph $A = A(\mathbf{a})$. If A contains a maximal complete subgraph of order at least 3, then this order is r, the complete subgraph spanned by $\{u_1, \ldots, u_r\}$, and so $\mathbf{a}$ is determined. If $r = 2$, a vertex of maximal degree could be chosen as u_1 (unique up to isomorphism), and if there is another vertex of degree at least 2, then that is u_2. If there is no such vertex, then A is a star, and $\mathbf{a} = (n-1, 1)$ or (n).
In this way we already have $S(n) - 1$ different n-subgraphs of G_N; we shall find at least two more (in fact, $\lceil n/3 \rceil - 10$) graphs.
For $10 \leq i \leq n/3$ let $V_1' \subset V_1, V_2' \subset V_2, V_3' \subset V_3$, with $|V_1'| = i, |V_2'| = n - i - 12$, and $|V_3'| = 10$. Consider the graph H_i spanned by $\{u_1, u_2\} \cup V_1' \cup V_2' \cup V_3'$. These graphs H_i are different from any graph $A(\mathbf{a})$ for any $\mathbf{a}$ as they contain 10 isolated vertices (and $A(\mathbf{a})$ has none). Furthermore the graphs H_i are clearly different from each other.

(iv) This case is shown in Figure 1 (iv). Call a vector $\mathbf{a} = \langle a_i \rangle_1^r$ *good*, if $r \geq 3$, the second largest coordinate is at least 2 and the largest coordinate is at least 3. We claim that the graphs $A(\mathbf{a})$, where $\mathbf{a}$ is a good vector, are all different. To prove this, from a graph $A(\mathbf{a})$ we shall reconstruct the good vector $\mathbf{a}$ that defines it. First, observe that if $x \in V_1'$ and $y \in V_2'$ then the set $\{u_1, u_2, x, y\}$ spans a 4-cycle in A. This implies that A has a unique covering by two cliques, which determines r, and so the vector $\mathbf{a}$ is determined, proving the claim.
The number of good vectors is at least $S(n) - 3n$; hence, it is sufficient to find $3n$ more n-subgraphs of G_N.
For $10 \leq i \leq n/6+11 < j \leq n/3+12$, select sets $V_1' \subset V_1, V_2' \subset V_2, V_3' \subset V_3, V_4' \subset V_4$ with $|V_1'| = i, |V_2'| = j, |V_3'| = n - i - j - 13$ and $|V_4'| = 10$. Consider the graphs $H_{i,j}$ spanned by $\{u_1, u_2, u_3\} \cup V_1' \cup V_2' \cup V_3' \cup V_4'$. As for every i, j, the graph $H_{i,j}$ has a unique covering with two cliques, where the smallest clique of order 3 consists of $\{u_1, u_2, u_3\}$, and the largest clique contains vertices not joined to the smaller cliques, the graphs $H_{i,j}$ are different from $A(\mathbf{a})$. It is clear that the graphs $H_{i,j}$'s are different from each other, giving at least $n^2/36$ new n-subgraphs of G_N.

(i) (ii)

FIGURE 3. The two Cases of A.3

A.3. Here we have two possibilities, case (i) in Figure 3 when both $U = \{u_2, \ldots, u_t\}$ and $V_2 \cup \ldots \cup V_t$ span independent sets, and case (ii) in Figure 3 when $U = \{u_2, \ldots, u_t\}$ spans a complete graph, and $V_2 \cup \ldots \cup V_t$ spans an independent set.

Our strategy is similar to the ones we applied in cases of A.1., but there are significant differences in the details. One is that u_1 is not included (in order to use the "complementary pairs"), and V_1 is omitted for technical reasons. Another difference is that we shall use a different set of vectors. Here we consider vectors $\mathbf{a} = \langle a_2, \ldots, a_r \rangle$ with integer coordinates, such that $r \geq 3$, $a_2 + \ldots + a_r = n$, $a_2 \geq 2$ and $a_r \geq 3$. The number of these vectors is larger than $S(n)$ (for $n > 100$), as here the coordinates are ordered.
Let A denote the graph spanned by $\{u_2, \ldots, u_r\} \cup V_2' \cup \ldots \cup V_r'$ where $V_i' \subset V_i$ for every $2 \leq i \leq r$.
Again, we claim that a graph $A = A(\mathbf{a})$ determines the vector $\mathbf{a}$. We shall check this in both cases.

