COCHARACTERS FOR THE WEAK POLYNOMIAL IDENTITIES OF THE LIE ALGEBRA OF 3×3 SKEW-SYMMETRIC MATRICES

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ABSTRACT. Let $so_3(K)$ be the Lie algebra of 3×3 skew-symmetric matrices over a field K of characteristic 0. The ideal $I(M_3(K), so_3(K))$ of the weak polynomial identities of the pair $(M_3(K), so_3(K))$ consists of the elements $f(x_1,\dots,x_n)$ of the free associative algebra $K\langle X\rangle$ with the property that $f(a_1,\dots,a_n)=0$ in the algebra $M_3(K)$ of all 3×3 matrices for all $a_1,\dots,a_n\in so_3(K)$. The generators of $I(M_3(K), so_3(K))$ were found by Razmyslov in the 1980s. In this paper the cocharacter sequence of $I(M_3(K), so_3(K))$ is computed. In other words, the $\mathrm{GL}_p(K)$-module structure of the algebra generated by p generic skew-symmetric matrices is determined. Moreover, the same is done for the closely related algebra of $\mathrm{SO}_3(K)$-equivariant polynomial maps from the space of p-tuples of 3×3 skew-symmetric matrices into $M_3(K)$ (endowed with the conjugation action). In the special case $p=3$ the latter algebra is a module over a 6-variable polynomial subring in the algebra of $\mathrm{SO}_3(K)$-invariants of triples of 3×3 skew-symmetric matrices, and a free resolution of this module is found. The proofs involve methods and results of classical invariant theory, representation theory of the general linear group and explicit computations with matrices.

1. INTRODUCTION

This paper can be considered as a relative of the well-known paper of Procesi [P2] whose abstract says that "In a precise way the ring of m generic 2×2 matrices and related rings are described." In the present work we also describe the ring of m generic 3×3 skew-symmetric matrices and a related ring in a precise way, but in somewhat different terms than [P2] (and we restrict to the case of a characteristic zero base field).

Take 3×3 generic skew-symmetric matrices

$$t_k=\begin{pmatrix}0 & t_{12}^{(k)} & t_{13}^{(k)}\\ -t_{12}^{(k)} & 0 & t_{23}^{(k)}\\ -t_{13}^{(k)} & -t_{23}^{(k)} & 0\end{pmatrix},\quad k=1,\dots,p,$$

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where $T_p = \{t_{ij}^{(k)} \mid i, j = 1, 2, 3;\ k = 1, \dots, p\}$ are commuting variables. Till the end of the paper we fix a field K of characteristic zero. Then $t_1, \dots, t_p$ are elements of the 3×3 matrix algebra $M_3(K[T_p])$ over the polynomial ring $K[T_p]$. As usual, we shall identify the elements of $K[T_p]$ with polynomial maps $so_3(K)^{\oplus p} \to K$ in the obvious way, where $so_3(K)$ is the space of 3×3 skew-symmetric matrices over K, which is also the Lie algebra of the special orthogonal group $\mathrm{SO}_3(K) = \{A \in K^{3\times 3} \mid AA^T = I,\ \det(A) = 1\}$. Accordingly, $M_3(K[T_p])$ is identified with the set of polynomial maps $so_3(K)^{\oplus p} \to M_3(K)$. Denote by $\mathcal{F}_p$ the associative K-subalgebra (with an identity element) of $M_3(K[T_p])$ generated by $t_1, \dots, t_p$ (so the identity matrix I is an element of $\mathcal{F}_p$ by definition):

$$\mathcal{F}_p = K\langle t_1, \dots, t_p\rangle \subset M_3(K[T_p])$$

The special orthogonal group $\mathrm{SO}_3(K)$ acts on $so_3(K)$ by conjugation (the adjoint action of $\mathrm{SO}_3(K)$ on its Lie algebra), and $\mathrm{SO}_3(K)$ acts by simultaneous conjugation on $so_3(K)^{\oplus p}$, the space of p-tuples of skew-symmetric 3×3 matrices. Also $\mathrm{SO}_3(K)$ acts on $M_3(K)$ by conjugation, and we write $\mathcal{E}_p$ for the subset of $M_3(K[T_p])$ consisting of the $\mathrm{SO}_3(K)$-equivariant polynomial maps $so_3(K)^{\oplus p} \to M_3(K)$. Clearly $\mathcal{E}_p$ is an associative K-subalgebra of $M_3(K[T_p])$, and $\mathcal{E}_p$ contains $\mathcal{F}_p$. If follows easily from known results (see Corollary 2.13) that

$$\mathcal{E}_p = K\langle t_1, \dots, t_p,\ \mathrm{tr}(t_i t_j)I,\ \mathrm{tr}(t_k t_l t_m)I \mid i \le j,\ k < l < m\rangle \subset M_3(K[T_p]),$$

where I stands for the 3×3 identity matrix throughout the paper.

In the present paper we aim at a combinatorial description of the algebras $\mathcal{F}_p$ and $\mathcal{E}_p$. The general linear group $\mathrm{GL}_p(K)$ acts on $\mathcal{E}_p$ via graded K-algebra automorphisms. Note first that $\mathrm{GL}_p(K)$ acts (from the right) on $so_3(K)^{\oplus p}$ as follows. For

$$g = \begin{pmatrix} g_{11} & \cdots & g_{1p} \\ \vdots & \ddots & \vdots \\ g_{p1} & \cdots & g_{pp} \end{pmatrix},\ \text{and } a = (a_1, \dots, a_p) \in so_3(K)^{\oplus p}$$

we have

$$(a_1, \dots, a_p) \cdot g = (\sum_{i=1}^p g_{i1} a_i, \sum_{i=1}^p g_{i2} a_2, \dots, \sum_{i=1}^p g_{ip} a_i).$$

This induces a left action (via graded K-algebra automorphisms) of $\mathrm{GL}_p(K)$ on the algebra $K[T_p]$ (respectively $M_3(K[T_p])$) of polynomial maps $so_3^{\oplus p} \to K$ (respectively $so_3^{\oplus p} \to M_3(K[T_p])$) in the standard way (for a function f, we have $(g \cdot f)(a) = f(a \cdot g)$). More explicitly, $g \cdot t_{ij}^{(k)} = \sum_{l=1}^p g_{lk} t_{ij}^{(l)}$, and for a matrix $m = (m_{ij})_{i,j=1}^3 \in M_3(K[T_p])$, we have $g \cdot m = (g \cdot m_{ij})_{i,j=1}^3$. The $\mathrm{GL}_3(K)$-action on $so_3(K)^{\oplus p}$ commutes with the $\mathrm{SO}_3(K)$-action, hence $\mathcal{E}_p$ is a $\mathrm{GL}_p(K)$-submodule of $M_3(K[T_p])$. Obviously $\mathcal{F}_p$ is a $\mathrm{GL}_p(K)$-submodule in $\mathcal{E}_p$.

We shall determine the $\mathrm{GL}_p(K)$-module structure both for $\mathcal{F}_p$ and $\mathcal{E}_p$. Our Theorem 3.7 (see also Theorem 4.1) and Theorem 4.2 (i) (together with Lemma 3.3) give the multiplicities of the irreducible $\mathrm{GL}_p(K)$-modules as summands in $\mathcal{F}_p$ and in $\mathcal{E}_p$. In fact, in the course of the proofs highest weight vectors for each irreducible summand are explicitly provided. In the case of $\mathcal{E}_p$ results from classical invariant theory allow to compute these multiplicities. Then with explicit constructions we show that for almost all irreducibles these upper bounds are achieved even in $\mathcal{F}_p$.

