Miskolc Mathematical Notes
Vol. 21 (2020), No. 1, pp. 451–461
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2020.3057

A SHARP IMPROVEMENT OF FIXED POINT RESULTS FOR QUASI-CONTRACTIONS IN b-METRIC SPACES

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Received 30 September, 2019

Abstract. In this paper, a general fixed point theorem for quasi-contractions in b-metric spaces, which is a sharp improvement of Amini-Harandi's result, Mitrovic and Hussain's result, and is a generalization of many b-metric fixed point theorems in the literature, is proved. The technique overcomes some limits in b-metric fixed point theory compared to metric fixed point theory. The obtained results are also supported by examples.

2010 *Mathematics Subject Classification:* 47H10; 54H25; 54D99; 54E99

Keywords: quasi-contraction, b-metric, fixed point

1. Introduction and preliminaries

There have been many types of contraction conditions in metric spaces and generalized metric spaces [12], [19]. One of the most interesting types is the quasi-contraction [7]. Quasi-contractions have been studied and many nice results have been proved. In [4], Bessenyei studied nonlinear quasicontractions in complete metric spaces. In [5], Bessenyei studied weak φ-quasi-contractions and presented an elementary proof for known fixed point results in the literature. In [2], Amini-Harandi proved a fixed point theorem for quasi-contraction maps in b-metric spaces. Recently, Mitrović and Hussain [17] established fixed point results for weak φ-quasi-contractions involving comparison function in b-metric spaces.

Recall that the b-metric space is a generalization of a metric space. One of the main differences between a b-metric space and a metric space is that the modulus of concavity $\kappa \geq 1$ in the generalized triangle inequality, see Definition 1. (3) below. It implies that a b-metric is not necessarily continuous, see [3, Example 3.10] for example. It also implies that the contraction constants in certain b-metric fixed point theorems are in $[0; \frac{1}{\kappa})$ instead of $[0;1)$, see [9, Remark 2.7] and [17, Corollary 3.5] for example. So, in many b-metric fixed point theorems, certain additional assumptions have been added to overcome the above difference such as the Fatou property in [2],

The work is supported by the project SPD2018.01.27 of Dong Thap University, Vietnam.

the bounded orbit in [5] and [17]. For some recent improvements in b-metric fixed point theory, the reader may refer to [9], [18], [10], [14], [16], [15], [21].

In this paper, we are interested to improve the main results of [2] and [17]. By using a technical calculation that $\lambda^n \in [0;\frac{1}{\kappa})$ for all $\lambda \in [0;1)$ and all n large enough, we prove a fixed point theorem for quasi-contractions in b-metric spaces which is a sharp improvement of the main results in [2] and [17], and is a generalization of many b-metric fixed point theorems in the literature. We also construct examples to support the obtained results.

Now we recall notions and results which will be useful in the next.

Definition 1 ([8], page 263)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D : X \times X \to [0;\infty)$ be a function such that for all $x,y,z \in X$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,z) \leq \kappa[D(x,y) + D(y,z)]$.

Then

(1) D is called a *b-metric* on X and (X,D,κ) is called a *b-metric space*. Without loss of generality we may assume that κ is the smallest possible value, and it is called the *modulus of concavity* of the given b-metric.
(2) The sequence $\{x_n\}$ is called *convergent* to x if $\lim\limits_{n\to\infty} D(x_n,x) = 0$, written by $\lim\limits_{n\to\infty} x_n = x$.
(3) The sequence $\{x_n\}$ is called *Cauchy* if $\lim\limits_{n,m\to\infty} D(x_n,x_m) = 0$.
(4) The b-metric space (X,D,κ) is called *compete* if every Cauchy sequence is a convergent sequence.

Definition 2 ([12], Definition 12.7)**.** Let X be a nonempty set, $\kappa \geq 1$ and $D : X \times X \to [0,\infty)$ be a function such that for all $x,y,z \in X$,

(1) $D(x,y) = 0$ if and only if $x = y$.
(2) $D(x,y) = D(y,x)$.
(3) $D(x,z) \leq D(x,y) + \kappa D(y,z)$.