Case (i) of Figure 3. As $|V_r'| \geq 2$, for every i, $2 \leq i \leq r$, the vertex u_i has degree at least 2 in the graph A. Every vertex $v \in V_3' \cup \ldots \cup V_r'$ is joined to u_2 and u_3, hence V_2' is the non-empty set of vertices of degree 1. The vertices in V_2' have one common neighbor, u_2. Clearly, u_2 is joined to all vertices, but those in $\{u_3, \ldots, u_r\}$, so we can find this set. Knowing the set $\{u_3, \ldots, u_r\}$, the labelling of the vertices can be determined up to isomorphism, since $d(u_i) < d(u_j)$ implies $i < j$, and $d(u_i) = d(u_j)$ implies that $\Gamma(u_i) = \Gamma(u_j)$. We can obtain the partition of the rest the vertices from the fact that the set of vertices of degree $i - 1$ is V_i'. This proves the claim.

Case (ii) of Figure 3. As $r \geq 3$ and $|V_r'| \geq 2$, the non-empty set of vertices of degree 1 is V_2'. The vertices in V_2' have one common neighbor, u_2. In the graph $A - \{u_2\} - V_2'$, one of the vertices which are joined to every other is u_3, and as these vertices are interchangeable, it does not matter which one is chosen as u_3. The set of vertices joined only to u_3 is V_3' (note that if we had more than one choice for u_3 then this

set would be empty). Proceeding in this way, we can find all u_i and V_i' (up to isomorphism). Hence the vector $\mathbf{a}$ is indeed determined by the graph A.
This proves Theorem 1 (ii) provided case A holds for $\mathcal{P}$.

Case B. To complete our proof, by Section 5 and Theorem 19 of [6], we may assume that $\mathcal{P}$ has the following property: there is an integer k (depending only on $\mathcal{P}$) such that for every integer $N > 50k^2$, there is a graph $G_N \in \mathcal{P}^N$ whose edge set is the symmetric difference of a Hamiltonian path P_N and a homogeneous t-part graph H^t with $t \leq k$, and each class of H^t is of order of at least $50k$.

In what follows, when we talk about a homogeneous t-part graph, we always assume that t is minimal, i.e., the graph is not a homogeneous $(t-1)$-part graph.

The strategy that we shall use is the same as in Case A. Let $n \geq 100k^2$ be an integer. We shall show that for N large enough, the graph G_N has at least $S(n)$ different n-subgraphs.

First, observe that if $t = 1$ then G_N is either a path P_N or its complement. In the first case $\mathcal{P}^n$ contains all the path forests of order of n, in the second case every graph of order of n which is the complement of a path forest. In both cases, $|\mathcal{P}_n| \geq S(n)$.

Hence, we may assume that $t \geq 2$. Let P_N be a Hamiltonian path and H_N^t a homogeneous t-part graph with vertex classes $V_1, \ldots, V_t$ such that $E(G_N) = E(P_N) \Delta E(H_N^t)$.

Let G_n be a subgraph of G_N. We say that a path forest F_n and a homogeneous s-part graph D_n (each of order n) *cover* G_n if $E(G_n) = E(F_n) \Delta E(D_n)$. For every n-subgraph G_n of G_N there is a pair (F_n, D_n) covering it, as $V(G_n)$ spans a path forest in P_N and a homogeneous s-part graph in H^t (with some $s \leq t$).

Lemma 3. *Let G_N, P_N, H^t and $V_1, \ldots, V_t$ be as above. Let G_n be a subgraph of G_N such that, for each i, $1 \leq i \leq k$, $|V(G_n) \cap V_i|$ is either 0 or at least 12. Then there is exactly one choice for a pair (F_n, D_n) covering G_n, with D_n satisfying $U_1, \ldots, U_s$ are the maximal homogeneous classes of D_n then $|U_i| \geq 12$ for every i.*

Proof. Let (F_n, D_n) be a cover of G_n so that $E(G_n) = E(F_n) \Delta E(D_n)$, and let $U_1, \ldots, U_s$ be the maximal homogeneous classes of D_n.
If $u, v \in V(G_n) \cap U_i$ for some $1 \leq i \leq s$ then $|\Gamma_{G_n}(u) \Delta \Gamma_{G_n}(v)| \leq 3 + 3$.
If $u \in V(G_n) \cap U_i$ and $v \in V(G_n) \cap U_j$ for some $1 \leq i < j \leq s$ then $|\Gamma_{G_n}(u) \Delta \Gamma_{G_n}(v)| \geq 12 - 2 - 3 = 7$.