In particular, our Theorem 4.2 (ii) shows exactly the difference between $\mathcal{E}_p$ and its subspace $\mathcal{F}_p$; namely, the $\mathrm{GL}_p(K)$-module $\mathcal{E}_p$ has the direct sum decomposition

$$\mathcal{E}_p = \mathcal{F}_p \oplus \bigoplus_{k=1}^{\infty} \langle \mathrm{tr}(t_1^{2k}) I \rangle_{\mathrm{GL}_p(K)}.$$

Here and later as well, given a subset U of a $\mathrm{GL}_p(K)$-module we write $\langle U \rangle_{\mathrm{GL}_p(K)}$ for the $\mathrm{GL}_p(K)$-submodule generated by U. For $p \leq q$, $\mathcal{E}_p$ is a subalgebra of $\mathcal{E}_q$ and $\mathcal{F}_p$ is a subalgebra of $\mathcal{F}_q$. It follows from general principles that for $p \geq 3$, we have

$$\mathcal{F}_p = \langle \mathcal{F}_3 \rangle_{\mathrm{GL}_p(K)} \text{ and } \mathcal{E}_p = \langle \mathcal{E}_3 \rangle_{\mathrm{GL}_p(K)}$$

(see Section 2.1 and Corollary 2.8). Therefore to a large extent, the combinatorial study of $\mathcal{E}_p$ and $\mathcal{F}_p$ can be reduced to the special case $p = 3$. We shall present the 3-variable Hilbert series (i.e. the formal $\mathrm{GL}_3(K)$-character) of $\mathcal{E}_3$ as a rational function (see Proposition 5.1). Furthermore, $\mathcal{E}_3$ is a module over the algebra $K[T_3]^{\mathrm{SO}_3(K)}$ of polynomial $\mathrm{SO}_3(K)$-invariants on $so_3(K)$, and we shall determine the structure of this module (see Theorem 5.6).

The algebra $\mathcal{F}_p$ is isomorphic to the factor of the free associative algebra $K\langle X_p \rangle = K\langle x_1, \dots, x_p \rangle$ modulo $I(M_3(K), so_3(K)) \cap K\langle X_p \rangle$, the ideal of p-variable weak polynomial identities of the pair $(M_3(K), so_3(K))$. Therefore the computation of the $\mathrm{GL}_p(K)$-module structure of $\mathcal{F}_p$ is the same thing as the computation of the cocharacter sequence of the ideal $I(M_3(K), so_3(K))$ of weak polynomial identities of the pair $(M_3(K), so_3(K))$. In fact this was our original motivation for the present work, since weak polynomial identities play a significant role in the theory of PI-algebras. An overview of some relevant results on weak polynomial identities is given in Section 2.1.

We note that our computation is independent of the base field and the *form* of the Lie algebra. In particular, Theorem 3.7 can be interpreted as the computation of the cocharacter sequence for the weak polynomial identities of the pair $(M_3(K), \mathrm{ad}(sl_2(K)))$, where ad stands for the adjoint representation of the Lie algebra $sl_2(K)$.

2. Preliminaries

For a background on the mathematics used in this paper we recommend:

- On trace identities the paper by Procesi [P1] and the book by Razmyslov [Ra4, Chapter IV];
- On invariant theory the book by Weyl [W];
- On representation theory of the general linear group the book by Macdonald [Mc] and for the applications to algebras with polynomial identities the book by one of the authors [Dr2, Chapter 12].

2.1. Weak polynomial identities.

Definition 2.1. Let R be an associative algebra over a field K and let $R^{(-)}$ be the Lie algebra with respect to the operation $[r_1, r_2] = r_1 r_2 - r_2 r_1,\ r_1, r_2 \in R$. Let L be a Lie subalgebra of $R^{(-)}$ which generates R as an associative algebra, i.e., R is an associative enveloping algebra of L. The polynomial $f(x_1, \dots, x_n)$ of the free associative algebra $K\langle X \rangle = K\langle x_1, x_2, \dots \rangle$ is called a weak polynomial identity for the pair (R, L) if $f(a_1, \dots, a_n) = 0$ in R for all $a_1, \dots, a_n \in L$. The ideal $I(R, L)$ of the weak polynomial identities of (R, L) is generated by the system

$B = \{f_j(x_1,\dots,x_{n_j}) \mid j \in J\}$ (and B is called a basis of the weak polynomial identities of the pair (R,L)) if $I(R,L)$ is the minimal ideal of weak polynomial identities containing B. Then $I(R,L)$ is generated as an ideal by the polynomials $f_j(u_1,\dots,u_{n_j})$, $j \in J$, where $u_1,\dots,u_{n_j}$ are Lie elements in $K\langle X\rangle$. We shall also use the expression '$f = g$ is a weak polynomial identity for (R,L)' with some $f,g \in K\langle X\rangle$ if $f - g \in I(R,L)$.

Weak polynomial identities were introduced by Razmyslov [Ra1, Ra2] as a powerful tool in the solution of two important problems in the theory of PI-algebras. In [Ra1] Razmyslov found bases, over a field K of characteristic 0, of the weak polynomial identities of the pair $(M_2(K), sl_2(K))$, the polynomial identities of the Lie algebra $sl_2(K)$ of traceless 2×2 matrices, and the polynomial identities of the associative algebra $M_2(K)$ of 2×2 matrices. Up till now, in the case of characteristic 0, the algebras $sl_2(K)$ and $M_2(K)$ are the only nontrivial simple Lie and associative algebras with known bases of their polynomial identities. (Another proof for the basis of the weak polynomial identities of $(M_2(K), sl_2(K))$ is given in [DrK]). In [Ra2] Razmyslov constructed, using weak polynomial identities of the pair $(M_q(K), sl_q(K))$, a central polynomial for the algebra $M_q(K)$ of $q\times q$ matrices, solving an old problem of Kaplansky [K1, K2]. The existence of central polynomials for $M_q(K)$ was established independently with other methods by Formanek [F]. (For more information on the polynomial identities and central polynomials for matrices see, e.g., [Dr2, DrF].)

Let $\mathfrak{g} \cong sl_2(\mathbb{C})$ be the three-dimensional complex simple Lie algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. In [Ra3] Razmyslov showed that the ideal $I(U(\mathfrak{g}),\mathfrak{g})$ satisfies the Specht property: it is finitely generated and the same holds for any ideal of weak polynomial identities which contains it. Later, in [Ra4, Theorem 38.1] (page 251 in the Russian original and page 181 in the English translation) he found an explicit basis of the weak polynomial identities of the pair $(M_q(\mathbb{C}), \varrho(\mathfrak{g}))$, where $\varrho : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V_q) \cong M_q(\mathbb{C})$ is a q-dimensional irreducible representation of $\mathfrak{g}$. The basis consists of three weak polynomial identities:

$$s_3(x_1,x_2,x_3)x_4 = x_4 s_3(x_1,x_2,x_3),$$

where

$$s_3(x_1,x_2,x_3) := \sum_{\sigma\in S_3} \mathrm{sign}(\sigma) x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$$

is the standard polynomial of degree 3,

$$\delta \sum_{\sigma\in S_3} \mathrm{sign}(\sigma)[x_4, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = 2x_4 s_3(x_1,x_2,x_3),$$

where the commutators are left normed, e.g., $[x_1,x_2,x_3] = [[x_1,x_2],x_3]$, and $\delta = (q^2-1)/4$ is the value of the Casimir element in the representation ϱ, and one more identity in two variables

$$\mathrm{ART}_q(x_1,x_2) := \mathrm{ad}x_2 \prod_{i=1}^{q-1} \left(L_{x_2} - \left(i - \frac{1-q}{2} \right) \mathrm{ad}x_2 \right) x_1 = 0.$$

Here $L_r : R \to R$, $r \in R$, is the operator of left multiplication of the algebra R, defined by $r' \to rr'$, $r' \in R$, and $\mathrm{ad}r(r') = [r,r']$, $r,r' \in R$. For $q = 2$ this gives that the weak polynomial identities of the pair $(M_2(\mathbb{C}), sl_2(\mathbb{C}))$ follow from the weak identity $[x_1^2, x_2] = 0$, which was established already in [Ra1]. The Lie algebra

$sl_2(\mathbb{C})$ is isomorphic to the Lie algebra $so_3(\mathbb{C})$ of 3×3 skew-symmetric matrices and after easy computations the result from [Ra4, Theorem 38.1] gives:

Theorem 2.2. *The weak polynomial identities of the pair $(M_3(\mathbb{C}), so_3(\mathbb{C}))$ follow from its weak polynomial identities*

$$s_3(x_1, x_2, x_3)x_4 = x_4 s_3(x_1, x_2, x_3),$$

$$\sum_{\sigma \in S_3} \operatorname{sign}(\sigma)[x_4, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = x_4 s_3(x_1, x_2, x_3),$$

and

$$x_1[x_1, x_2]x_1 = 0.$$

(The result in [Ra4, Theorem 38.1] gives explicit bases of the weak polynomial identities also in the infinite dimensional cases.)