Then D is called a *strong b-metric* on X and (X,D,κ) is called a *strong b-metric space*.

Note that every strong b-metric is continuous, and the convergence and completeness in strong b-metric spaces are defined as in b-metric spaces.

Definition 3 ([2], Definition 2.4)**.** A b-metric space (X,D,κ) is called to have *Fatou property* if for all $x,y \in X$ and $\lim\limits_{n\to\infty} x_n = x$ we have

$$D(x,y) \leq \liminf_{n\to\infty} D(x_n,y).$$

Theorem 1 ([2], Theorem 2.8)**.** *Let (X,D,κ) be a complete b-metric space having Fatou property and $f : X \to X$ be a map such that for some $\lambda \in [0;\frac{1}{\kappa})$ and all $x,y \in X$,*

$$\begin{aligned} &D(f(x),f(y)) \\ &\leq \lambda \max\{D(x,y),D(x,f(x)),D(y,f(y)),D(x,f(y)),D(y,f(x))\}. \end{aligned} \tag{1.1}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Theorem 2 ([5], Theorem on page 289)**.** *Assume that*

(1) *(X,D,κ) is a complete metric space and $f : X \to X$ is a map such that for all $x,y \in X$,*
$$D(f(x),f(y)) \leq \varphi(\mathrm{diam}O(x,y)) \tag{1.2}$$
where $\varphi : [0;\infty) \to [0;\infty)$ is an increasing, upper semicontinuous function, $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$, and
$$O(x,y) = \{f^n(x), f^n(y) : n \in \mathbb{N} \cup \{0\}\}.$$

(2) *Each orbit of f is bounded.*

Then f has unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Theorem 3 ([17], Theorem 3.3)**.** *Assume that*

(1) *(X,D,κ) is a complete b-metric space and $f : X \to X$ is a map such that for some $\lambda \geq 0$ and all $x,y \in X$,*
$$D(f(x),f(y)) \leq \lambda \mathrm{diam}O(x,y). \tag{1.3}$$

(2) *$\lambda \in [0;1)$ and each orbit of f is bounded.*

Then we have

(1) *There exists $x^* \in X$ such that $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.*

(2) *f has unique fixed point x^* if one of the following holds*
  (a) *f is continuous at x^*.*
  (b) *D is continuous.*

Theorem 4 ([17], Corollary 3.5)**.** *Let (X,D,κ) be a complete b-metric space and $f : X \to X$ be a map such that for some $\lambda \in [0;\frac{1}{\kappa})$ and all $x,y \in X$,*

$$\begin{aligned} &D(f(x),f(y)) \\ &\leq \lambda \max\left\{D(x,y),D(x,f(x)),D(y,f(y)),\frac{D(x,f(y))}{2\kappa},\frac{D(y,f(x))}{2\kappa}\right\}. \end{aligned} \tag{1.4}$$

Then f has a unique fixed point.

Theorem 5 ([17], Corollary 3.6)**.** *Let (X,D,κ) be a complete strong b-metric space and $f : X \to X$ be a map such that for some $\lambda \in [0;1)$ and all $x,y \in X$,*

$$D(f(x),f(y)) \leq \lambda \max\{D(x,y),D(x,f(x)),D(y,f(y)),D(x,f(y)),D(y,f(x))\}.$$

Then f has a unique fixed point.