This implies the uniqueness of D_n and for a given G_n and D_n, the path forest F_n is unique as well. □

Let G_n be an n-subgraph with a unique cover (F_n, D_n). Let S^n be the set of decreasing vectors with at least $12k$ coordinates. Thus $\mathbf{a} = \langle a_i \rangle_1^r \in S^n$ if $r \geq 12k$ and $a_1 \geq a_2 \geq \ldots \geq a_r \geq 1$ (and $a_1 + \ldots + a_r = n$). Clearly, $|S^n| > S(n) - n^{12k}$. We claim that for each $\mathbf{a} \in S^n$ there is an n-subgraph G_n of G_N such that $|V(G_n) \cap V_i| \geq 12$ for every i, $1 \leq i \leq t \leq k$, and the component order sequence of the path forest spanned by $V(G_n)$ in P_N is the vector $\mathbf{a}$. This is an immediate consequence of Lemma 9 of [6] and is also easily shown directly (the starting point of each of the $\geq 12k$ components can be chosen arbitrarily). Consequently, by Lemma 3, the sequences in S^n give us $|S^n| > S(n) - n^{12k}$ non-isomorphic induced n-subgraphs of G_N. To complete the proof of Theorem 1 in case B, we need to find n^{12k} more n-subgraphs.

We shall construct many n-subgraphs G_n of G_N such that $|V(G_n) \cap V_i| \geq 12$ for every $1 \leq i \leq t$, and in the unique (F_n, D_n)-covering of G_n the order sequence of the components of F_n is the vector $\mathbf{a}$, given below. First observe that if there is a class V_i of G_N which contains a $2n$-subpath of P_N then $\mathcal{T}^n \subset \mathcal{P}^n$ or $\overline{\mathcal{T}^n} \subset \mathcal{P}^n$. Hence we may assume that there is a class, say V_1, where P_N enters at least $N/(2nk)$ times.

Now we can start to build a graph G_n. To this end, set $r = \lfloor \frac{\sqrt{n}}{2} \rfloor$ and $\mathbf{a} = \langle 1, 20, 21, \ldots, r+17, n-(r+17)-(r+16)-\ldots-20-1 \rangle$ (thus $\mathbf{a}$ has r coordinates). The component orders $20, 21, \ldots, 19+12k$ are used to make sure that $|V(G_n) \cap V_i| \geq 12$ for every $1 \leq i \leq t$, just as earlier. This will guarantee the uniqueness of the (F_n, D_n)-covering of G_n. Choose a vertex from V_1 as a component of order 1 of F_n: this makes the class V_1 to be distinguishable from the others. The following observation helps us to construct many different n-subgraphs.

Fix an even integer i where $20+12k \leq i \leq r/2$. We claim that there are at least 2 different ways to choose subpaths of orders i and $2i-1$. Let us start a subpath of P_N from V_1. If the i^{th} vertex of a subpath is in V_1 then there are (at least) two different ways to construct the subpath of order i, because if we started the path outside of V_1, then we obtain an i-path with different type. Otherwise, we may assume that every i-subpath of P_N has exactly one endvertex in V_1. In that case the $(2i-1)$st vertex must be in V_1, and then there is a $(2i-1)$-subpath with both ends in V_1. As there is a $(2i-1)$-subpath with (at least) one endvertex outside V_1, there are two different ways to choose the subpath of order $2i-1$.

As we have (at least) two choices for the pair of paths (P_i, P_{2i-1}), for every even i between $20+12k$ and $r/2$, the number of ways of choosing

G_n is at least $2^{r/4-(20+6k)}$ which is much larger than n^{12k} for n large enough.

This proves that either $\mathcal{T}^n \subset \mathcal{P}^n$ or $\overline{\mathcal{T}^n} \subset \mathcal{P}^n$ or $S(n) < |\mathcal{P}^n|$.

If for infinitely many integers n we have $\mathcal{T}^n \subset \mathcal{P}^n$ then, as $\mathcal{P}$ is a hereditary property, this is true for every n. Furthermore, if for infinitely many integers n we have $S(n) < |\mathcal{P}_n|$, then this holds for every (large) n. This implies that if $S(n) = |\mathcal{P}_n|$ infinitely often, then for every n large enough $S(n) = |\mathcal{P}_n|$. Otherwise, we have $S(n) < |\mathcal{P}_n|$ for every large n. This completes the proof of part B. □

4. Remarks

4.1. **Lower order behavior.** In the first possibility given by Theorem 1, we have that $|\mathcal{P}_n| = cn^k + O(n^{k-1})$ for some nonnegative integer k and positive c. With a more careful counting, it is possible to describe the lower order behavior more precisely. A function f defined on the set of natural numbers is *a periodic polynomial of degree at most* d provided that there is a positive integer t and polynomials $p_0, p_1, \dots, p_{t-1}$, each of degree at most d, such that for each $j \in \{0, \dots, t-1\}$, $f(n) = p_j(n)$ for all $n \equiv j \bmod t$. Then the first part of Theorem 1 can be refined to say that $|\mathcal{P}_n| = cn^k + f(n) + g(n)$ where $f(n)$ is a periodic polynomial of degree at most $k-1$ and $g(n) \neq 0$ for only finitely many n.