As in the case of ordinary polynomial identities the symmetric group S_n of degree n acts from the left on the vector space $P_n \subset K\langle X\rangle$ of multilinear polynomials of degree n and for any ideal $I(R, L)$ of weak polynomial identities $P_n \cap I(R, L)$ is an S_n-submodule of P_n. The sequence of S_n-characters $\chi_n(R, L)$ of $P_n/(P_n \cap I(R, L))$, $n = 0, 1, 2, \ldots$, is called the cocharacter sequence of $I(R, L)$. Then

$$\chi_n(R, L) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible character of S_n indexed by the partition λ of n and the nonnegative integer m_λ is the multiplicity of χ_λ in $\chi_n(R, L)$. By a result of Berele [B] and one of the authors [Dr1] the multiplicity m_λ, $\lambda = (\lambda_1, \ldots, \lambda_p) \vdash n$, is the same as the multiplicity of the irreducible polynomial $\mathrm{GL}_p(K)$-module $W_p(\lambda)$ in the $\mathrm{GL}_p(K)$-module

$$F_p(R, L) = K\langle X_p\rangle/(K\langle X_p\rangle \cap I(R, L)) \cong \sum_{\lambda} m_\lambda W_p(\lambda),$$

where $K\langle X_p\rangle = K\langle x_1, \ldots, x_p\rangle$, the general linear group $\mathrm{GL}_p(K)$ acts canonically on the vector space KX_p with basis $X_p = \{x_1, \ldots, x_p\}$ and this action is extended diagonally on the whole algebra $K\langle X_p\rangle$.

Our Theorem 3.7 gives explicitly the cocharacter sequence $\chi_n(M_3(K), so_3(K))$, $n = 0, 1, 2, \ldots$. The proof is based on a combination of classical invariant theory and representation theory of the general linear group. By standard arguments due to Regev [Re], since $\dim(so_3(K)) = 3$, we work in the algebra $F_3(M_3(K), so_3(K))$ considered as a $\mathrm{GL}_3(K)$-module instead to work with $P_n(M_3(K), so_3(K))$ and representations of S_n. Using classical results from invariant theory we give upper bounds for the multiplicities m_λ, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ depending on the parity of the differences $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3$. Then with explicit constructions we show that these upper bounds are achieved.

2.2. **Invariant theory of $\mathrm{SO}_3(K)$.** The general linear group $\mathrm{GL}_d(K)$ acts on the space $M_d(K)^{\oplus p}$ of p-tuples of $d \times d$ matrices by simultaneous conjugation:

$$g \cdot (r_1, \ldots, r_p) = (gr_1g^{-1}, \ldots, gr_pg^{-1}), \quad g \in \mathrm{GL}_d(K), \quad r_1, \ldots, r_p \in M_d(K).$$

The polynomial algebra corresponding to this action is in pd^2 variables,

$$K[Z_p] = K[z_{ij}^{(k)} \mid 1 \le i, j \le d; \quad k = 1, \ldots, p].$$

The action of $\mathrm{GL}_d(K)$ is defined in terms of generic $d \times d$ matrices

$$z_k = \begin{pmatrix} z_{11}^{(k)} & \dots & z_{1d}^{(k)} \\ \vdots & \ddots & \vdots \\ z_{d1}^{(k)} & \dots & z_{dd}^{(k)} \end{pmatrix}, \quad k = 1, \dots, p.$$

If

$$g^{-1}\left(z_{ij}^{(k)}\right)g = \left(w_{ij}^{(k)}\right), \quad g \in \mathrm{GL}_d(K), \quad k = 1, \dots, p,$$

then under the action of g the variable $z_{ij}^{(k)}$ goes to $w_{ij}^{(k)}$.

The algebra of invariants of the orthogonal group $\mathrm{O}_d(K) \subset \mathrm{GL}_d(K)$ is described by Sibirskii [S] and Procesi [P1, Theorem 7.1]:

Theorem 2.3. *The algebra $K[Z_p]^{\mathrm{O}_d(K)}$ of invariants of the group $\mathrm{O}_d(K)$ acting by simultaneous conjugation on p copies of $M_d(K)$ is generated by the traces*

$$\mathrm{tr}(u_{k_1} \cdots u_{k_n}), \quad 1 \le k_1, \dots, k_n \le p,$$

where $u_{k_r} = z_{k_r}$ or $u_{k_r} = z'_{k_r}$, the transpose of z_{k_r}, $r = 1, \dots, n$.

The generators of the algebra $K[Z_m]^{\mathrm{SO}_d(K)}$ of invariants of $\mathrm{SO}_d(K)$ are given by Aslaksen, Tan, and Zhu [ATZ, Theorem 3].

Theorem 2.4. (i) *For d odd the algebra $K[Z_p]^{\mathrm{SO}_d(K)}$ of $\mathrm{SO}_d(K)$-invariants coincides with the algebra $K[Z_p]^{\mathrm{O}_d(K)}$ of $\mathrm{O}_d(K)$-invariants.*

(ii) *For d even $K[Z_p]^{\mathrm{SO}_d(K)}$ is generated by the generators of $K[Z_p]^{\mathrm{O}_d(K)}$ and the so called polarized Pfaffians.*

The well-known generating system of the algebra of invariants $K[T_p]^{\mathrm{SO}_3(K)}$ of the special orthogonal group $\mathrm{SO}_3(K)$ acting by simultaneous conjugation on p copies of the Lie algebra $so_3(K)$ of 3×3 skew-symmetric matrices can be obtained as a consequence of the special case $d = 3$ of Theorems 2.3 and 2.4. For the rest of this section we assume $d = 3$.

Corollary 2.5. *The algebra $K[T_p]^{SO_3(K)}$ is generated by the traces*

$$\mathrm{tr}(t_{k_1} \cdots t_{k_n}), \quad 1 \le k_1, \dots, k_n \le p.$$

Proof. Since $so_3(K)^{\oplus p}$ is an $\mathrm{SO}_3(K)$-module direct summand of $M_3(K)^{\oplus p}$, the substitution $z_k \mapsto t_k$, $k = 1, \dots, p$ induces a K-algebra surjection $K[Z_p]^{\mathrm{SO}_3(K)} \to K[T_p]^{\mathrm{SO}_3(K)}$. Since z'_k is mapped to $-t_k$, a generator $\mathrm{tr}(u_{k_1} \cdots u_{k_n})$ from Theorem 2.3 is mapped to $\pm\mathrm{tr}(t_{k_1} \cdots t_{k_n})$. □

In the sequel we shall refer to the algebra $K[T_p]^{SO_3(K)}$ generated by traces of products of 3×3 generic skew-symmetric matrices as the generic trace algebra.

Remark 2.6. The algebra of $\mathrm{SL}_2(K)$-invariants under the adjoint action on $sl_2(K)^{\oplus p}$ is generated by traces of monomials in the 2×2 matrix components.