2. The main result

The main result is Theorem 6 below. Note that

(1) Theorem 6 is an improvement of Theorem 1 in the sense that the assumption of Fatou property is omitted, and the contraction constant $\lambda \in [0;1)$.
(2) Theorem 6 is an improvement of Theorem 4 in the sense that the right side of (2.1) is greater than that of (1.4), and the contraction constant $\lambda \in [0;1)$.
(3) Theorem 6 is an improvement of Theorem 5 in the sense that the strong b-metric is replaced by a continuous b-metric.
(4) Theorem 6 is a generalization of many b-metric fixed point theorems in the literature such as [1, Theorem 2.1], [1, Theorem 3.1], [11, Corollary 3.12], [20, Corollary 2.6].
(5) Recently, an analogue of Reich contraction in b-metric spaces was proved [13, Theorem 3.1]. In the proof on [13, page 85], the author claimed $\lim\limits_{n\to\infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*)$ provided that $\lim\limits_{n\to\infty} x_n = x^*$. Unfortunately, this claim does not hold since the b-metric d is not necessarily continuous . In fact, the conclusion in [13, Theorem 3.1] does not hold which was proved in [9, Remark 2.7].

Theorem 6. *Assume that*

(1) *(X, D, κ) is a complete b-metric space and $f : X \to X$ is a map such that for some $\lambda \geq 0$ and all $x, y \in X$,*

$$\begin{aligned} &D(f(x), f(y)) \\ \leq &\lambda \max\big\{D(x,y), D(x, f(x)), D(y, f(y)), D(x, f(y)), D(y, f(x))\big\}. \end{aligned} \tag{2.1}$$

(2) *One of the following holds*
   (a) *D is continuous and $\lambda \in [0;1)$.*
   (b) *$\lambda \in [0; \frac{1}{\kappa})$.*

Then f has a unique fixed point x^* and $\lim\limits_{n\to\infty} f^n(x) = x^*$ for all $x \in X$.

Proof. For $m \leq i \leq n-1$ and $m \leq j \leq n$, from (2.1) we find that

$$\begin{aligned} &D(f^i(x), f^j(x)) \\ =&D(ff^{i-1}(x), ff^{j-1}(x)) \\ \leq&\lambda \max\big\{D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), ff^{i-1}(x)), D(f^{j-1}(x), ff^{j-1}(x)), \\ &D(f^{i-1}(x), ff^{j-1}(x)), D(f^{j-1}(x), ff^{i-1}(x))\big\} \\ =&\lambda \max\big\{D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), f^i(x)), D(f^{j-1}(x), f^j(x)), \\ &D(f^{i-1}(x), f^j(x)), D(f^{j-1}(x), f^i(x))\big\}. \end{aligned} \tag{2.2}$$

From (2.2), we get

$$\max\{D(f^i(x), f^j(x)) : m \leq i, j \leq n\}$$

$$\begin{aligned}&\leq \lambda \max\{D(f^i(x),f^j(x)) : m-1 \leq i,j \leq n\}\\ &\leq \dots\\ &\leq \lambda^m \max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq n\}.\end{aligned} \tag{2.3}$$

It implies that

$$\max\{D(f^i(x),f^j(x)) : 1 \leq i,j \leq n\} \leq \lambda \max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq n\}.$$

Since $0 \leq \lambda < 1$, we see that

$$\max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq n\} = \max\{D(x,f^i(x)) : 1 \leq i \leq n\}.$$

So there exists $1 \leq k_n(x) \leq n$ such that

$$D(x,f^{k_n(x)}(x)) = \max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq n\}.$$

Since $\lambda \in [0;1)$, there exists n_0 such that $\lambda^{n_0} < \frac{1}{\kappa}$. If $k_n(x) \leq n_0$, then

$$D(x,f^{k_n(x)}(x)) \leq \max\{D(x,f^i(x)) : 0 \leq i \leq n_0\}. \tag{2.4}$$

If $k_n(x) > n_0$, then using (2.3) we find that

$$\begin{aligned}&D(x,f^{k_n(x)}(x))\\ \leq&\kappa[D(x,f^{n_0}(x)) + D(f^{n_0}(x),f^{k_n(x)}(x))]\\ \leq&\kappa[D(x,f^{n_0}(x)) + \lambda^{n_0}\max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq k_n(x)\}]\\ \leq&\kappa[D(x,f^{n_0}(x)) + \lambda^{n_0}\max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq n\}]\\ =&\kappa[D(x,f^{n_0}(x)) + \lambda^{n_0}D(x,f^{k_n(x)}(x))].\end{aligned}$$