The proof of this result is rather involved. One needs a refinement of Theorem C which allows one to write $\mathcal{P}$ as a *disjoint union* of a finite number of structured classes of graphs. For each of these classes, each graph in the set is describable by a multiset on a particular ground set associated to the class. A multiset is admissible (for this class) if it corresponds to a graph in the class. The properties of each class ensure that (1) the set of admissible multisets is *order convex*, i.e. if a, b, c are multisets on the ground set and $a \le b \le c$ (where $\le$ is the product order) and a, c are admissible then so is b and (2) there is a group action on the ground set, such that two admissible multisets correspond to isomorphic graphs if and only if they are equivalent under the group action. It follows that counting the number of distinct isomorphism types of graphs in the class is equivalent to counting the number of equivalence classes of multisets under a given group action, in a given order convex subset of multisets. Using standard counting arguments (generating functions, Pólya counting, and inclusion-exclusion) one can show that the number of such equivalence classes of multisets is given by a periodic polynomial for all sufficiently large n. The desired result then follows by summing over all of the structured classes in the partition of $\mathcal{P}$.

4.2. **Oscillation.** There are hereditary properties of graphs whose labelled speeds oscillate: such oscillation was studied in [4] and [5]. This phenomenon can arise in other combinatorial structures as well. A fundamental example of a hereditary graph property whose unlabelled speed oscillates is the following.
Fix a positive integer $t > 3$. Let us build a *monotone* property $\mathcal{P}$ simultaneously with a sequence $n_1 = 1 < n_2 < \dots$ as follows. Having constructed n_{i-1}, $i \geq 1$, if n is large enough, there is a graph of order n, girth at least $n_{i-1} + 1$, and size $e = \lfloor n^{1+1/(2n_{i-1})} \rfloor$ [7, pp 233]. Furthermore, this n can be chosen to satisfy $2^e/(n!) > n^{tn}$; choose such an n for n_i. For $n = n_i$, let $\mathcal{P}_n$ consist of all graphs of girth at least $n_{i-1} + 1$. Our choice of n_i implies that $|\mathcal{P}_{n_i}| \geq 2^e/(n!) \geq n_i^{tn_i}$. For $n_i < n < n_{i+1}$, let $\mathcal{P}_n$ consist of all subgraphs of $\mathcal{P}_{n_{i+1}}$. Note that $\mathcal{P} = \cup_{n=1}^{\infty} \mathcal{P}_n$ is a monotone property. Also, $\mathcal{P}_{n_i+1}$ consists of all forests and the cycle C_{n_i+1}, so $|\mathcal{P}_{n_i+1}| < 3^{n_i}$ if n_i is large enough. (Otter [10] proved that the number of unlabelled trees of order n is approximately $0.4399237(2.95576)^n/n^{3/2}$.) Consequently $|\mathcal{P}_n|$ is less than 3^n infinitely often, and greater than n^{tn} infinitely often.

4.3. **High Range.** Trivially, the maximum labelled speed of a graph property is $2^{\binom{n}{2}}$; accordingly, we say that the labelled speed of a graph property is in the *high range* if it is at least 2^{cn^2} for some $c > 0$. Defining c_n by $|\mathcal{P}^n| = 2^{c_n \binom{n}{2}}$, it is shown in [1] and [8], that $c = \lim_{n\to\infty} c_n$ exists. This result says very little about a property with $|\mathcal{P}^n| = 2^{o(n^2)}$. In [9] it is proved that the set of limit points $c = \lim_{n\to\infty} c_n$ is $\{0, 1/2\} \cup \{(1 - 1/t)/2 : t \in \mathbb{N}\}$. Since $|P_n| = 2^{(c_n + o(1))\binom{n}{2}}$, we have the same results for the unlabelled speed.

Acknowledgements.

This work was done in part while the latter three authors were visiting Microsoft Research in Redmond Washington.

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