Consider the space $so_3(K)^{\oplus p} \oplus M_3(K)$, on which $\mathrm{SO}_3(K)$ acts by simultaneous conjugation, and $\mathrm{GL}_p(K)$ acts on the right by

$$(a_1, \dots, a_p, b) \cdot g = (\sum_{i=1}^{3} g_{i1} a_i, \dots, \sum_{i=1}^{3} g_{ip} a_i, b)$$

for $g = (g_{ij})_{i,j=1}^{p} \in \mathrm{GL}_p(K)$. The coordinate ring of $so_3(K)^{\oplus p} \oplus M_3(K)$ is $K[T_p, Z]$, where $Z = \{z_{ij} \mid 1 \le i, j \le 3\}$ is a set of commuting indeterminates over $K[T_p]$. We have the K-linear embedding

(1) $$M_3(K[T_p]) \to K[T_p, Z]^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)} \text{ given by } f \mapsto \mathrm{tr}(fz),$$

where $z = (z_{ij})_{i,j=1}^{3}$ is a generic 3×3 matrix as in Section 2.2, and $(\mathbb{N}_0, \dots, \mathbb{N}_0, 1)$ in the exponent means that we take the component of $K[T_p, Z]$ consisting of the polynomial functions that are linear on the summand $M_3(K)$ of $so_3(K)^{\oplus p} \oplus M_3(K)$.

Proposition 2.7. (i) *The K-subalgebra $\mathcal{E}_p$ of $M_3(K[T_p])$ is generated by the generic skew-symmetric matrices $t_1, \dots, t_p$ and the scalar matrices $\mathrm{tr}(t_{k_1} \cdots t_{k_n})I$ ($n \ge 2$, $1 \le k_1, \dots, k_n \le p$).*

(ii) *The map $f \mapsto \mathrm{tr}(fz)$ gives a $\mathrm{GL}_p(K)$-module isomorphism*

$$\iota : \mathcal{E}_p \xrightarrow{\cong} (K[T_p, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)}.$$

Proof. By standard properties of the trace, the restriction of the embedding (1) maps $\mathcal{E}_p$ into $(K[T_p, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)}$. On the other hand, $\mathcal{E}_p$ contains the K-subalgebra generated by $t_1, \dots, t_p$, $\mathrm{tr}(t_{k_1} \dots t_{k_n})I$ ($1 \le k_1, \dots, k_n \le p$), and the images of the elements of this subalgebra already exhaust $(K[T_p, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)}$ (and hence both (i) and (ii) hold): indeed, similarly to the proof of Corollary 2.5, the specialization $z_k \mapsto t_k$ ($k = 1, \dots, p$), $z_{p+1} \mapsto z$ maps the generators of $K[Z_{p+1}]^{\mathrm{SO}_3(K)}$ given in Theorem 2.3 to generators of $K[T_p, Z]^{\mathrm{SO}_3(K)}$. If such a generator is linear in z, then up to sign, it is of the form $\mathrm{tr}(t_{k_1} \cdots t_{k_n} z)$, since a matrix and its transpose have equal trace, therefore $\mathrm{tr}(t_{k_1} \cdots t_{k_n} z') = \mathrm{tr}(z'' t'_{k_n} \cdots t'_{k_1}) = (-1)^n \mathrm{tr}(t_{k_n} \cdots t_{k_1} z)$. Taking into account Corollary 2.5 we conclude that the above subalgebra of $\mathcal{E}_p$ is mapped by ι onto $(K[T_p, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)}$. □

Corollary 2.8. *For $p \ge 3$ we have $\mathcal{E}_p = \langle \mathcal{E}_3 \rangle_{\mathrm{GL}_p(K)}$.*

Proof. Since $\dim_K(so_3(K)) = 3$, by Weyl's Theorem on polarizations (derived from Capelli's identities in [W]) we have $K[T_p, Z] = \langle K[T_3, Z] \rangle_{\mathrm{GL}_p(K)}$, hence

$$(K[T_p, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)} = \langle (K[T_3, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)} \rangle_{\mathrm{GL}_p(K)}.$$

So the statement follows by the isomorphism ι in Proposition 2.7. □

We need some facts on $K[T_p]^{\mathrm{SO}_3(K)}$ contained in the theorems on the vector invariants of the special orthogonal group, that we recall now. Let V_d be the d-dimensional K-vector space with basis $\{v_1, \dots, v_d\}$ with the canonical action of the group $\mathrm{GL}(V_d)$ identified in the usual way with $\mathrm{GL}_d(K)$. The action of $\mathrm{GL}_d(K)$ on V_d induces an action on the algebra $K[X_d] = K[x_1, \dots, x_d]$ of polynomial functions on V_d (here $x_1, \dots, x_d$ is the dual basis in V_d^* to the basis chosen in V_d). If

$$v = \alpha_1 v_1 + \dots + \alpha_d v_d \in V_d, \quad f = f(X_d) \in K[X_d], \quad g \in \mathrm{GL}_d(K),$$

then

$$f(v) = f(\alpha_1, \dots, \alpha_d) \text{ and } (g(f))(v) = f(g^{-1}(v)).$$

For any subgroup G of $\mathrm{GL}_d(K)$ the algebra $K[X_d]^G$ of G-invariants consists of all $f(X_d) \in K[X_d]$ with the property $g(f) = f$ for all $g \in G$.

We equip the vector space V_d with a nondegenerate symmetric bilinear form. If

$$v' = \alpha_1 v_1 + \dots + \alpha_d v_d, \quad v'' = \beta_1 v_1 + \dots + \beta_d v_d,$$

then

$$\langle v', v'' \rangle = \alpha_1\beta_1 + \cdots + \alpha_d\beta_d.$$

The special orthogonal group $\mathrm{SO}_d(K)$ acts canonically on the vector space V_d and consists of all matrices with determinant equal to 1 which preserve the symmetric bilinear form. The action of $\mathrm{SO}_d(K)$ can be extended to the direct sum $V_d^{\oplus p}$ of p copies of V_d. Write $\{v_{i1}, \dots, v_{id}\}$ for the basis of the ith direct summand of $V_d^{\oplus p}$ corresponding to the chosen basis $\{v_1, \dots, v_n\}$ of V_d, and let y_{ik} be the polynomial (in fact linear) function which sends the vector v_{ik} to 1 and to 0 all other vectors of the fixed basis of $V_d^{\oplus p}$. Set $y_i = (y_{i1}, \dots, y_{id})$, $i = 1, \dots, p$, and consider the scalar products

$$\langle y_i, y_j \rangle = y_{i1}y_{j1} + \cdots + y_{id}y_{jd}, \quad 1 \le i, j \le p,$$

the determinant

$$\Delta_d(y_{j_1}, \dots, y_{j_d}) = \det(y_{j_1}, \dots, y_{j_d}) = \begin{vmatrix} y_{1j_1} & y_{1j_2} & \cdots & y_{1j_d} \\ y_{2j_1} & y_{2j_2} & \cdots & y_{2j_d} \\ \vdots & \vdots & \ddots & \vdots \\ y_{dj_1} & y_{dj_2} & \cdots & y_{dj_d} \end{vmatrix},$$

$1 \le j_1 < \cdots < j_d \le p$, and the Gram determinant

$$\Gamma_k(y_{i_1}, \dots, y_{i_k} \mid y_{j_1}, \dots, y_{j_k}) = \det(\langle y_{i_r}, y_{j_s} \rangle) = \begin{vmatrix} \langle y_{i_1}, y_{j_1} \rangle & \dots & \langle y_{i_1}, y_{j_k} \rangle \\ \vdots & \ddots & \vdots \\ \langle y_{i_k}, y_{j_1} \rangle & \dots & \langle y_{i_k}, y_{j_k} \rangle \end{vmatrix},$$

$1 \le i_1 < \cdots < i_k \le p$, $1 \le j_1 < \cdots < j_k \le p$.