Note that $\lambda^{n_0} < \frac{1}{\kappa}$. So we get

$$D(x,f^{k_n(x)}(x)) \leq \frac{\kappa}{1-\kappa\lambda^{n_0}} D(x,f^{n_0}(x)). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\max\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq n\} \leq \frac{\kappa}{1-\kappa\lambda^{n_0}} \max\{D(x,f^i(x)) : 0 \leq i \leq n_0\}$$

for all n. So

$$\sup\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq \infty\} \leq M < \infty$$

where

$$M = \frac{\kappa}{1-\kappa\lambda^{n_0}} \max\{D(x,f^i(x)) : 0 \leq i \leq n_0\}.$$

By (2.3) we have

$$\begin{aligned}\sup\{D(f^i(x),f^j(x)) : m \leq i,j \leq \infty\} &\leq \lambda \sup\{D(f^i(x),f^j(x)) : m-1 \leq i,j \leq \infty\}\\ &\leq \dots \leq \lambda^m \sup\{D(f^i(x),f^j(x)) : 0 \leq i,j \leq \infty\} \leq \lambda^m M.\end{aligned}$$

Then

$$\lim_{m\to\infty} \sup\{D(f^i(x),f^j(x)) : m \leq i,j \leq \infty\} = 0.$$

Therefore the sequence $\{f^n(x)\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} f^n(x) = x^*. \tag{2.6}$$

By (2.1) we get

$$\begin{aligned} &D(f^{n+1}(x), f(x^*)) \\ =&D(ff^n(x), f(x^*)) \\ \le&\lambda \max\big\{D(f^n(x),x^*), D(f^n(x), f^{n+1}(x)), D(x^*, f(x^*)), \\ &D(f^n(x), f(x^*)), D(x^*, f^{n+1}(x))\big\}. \end{aligned} \tag{2.7}$$

We consider two following cases.

Case 1. *D is continuous and $\lambda \in [0;1)$.*

Using (2.7) and the continuity of D, we obtain

$$\begin{aligned} D(x^*, f(x^*)) \le& \lambda \max\big\{0, 0, D(x^*, f(x^*)), D(x^*, f(x^*)), 0\big\} \\ =&\lambda D(x^*, f(x^*)). \end{aligned}$$

Since $\lambda \in [0;1)$, we have $D(x^*, f(x^*)) = 0$. So x^* is a fixed point of f.

Case 2. $\lambda \in [0; \frac{1}{\kappa})$.

It follows from (2.7) that

$$\begin{aligned} &\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \\ \le&\lambda \max\big\{0, 0, D(x^*, f(x^*)), \liminf_{n\to\infty} D(f^n(x), f(x^*)), 0\big\} \\ =&\lambda \max\big\{D(x^*, f(x^*)), \liminf_{n\to\infty} D(f^n(x), f(x^*))\big\}. \end{aligned} \tag{2.8}$$

By (2.8), we consider two following subcases.

Subcase 2.1. $\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \le \lambda D(x^*, f(x^*))$

We find that

$$D(x^*, f(x^*)) \le \kappa[D(x^*, f^{n+1}(x)) + D(f^{n+1}(x), f(x^*))]. \tag{2.9}$$

From (2.6) and (2.9) we deduce that

$$\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \ge \frac{1}{\kappa} D(x^*, f(x^*)). \tag{2.10}$$

On the contrary, suppose that $x^* \ne f(x^*)$. Note that $0 \le \lambda < \frac{1}{\kappa}$. Then

$$\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \le \lambda D(x^*, f(x^*)) < \frac{1}{\kappa} D(x^*, f(x^*)).$$

This is a contradiction with (2.10). Therefore $x^* = f(x^*)$.

Subcase 2.2. $\liminf_{n\to\infty} D(f^{n+1}(x), f(x^*)) \le \lambda \liminf_{n\to\infty} D(f^n(x), f(x^*))$.

From $\liminf_{n\to\infty} D(f^n(x), f(x^*)) = \liminf_{n\to\infty} D(f^{n+1}(x), f(x^*))$ and $0 \leq \lambda < \frac{1}{\kappa}$ we have $\liminf_{n\to\infty} D(f^n(x), f(x^*)) = 0$. So there exists a subsequence $\{f^{k_n}(x)\}$ of $\{f^n(x)\}$ such that

$$\lim_{n\to\infty} f^{k_n}(x) = f(x^*). \tag{2.11}$$

Note that

$$D(x^*, f(x^*)) \leq \kappa[D(x^*, f^{k_n}(x)) + D(f^{k_n}(x), f(x^*))]. \tag{2.12}$$

Letting $n \to \infty$ in (2.12) and using (2.11), (2.6) we obtain $D(x^*, f(x^*)) = 0$. Then $x^* = f(x^*)$.

By the conclusions of Case 1 and Case 2, we find that f has a fixed point x^* and by (2.6), $\lim_{n\to\infty} f^n(x) = x^*$.

Finally, we prove the uniqueness of the fixed point of f. Indeed, let x^*, y^* be two fixed points of f. From (2.1) we have

$$\begin{aligned}
&D(x^*, y^*)\\
=&D(f(x^*), f(y^*))\\
\leq&\lambda \max\left\{D(x^*, y^*), D(x^*, f(x^*)), D(y^*, f(y^*)), D(x^*, f(y^*)), D(y^*, f(x^*))\right\}\\
=&\lambda D(x^*, y^*).
\end{aligned}$$

Since $\lambda \in [0;1)$, we obtain $D(x^*, y^*) = 0$, that is, $x^* = y^*$. Then the fixed point of f is unique. □

Next we present some examples to illustrate the obtained result. The following example shows there exists the map $f : X \to X$ so that Theorem 6 is applicable but Theorem 1, Theorem 4 and Theorem 5 are not.

Example 1. Let $X = \mathbb{R}$, and $D(x,y) = |x-y|^2$ for all $x, y \in X$, and the map $f : X \to X$ be defined by $f(x) = \frac{3}{4}x$ for all $x \in X$. Then

(1) (X, D, κ) is a complete b-metric space with the modulus of concavity $\kappa = 2$, D is continuous, and the condition (2.1) holds for all $\lambda \in [\frac{3}{4}, 1)$. Then Theorem 6 is applicable to f.

(2) The conditions (1.1) and (1.4) do not hold for all $\lambda \in [0, \frac{1}{\kappa})$. Then Theorem 1 and Theorem 4 are not applicable to f.

(3) D is not a strong b-metric. Then Theorem 5 is not applicable to f.

Proof. (1). It is easy to check that (X, D, κ) is a complete b-metric space with the modulus of concavity $\kappa = 2$, D is continuous, and the condition (2.1) holds for all $\lambda \in [\frac{3}{4}, 1)$. Then Theorem 6 is applicable to f.

(2). For $x = 0, y = 1$ and $\lambda \in [0, \frac{1}{\kappa}) = [0, \frac{1}{2})$, we find that

$$\begin{aligned}
&D(f(0), f(1))\\
=&\frac{9}{16}
\end{aligned}$$

$$\geq \lambda$$
$$= \lambda \max \{D(0,1), D(0,f(0)), D(1,f(1)), D(0,f(1)), D(1,f(0))\}.$$

This proves that conditions (1.1) and (1.4) do not hold for all $\lambda \in [0, \frac{1}{\kappa})$. Then Theorem 1 and Theorem 4 are not applicable to f.