The following classical theorems, see, e.g., [W, Theorems 2.9.A and 2.17.A], describe the generating set and the defining relations of the algebra

$$K[Y_{pd}]^{\mathrm{SO}_d(K)} = K[y_{ik} \mid i = 1, \dots, p;\ k = 1, \dots, d]^{\mathrm{SO}_d(K)}$$

of $\mathrm{SO}_d(K)$-invariants of $V_d^{\oplus p}$.

Theorem 2.9 (First fundamental theorem for the invariants of $\mathrm{SO}_d(K)$)**.** (i) *The algebra $K[Y_{pd}]^{\mathrm{SO}_d(K)}$ is generated by the scalar products $\langle y_i, y_j \rangle$, $1 \le i, j \le p$, and by the determinants $\Delta_d(y_{j_1}, \dots, y_{j_d})$, $1 \le j_1 < \cdots < j_d \le p$.*

(ii) *The elements of $K[Y_{pd}]^{\mathrm{SO}_d(K)}$ are linear combinations of products*

$$\langle y_{i_1}, y_{j_1} \rangle \cdots \langle y_{i_n}, y_{j_n} \rangle \text{ and } \Delta_d(y_{k_1}, \dots, y_{k_d})\langle y_{i_1}, y_{j_1} \rangle \cdots \langle y_{i_n}, y_{j_n} \rangle,$$
$$1 \le i_r, j_r \le p, \quad r = 1, \dots, n, \quad 1 \le k_1 < \cdots < k_d \le p.$$

Theorem 2.10 (Second fundamental theorem for the invariants of $\mathrm{SO}_d(K)$)**.** *The defining relations of the algebra $K[Y_{pd}]^{\mathrm{SO}_d(K)}$ consist of*

$$\Gamma_{d+1}(y_{i_0}, y_{i_1}, \dots, y_{i_d} \mid y_{j_0}, y_{j_1}, \dots, y_{j_d}) = 0,$$
$$1 \le i_0 < i_1 < \cdots < i_d \le p, \quad 1 \le j_0 < j_1 < \cdots < j_d \le p,$$
$$\Delta_d(y_{i_1}, \dots, y_{i_d})\Delta_d(y_{j_1}, \dots, y_{j_d}) - \Gamma_d(y_{i_1}, \dots, y_{i_d} \mid y_{j_1}, \dots, y_{j_d}) = 0,$$
$$1 \le i_1 < \cdots < i_d \le p, \quad 1 \le j_1 < \cdots < j_d \le p,$$
$$\sum_{r=0}^{d} (-1)^r \langle y_i, y_{j_r} \rangle \Delta_d(y_{j_0}, \dots, \hat{y}_{j_r}, \dots, y_{j_d}) = 0,$$
$$1 \le i \le p, \quad 1 \le j_0 < j_1 < \cdots < j_d \le p,$$

where $\hat{y}_{j_r}$ means that y_{j_r} does not participate in the expression.

In [DoDr] we found a Gröbner basis of the ideal of defining relations of the algebra $K[Y_{pd}]^{\mathrm{SO}_d(K)}$.

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We define an $\mathbb{N}_0^p$-grading on the polynomial algebras $K[Y_{p3}]$, $K[T_p]$ and on the algebra $\mathcal{F}_p$ assuming that the variables y_{kj}, $t_{ij}^{(k)}$ and the matrix t_k are of degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ (the kth coordinate is equal to 1 and all other coordinates are equal to 0). The generic trace algebra is an $\mathbb{N}_0^p$-graded subalgebra of $K[T_p]$.

Proposition 2.11. *The algebras $K[Y_{p3}]^{\mathrm{SO}_3(K)}$ and $K[T_p]^{\mathrm{SO}_3(K)}$ are isomorphic as $\mathbb{N}_0^p$-graded algebras.*

Proof. The vector space $so_3(K)$ has a basis

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Denoting by e_1, e_2, e_3 the standard basis vectors in the space K^3 of column vectors, a straightforward calculation shows that the linear map

$$so_3(K) \to K^3, \quad a_1 \mapsto e_3, \ a_2 \mapsto -e_2, \ a_3 \mapsto e_1$$

is an isomorphism between the $\mathrm{SO}_3(K)$-modules $so_3(K)$ and K^3, where $\mathrm{SO}_3(K)$ acts via conjugation on $so_3(K)$ and via matrix multiplication on K^3. This isomorphism induces an isomorphism of the $\mathrm{SO}_3(K)$-modules $so_3(K)^{\oplus p} \cong (K^3)^{\oplus p}$, their coordinate rings $K[T_p] \cong K[Y_{p3}]$, and finally the $\mathbb{N}_0^p$-graded subalgebras $K[T_p]^{\mathrm{SO}_3(K)} \cong K[Y_{p3}]^{\mathrm{SO}_d(K)}$ of $\mathrm{SO}_3(K)$-invariants. For sake of completeness of the picture, we mention that the basis $\{a_1, a_2, a_3\}$ in $so_3(K)$ is orthonormal with respect to the nondegenerate, symmetric, $\mathrm{SO}_3(K)$-invariant bilinear form defined by

$$\langle a, b \rangle = -\frac{1}{2}\mathrm{tr}(ab), \quad a, b \in so_3(K).$$

□

For a skew-symmetric 3×3 matrix a and a symmetric 3×3 matrix b we have $\mathrm{tr}(ab) = 0$. It follows that $\mathrm{tr}(t_1t_2t_3) + \mathrm{tr}(t_2t_1t_3) = \mathrm{tr}((t_1t_2 + t_2t_1)t_3) = 0$, so for any permutation $\pi \in S_3$ we have

$$\mathrm{tr}(t_{\pi(1)}t_{\pi(2)}t_{\pi(3)}) = \mathrm{sign}(\pi)\mathrm{tr}(t_1t_2t_3).$$

Corollary 2.12. (i) *The algebra $K[T_p]^{\mathrm{SO}_3(K)}$ is generated by the elements $\mathrm{tr}(t_it_j)$, $1 \leq i \leq j \leq p$, and $\mathrm{tr}(t_kt_lt_m)$, $1 \leq k < l < m \leq p$.*

(ii) *The algebra $K[T_3]^{\mathrm{SO}_3(K)}$ is a rank two free module generated by $\mathrm{tr}(t_1t_2t_3)$ over its subalgebra generated by the algebraically independent elements $\mathrm{tr}(t_1^2)$, $\mathrm{tr}(t_2^2)$, $\mathrm{tr}(t_3^2)$, $\mathrm{tr}(t_1t_2)$, $\mathrm{tr}(t_1t_3)$, $\mathrm{tr}(t_2t_3)$.*

Proof. (i) is an immediate consequence of Theorem 2.9, Corollary 2.5, Proposition 2.11. Taking into account also Theorem 2.10, we get (ii). □

Corollary 2.13. *The K-subalgebra $\mathcal{E}_p$ of $M_3(K[T_p])$ is generated by the generic skew-symmetric matrices $t_1, \ldots, t_p$ and the scalar matrices $\mathrm{tr}(t_it_j)I$ $(1 \leq i \leq j \leq p)$, $\mathrm{tr}(t_kt_lt_m)I$ $(1 \leq k < l < m \leq p)$.*

Proof. This follows from Proposition 2.7 (i) and Corollary 2.12 (i). □

2.3. **Representation theory of** $\mathrm{GL}_p(K)$**.** In what follows we assume that the general linear group $\mathrm{GL}_p(K) = \mathrm{GL}(KX_p)$ acts canonically on the vector space KX_p with basis X_p. That is, for

$$g = (g_{ij})_{i,j=1}^p \in \mathrm{GL}_p(K) \text{ we have } g(x_j) = \sum_{i=1}^p g_{ij}x_i, \quad j = 1, \dots, p.$$