(3). On the contrary, suppose that D is a strong b-metric. Then there exists $K \geq 1$ such that for all $x, y, z \in X$,

$$|x-y|^2 \leq |x-z|^2 + K|z-y|^2. \tag{2.13}$$

For $n \in \mathbb{N}$, and $x_0 = \frac{1}{n}$, $y_0 = 1 + \frac{1}{n}$, $z_0 = 1$ we have

$$|x_0 - y_0|^2 = 1$$
$$|x_0 - z_0|^2 + K|z_0 - y_0|^2 = (\frac{1}{n} - 1)^2 + \frac{K}{n^2} = 1 + \frac{1 - 2n + K}{n^2}.$$

So for $n > K$ we have

$$|x_0 - z_0|^2 + K|z_0 - y_0|^2 < 1 = |x_0 - y_0|^2.$$

It is a contradiction to (2.13). Then D is not a strong b-metric, and Theorem 5 is not applicable to f. □

The following example shows that the continuity of D in Theorem 6. (2a) and the condition $\lambda \in [0; \frac{1}{\kappa})$ in Theorem 6. (2b) are essential.

Example 2. Let $X = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$, and

$$D(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0,1\} \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n = 1, 2, \ldots\} \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and let $f : X \to X$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{10n} & \text{if } x = \frac{1}{n}, n = 1, 2, \ldots \end{cases}$$

Then

(1) (X, D, κ) is a complete b-metric space with the modulus of concavity $\kappa = 4$.
(2) There exist $\lambda \geq 0$ such that the contraction condition (2.1) holds for all $x, y \in X$.
(3) D is not continuous and $\lambda \in [\frac{1}{\kappa}; 1)$.
(4) f is fixed point free.

Proof. By [9, Example 2.6], (X, D, κ) is a complete metric-type space with the modulus of concavity $\kappa = 4$. Then (X, D, κ) is also a complete b-metric on X with the modulus of concavity $\kappa = 4$. The remaining conclusions were proved in [9, Example 2.6] and [9, Remark 2.7]. □

The following example shows that the assumption of bounded orbit in Theorem 2 and Theorem 3 is essential. Moreover, for the case of unbounded orbit, the value $\operatorname{diam} O(x,y)$ cannot be replaced by $\max\{d(x,f(y)),d(y,f(x))\}$. However, the value $\operatorname{diam} O(x,y)$ can be replaced by $d(x,y)$ in the class of complete regular semimetric spaces, which is a generalization of the class of complete b-metric spaces, see [6, Theorem 1].

Example 3. Let $X = \{1,2,3,\ldots\}$, $d(x,y) = |x-y|$ for all $x,y \in X$, $f(x) = x+2$ for all $x \in X$ and $\varphi : [0;\infty) \to [0;\infty)$ be defined by

$$\varphi(t) = \begin{cases} \frac{2}{3}t & \text{if } t \in [0;3) \\ t-1 & \text{if } t \geq 3. \end{cases}$$

Then we have

(1) (X,d) is a complete metric space, and φ is an increasing, upper semicontinuous function, $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. In particular, (X,d) is also a complete b-metric space.
(2) Every orbit of f is unbounded. So $\operatorname{diam} O(x,y) = \infty$ and then (1.3) holds for all $x,y \in X$ and all $\lambda \in (0;1)$.
(3) $d(f(x),f(y)) \leq \varphi(\max\{d(x,f(y)),d(y,f(x))\})$ for all $x,y \in X$. Then (1.2) holds for all $x,y \in X$.
(4) f is fixed point free.

Proof. (1). It is clear that (X,d) is a complete metric space, and φ is an increasing, upper semicontinuous function, $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

(2). For all $x \in X$ we have $O(x) = \{x, x+2, x+4, \ldots\}$ which is an unbounded orbit of f.

(3). Let $x,y \in X$. We may assume that $x < y$. The we have

$$d(f(x),f(y)) = |x-y| = y-x$$

and

$$d(x,f(y)) = y-x+2 \geq 3.$$

Then we have

$$\varphi(\max\{d(x,f(y)),d(y,f(x))\}) = \varphi(d(x,f(y))) = y-x+1 > d(f(x),f(y)).$$

This proves that (1.2) holds for all $x,y \in X$.

(4). Since $f(x) = x+2$ for all $x \in X$, f is fixed point free. □

ACKNOWLEDGEMENTS

The author sincerely thanks the anonymous reviewers for their helpful comments during the preparation of the paper.

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