This action can be extended diagonally on the tensor algebra

$$T(KX_p) = \sum_{n\geq 0}(KX_p)^{\otimes n} \cong K\langle X_p\rangle.$$

In the sequel we shall identify $T(KX_p)$ with the free associative algebra $K\langle X_p\rangle$ and $(KX_p)^{\otimes n}$ with the homogeneous component $K\langle X_p\rangle^{(n)}$ of degree n of $K\langle X_p\rangle$. The standard $\mathbb{N}_0^p$-grading on $K\langle X_p\rangle$ corresponds to the decomposition of $K\langle X_p\rangle$ into the direct sum of the isotypic components under the action of the subgroup of diagonal matrices in $\mathrm{GL}_p(K)$. The $\mathrm{GL}_p(K)$-module $K\langle X_p\rangle$ is a direct sum of irreducible polynomial $\mathrm{GL}_p(K)$-modules. The irreducible polynomial $\mathrm{GL}_p(K)$-modules are indexed by partitions having not more than p parts and all they appear as summands in $K\langle X_p\rangle$. Let

$$\lambda = (\lambda_1, \dots, \lambda_p), \quad \lambda_1 \geq \dots \geq \lambda_p \geq 0, \quad \lambda_1 + \dots + \lambda_p = n,$$

be a partition of n and let $W_p(\lambda)$ be the corresponding $\mathrm{GL}_p(K)$-module. By Schur-Weyl duality (cf. [W] or [P3, page 256, (3.1.4)]), the homogeneous component $K\langle X_p\rangle^{(n)}$ of $K\langle X_p\rangle$ decomposes as

$$K\langle X_p\rangle^{(n)} \cong \sum_{\lambda\vdash n} \deg(\lambda)W_p(\lambda),$$

where $\deg(\lambda)$ is the degree of the irreducible S_n-character χ_λ. A non-zero element of $K\langle X_p\rangle^{(n)}$ is called a *highest weight vector of weight* λ if it is fixed by the subgroup of unipotent upper triangular matrices in $\mathrm{GL}_p(K)$, and it is multihomogeneous of $\mathbb{N}_0^p$-degree λ. Any $\mathrm{GL}_p(K)$-submodule $W \subset K\langle X_p\rangle^{(n)}$, $W \cong W_p(\lambda)$ contains a unique (up to non-zero scalar multiples) highest weight vector (necessarily having weight λ and generating W as a $\mathrm{GL}_p(K)$-module). The highest weight vectors in $K\langle X_p\rangle^{(n)}$ can be described in the following way. The symmetric group S_n acts from the right on $K\langle X_p\rangle^{(n)}$ by the rule

$$(x_{i_1}\cdots x_{i_n})^\tau = x_{i_{\tau(1)}}\cdots x_{i_{\tau(n)}}, \quad \tau \in S_n,$$

and this action commutes with the action of $\mathrm{GL}_p(K)$ introduced before. Let $[\lambda]$ be the Young diagram corresponding to the partition λ and let the lengths of the columns of $[\lambda]$ be $k_1, \dots, k_{\lambda_1}$. Consider the product of standard polynomials

$$w_\lambda(x_1, \dots, x_{k_1}) = \prod_{j=1}^{\lambda_1} s_{k_j}(x_1, \dots, x_{k_j}) = \prod_{j=1}^{\lambda_1}\left(\sum_{\sigma_j\in S_{k_j}} \mathrm{sign}(\sigma_j)x_{\sigma_j(1)}\cdots x_{\sigma_j(k_j)}\right).$$

Then every highest weight vector of weight λ is of the form

$$w = \sum_{\tau\in S_n} \alpha_\tau w_\lambda^\tau, \quad \alpha_\tau \in K.$$

A λ-tableau is the Young diagram $[\lambda]$ whose boxes are filled with positive integers. We say that the tableau is of content $(n_1, \dots, n_p)$ if $1, \dots, p$ appear in it $n_1, \dots, n_p$ times, respectively. The tableau is standard if its entries are the numbers $1, \dots, n$, without repetition, arranged in such a way that they increase in rows (reading them from left to right) and in columns (reading from top to bottom). It is semistandard if its entries (allowing repetitions) do not decrease in rows and increase in columns.

Given a partition λ of n, we set up a bijection between the set of λ-tableaux of content $(1,\ldots,1)$ and S_n as follows: we assign to the permutation $\varrho\in S_n$ the Young tableau $T_\lambda(\varrho)$ obtained by filling in the boxes of the first column of $[\lambda]$ with $\varrho^{-1}(1),\ldots,\varrho^{-1}(k_1)$, of the second column with $\varrho^{-1}(k_1+1),\ldots,\varrho^{-1}(k_1+k_2)$, etc. Then the highest weight vector w_λ^ϱ has skew-symmetries in the positions listed in the first column of $T_\lambda(\varrho)$, skew-symmetries in the positions listed in the second column of $T_\lambda(\varrho)$, etc. For example, for $n=5$, $\lambda=(2,2,1)$, and

$$\varrho^{-1}=\begin{pmatrix}1&2&3&4&5\\1&5&3&2&4\end{pmatrix},\text{ we have}$$

$$T_\lambda(\varrho)=\begin{array}{|c|c|}\hline 1&2\\\hline 5&4\\\hline 3\\\cline{1-1}\end{array}\ ,\quad w_\lambda^\varrho=\sum_{\sigma_1\in S_3,\sigma_2\in S_2}\text{sign}(\sigma_1)\text{sign}(\sigma_2)x_{\sigma_1(1)}x_{\sigma_2(1)}x_{\sigma_1(3)}x_{\sigma_2(2)}x_{\sigma_1(2)}.$$

It is known (see e.g. [Dr2]) that the set of all w_λ^ϱ corresponding to the standard λ-tableaux $T_\lambda(\varrho)$ is a basis of the vector space of the highest weight vectors of weight λ in $K\langle X_p\rangle^{(n)}$. We note also that if w_i $(i=1,2)$ is a highest weight vector of weight $\lambda^{(i)}\vdash n_i$ in $K\langle X_p\rangle^{(n_i)}$, then the product w_1w_2 is a highest weight vector of weight $\lambda^{(1)}+\lambda^{(2)}$ in $K\langle X_p\rangle^{(n_1+n_2)}$.

Proposition 2.14. (i) *There is a one-to-one correspondence between an arbitrary $\mathbb{N}_0^p$-graded basis of a $\mathrm{GL}_p(K)$-submodule of $K\langle X_p\rangle$ isomorphic to $W_p(\lambda)$ and the set of semistandard λ-tableaux filled in with $1,\ldots,p$, such that a basis vector of degree $(n_1,\ldots,n_p)$ corresponds to a semistandard tableau of content $(n_1,\ldots,n_p)$.*

(ii) *Let W be a polynomial $\mathrm{GL}_p(K)$-module (i.e. W is the direct sum of modules isomorphic to $W_p(\lambda)$ for various λ), endowed with the $\mathbb{N}_0^p$-grading given by the action of the subgroup of diagonal matrices in $\mathrm{GL}_p(K)$. Suppose that there exists a mapping π from an $\mathbb{N}_0^p$-graded basis of W into the set of semistandard λ-tableaux, such that a basis vector of degree $(n_1,\ldots,n_p)$ is mapped to a semistandard tableau of content $(n_1,\ldots,n_p)$, and for each partition λ, there exists a non-negative integer m_λ such that that every semistandard λ-tableau is the image of exactly m_λ basis elements. Then W decomposes as*

$$W=\sum_\lambda m_\lambda W_p(\lambda).$$

Proof. The statement (i) follows immediately from the fact that the dimension of the homogeneous component $W_p^{(n_1,\ldots,n_p)}(\lambda)$ of degree $(n_1,\ldots,n_p)$ is equal to the coefficient of $\xi_1^{n_1}\cdots\xi_d^{n_p}$ of the Schur function $S_\lambda(\xi_1,\ldots,\xi_p)$. On the other hand this coefficient is equal to the number of semistandard λ-tableaux of content $(n_1,\ldots,n_d)$. For (ii) it is sufficient to apply the fact that the Schur function plays the role of character of the representation of $\mathrm{GL}_p(K)$ corresponding to the $\mathrm{GL}_p(K)$-module $W_p(\lambda)$ and that the character of the direct sum of polynomial representations determines the decomposition of the corresponding $\mathrm{GL}_p(K)$-module W_p. □

The decomposition of the $\mathrm{GL}_p(K)$-module structure of the algebra of invariants of $\mathrm{SO}_3(K)$ acting on p copies of V_3 is given for example in [P2, Section 1.2] or in [LB, Chapter I, Theorem 4.3] in terms of semistandard tableaux.

Theorem 2.15. *The algebra $K[Y_{p3}]^{\mathrm{SO}_3(K)}$ has an $\mathbb{N}_0^p$-graded basis indexed (via a mapping π as in Proposition 2.14 (ii)) by all semistandard λ-tableaux for all $\lambda = (2\mu_1, 2\mu_2, 2\mu_3)$ and $\lambda = (2\mu_1 + 1, 2\mu_2 + 1, 2\mu_3 + 1)$, where $\mu_1, \mu_2, \mu_3 \in \mathbb{N}_0$.*

As an immediate consequence of Propositions 2.11 and 2.14 and Theorem 2.15 we obtain:

Corollary 2.16. *As a $\mathrm{GL}_p(K)$-module the algebra $K[T_p]^{\mathrm{SO}_3(K)}$ of invariants of the action by simultaneous conjugation of $\mathrm{SO}_3(K)$ on p copies of 3×3 skew-symmetric matrices decomposes as*

$$K[T_p]^{\mathrm{SO}_3(K)} \cong \sum W_p(2\mu_1 + \delta, 2\mu_2 + \delta, 2\mu_3 + \delta),$$

where the summation runs on all partitions (μ_1, μ_2, μ_3) and $\delta = 0$ or 1.

3. The cocharacter sequence and highest weight vectors

Since $\dim(so_3(K)) = 3$, by a theorem of Regev [Re] the cocharacter sequence of $I(M_3(K), so_3(K))$ is of the form

$$\chi_n(M_3(K), so_3(K)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of n in not more than three parts. This allows to replace the problem for the cocharacter sequence with the problem of the decomposition into a direct sum of irreducible components of the $\mathrm{GL}_3(K)$-module $F_3(M_3(K), so_3(K))$. The map $x_i \mapsto t_i$ induces an isomorphism

$$F_3(M_3(K), so_3(K)) \cong \mathcal{F}_3. \tag{2}$$

The algebra $\mathcal{F}_3$ is contained in $\mathcal{E}_3$. By Proposition 2.7 we have

$$\mathcal{F}_3 \subseteq \mathcal{E}_3 \cong (K[T_3, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\dots,\mathbb{N}_0,1)}. \tag{3}$$

Thus the multiplicities of the irreducible $\mathrm{GL}_3(K)$-modules in $\mathcal{F}_3$ are bounded by their multiplicities in $\mathcal{E}_3$, and the latter can be computed using Corollary 2.16, thanks to Lemma 3.1 below.

To state Lemma 3.1 we need some notation. Let $(K[T_5]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)}$ be the component of the generic trace algebra $K[T_5]^{\mathrm{SO}_3(K)}$ which is linear in the generic skew-symmetric matrices t_4 and t_5. Embedding $\mathrm{GL}_3(K)$ into $\mathrm{GL}_5(K)$ by

$$\mathrm{GL}_3(K) \ni g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \mapsto \begin{pmatrix} g_{11} & g_{12} & g_{13} & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 \\ g_{31} & g_{32} & g_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_5(K)$$

we equip the vector space $(K[T_5]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)}$ with the structure of a $\mathrm{GL}_3(K)$-module.

Lemma 3.1. *The comorphism of the map*

$$\mu : so_3(K)^{\oplus 5} \to so_3(K)^{\oplus 3} \oplus M_3(K),$$
$$(a_1, a_2, a_3, a_4, a_5) \mapsto (a_1, a_2, a_3, a_4 \cdot a_5)$$

gives a $\mathrm{GL}_3(K)$*-module isomorphism*

$$\mu^* : (K[T_3, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)} \xrightarrow{\cong} (K[T_5])^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)}.$$

Proof. Observe that $so_3(K) \oplus so_3(K) \to M_3(K)$, $(a, b) \mapsto ab$ is the algebraic quotient map for the action of the multiplicative group $K^\times$ on $so_3(K) \oplus so_3(K)$ given by $c \cdot (a, b) = (ca, c^{-1}b)$. Indeed, the algebra $K[T_2]^{K^\times}$ of $K^\times$-invariants on $so_3(K) \oplus so_3(K)$ is generated by all products $t_{ij}^{(1)} t_{kl}^{(2)}$, and these products span the same K-subspace in $K[T_2]$ as the entries of the product $t_1 t_2$ of the generic skew-symmetric matrices t_1 and t_2. It follows that μ is the algebraic quotient map for the action of $K^\times$ on $so_3(K)^{\oplus 5}$ given by

$$c \cdot (a_1, a_2, a_3, a_3, a_5) = (a_1, a_2, a_3, ca_4, c^{-1}a_5), \quad c \in K^\times.$$

Therefore the comorphism μ^* of μ maps $K[T_3, Z]$ onto

$$\mu^*(K[T_3, Z]) = K[T_5]^{K^\times} = \bigoplus_{j=0}^{\infty} K[T_5]^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,j,j)},$$

where $(\mathbb{N}_0, \mathbb{N}_0, \mathbb{N}_0, j, j)$ in the exponent means that we take the sum of the multihomogeneous components with multidegree $(\alpha_1, \alpha_2, \alpha_3, j, j)$, $\alpha_1, \alpha_2, \alpha_3$ ranging over $\mathbb{N}_0$. As the action of $K^\times$ commutes with the action of $\mathrm{SO}_3(K)$ on $so_3^{\oplus 5}$, the comorphism μ^* is $\mathrm{SO}_3(K)$-equivariant, and we have

$$\mu^*(K[T_3, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,j)}) = (K[T_5]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,j,j)}, \quad j = 0, 1, 2, \ldots.$$

The restriction of μ^* to $K[T_3, Z]^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)}$ is injective, because the image of the multiplication map $so_3(K) \oplus so_3(K) \to M_3(K)$ spans $M_3(K)$ as a K-vectorspace, hence a linear function on $M_3(K)$ vanishing on all $\{ab \mid a, b \in so_3(K)\}$ must be the zero map. Thus the restriction of μ^* to $K[T_3, Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)}$ is a vector space isomorphism onto $(K[T_5])^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)}$. Moreover, it is a $\mathrm{GL}_3(K)$-module homomorphism, because μ is obviously a $\mathrm{GL}_3(K)$-module homomorphism, and the actions of $\mathrm{SO}_3(K)$ and $K^\times$ both commute with the action of $\mathrm{GL}_3(K)$. □

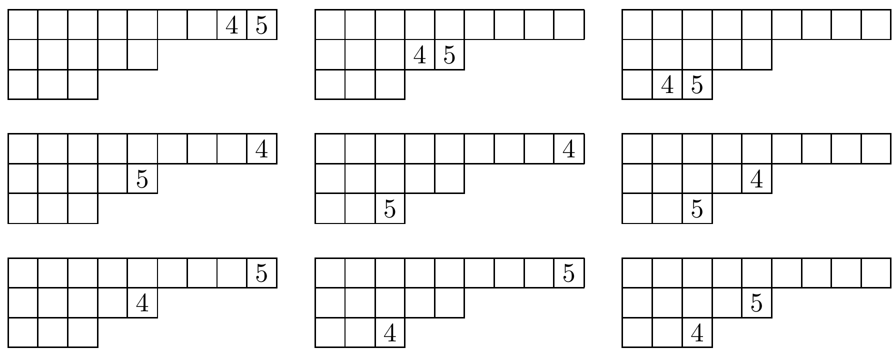
Lemma 3.2. *Let* $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (2\mu_1 + \delta, 2\mu_2 + \delta, 2\mu_3 + \delta)$*,* $\delta = 0$ *or* 1*. Consider the set of semistandard* λ*-tableaux of content* $(n_1, n_2, n_3, 1, 1)$*. Deleting the boxes containing* 4 *and* 5 *from each tableau, we obtain a multiset of semistandard tableaux of content* (n_1, n_2, n_3)*.* (i) *The multiplicity of a semistandard* ν*-tableau of content* (n_1, n_2, n_3) *in this multiset is non-zero if and only if* $\nu = (\nu_1, \nu_2, \nu_3)$*,* $\nu_1 \geq \nu_2 \geq \nu_3 \geq 0$*, and*

$$\nu \in \{(\lambda_1 - 2, \lambda_2, \lambda_3),\ (\lambda_1, \lambda_2 - 2, \lambda_3),\ (\lambda_1, \lambda_2, \lambda_3 - 2),$$
$$(\lambda_1 - 1, \lambda_2 - 1, \lambda_3),\ (\lambda_1 - 1, \lambda_2, \lambda_3 - 1),\ (\lambda_1, \lambda_2 - 1, \lambda_3 - 1)\}.$$

(ii) *Moreover, the multiplicity is* 2 *if* $\nu = (\lambda_1 - 1, \lambda_2 - 1, \lambda_3)$ *and* $\lambda_1 > \lambda_2$*, or* $\nu = (\lambda_1, \lambda_2 - 1, \lambda_3 - 1)$ *and* $\lambda_2 > \lambda_3$*, or* $\nu = (\lambda_1 - 1, \lambda_2, \lambda_3 - 1)$*.*

(iii) *All other positive multiplicities are equal to* 1*.*

Proof. If $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 \geq 2$, then the semistandard λ-tableaux of content $(n_1, n_2, n_3, 1, 1)$ are of the following form:



If $\lambda_1 = \lambda_2 > \lambda_3$ or $\lambda_2 = \lambda_3 > 0$, then 4 and 5 may appear in the same column, and 4 is necessarily above 5. These observations clearly yield the statements (i), (ii), (iii). □

Lemma 3.3. *The* $\mathrm{GL}_3(K)$*-module* $(K[T_5]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)}$ *decomposes as*

$$(K[T_5]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)} = \sum_{\nu} m_\nu W_3(\nu), \quad \nu = (\nu_1, \nu_2, \nu_3),$$

where:
(i) *If* $\nu_1 \equiv \nu_2 \equiv \nu_3 \pmod 2$*, then*

$$m_\nu = \begin{cases} 3, & \text{if } \nu_1 > \nu_2 > \nu_3; \\ 2, & \text{if } \nu_1 = \nu_2 > \nu_3; \\ 2, & \text{if } \nu_1 > \nu_2 = \nu_3; \\ 1, & \text{if } \nu_1 = \nu_2 = \nu_3; \end{cases}$$

(ii) *If* $\nu_1 \equiv \nu_2 \not\equiv \nu_3 \pmod 2$*, then*

$$m_\nu = \begin{cases} 2, & \text{if } \nu_1 > \nu_2; \\ 1, & \text{if } \nu_1 = \nu_2; \end{cases}$$

(iii) *If* $\nu_1 \equiv \nu_3 \not\equiv \nu_2 \pmod 2$*, then* $m_\nu = 2$*;*
(iv) *If* $\nu_1 \not\equiv \nu_2 \equiv \nu_3 \pmod 2$*, then*

$$m_\nu = \begin{cases} 2, & \text{if } \nu_2 > \nu_3; \\ 1, & \text{if } \nu_2 = \nu_3. \end{cases}$$

Proof. Combining Proposition 2.11 and Theorem 2.15 we obtain that as an $\mathbb{N}_0^5$-graded vector space the algebra $K[T_5]^{\mathrm{SO}_3(K)}$ has a graded basis indexed by all semistandard λ-tableaux for all $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, such that $\lambda_1 \equiv \lambda_2 \equiv \lambda_3 \pmod 2$. Hence the vector space $(K[T_5]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1,1)}$ has a basis indexed by the semistandard λ-tableaux of content $(n_1, n_2, n_3, 1, 1)$, $n_1, n_2, n_3 \in \mathbb{N}_0$. Deleting 4 and 5 from such a semistandard λ-tableau, we obtain the semistandard ν-tableaux of content (n_1, n_2, n_3) described in Lemma 3.2.

(i) If $\nu_1 \equiv \nu_2 \equiv \nu_3 \pmod 2$ and $\nu_1 > \nu_2 > \nu_3$ then we can obtain the ν-tableau only from the corresponding λ-tableaux for $\lambda = (\nu_1 + 2, \nu_2, \nu_3), (\nu_1, \nu_2 + 2, \nu_3), (\nu_1, \nu_2, \nu_3 + 2)$. By Proposition 2.14 (ii) we conclude $m_\nu = 3$. If $\nu_1 = \nu_2 > \nu_3$

we have to exclude the case $\lambda = (\nu_1, \nu_2 + 2, \nu_3)$ because $\nu_1 < \nu_2 + 2$. The other cases $\nu_1 > \nu_2 = \nu_3$ and $\nu_1 = \nu_2 = \nu_3$ are handled in a similar way.

(ii) If $\nu_1 \equiv \nu_2 \not\equiv \nu_3 \pmod 2$ and $\nu_1 > \nu_2$, then we can obtain the ν-tableau from the two λ-tableaux for $\lambda = (\nu_1 + 1, \nu_2 + 1, \nu_3)$, i.e., $m_\nu = 2$. When $\nu_1 = \nu_2$ there is only one λ-tableau $\lambda = (\nu_1 + 1, \nu_2 + 1, \nu_3)$ when 4 and 5 are in the most right column of the λ-tableau.

The proofs of the other two cases (iii) and (iv) are similar. □

By (2), (3) and Lemma 3.1, we get the following corollary:

Corollary 3.4. *The multiplicities of the irreducible components $W_3(\nu)$ in the decomposition of*

$$F_3(M_3(K), so_3(K)) \cong \sum_\nu m_\nu(M_3(K), so_3(K)) W_3(\nu)$$

are bounded from above by the integers m_ν in Lemma 3.3.

We turn to a construction of highest weight vectors in $K\langle X_3\rangle$ that are linearly independent modulo $I(M_3(K), so_3(K))$. As we shall see, for almost all partitions ν, there exist as many of those as the upper bound m_ν in Corollary 3.4 for the multiplicity of $W_3(\nu)$ in $F_3(M_3(K), so_3(K))$.

For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3) \vdash n$, a permutation $\varrho \in S_n$, and for certain $q \in \{1, \ldots, n\}$ we define operations $\iota_{1q}, \iota_2, \iota_3$ on the highest weight vector $w(x_1, x_2, x_3) = w_\lambda^\varrho(x_1, x_2, x_3) \in K\langle X_3\rangle$ that produce highest weight vectors in the degree $n+2$, $n+3$ or $n+4$ homogeneous components of $K\langle X_3\rangle$:

- If $\lambda_1 > \lambda_2$ and the integer q is at the rth position in the first row of the tableau $T_\lambda(\varrho)$, and $r > \lambda_2$, then $w(x_1, x_2, x_3)$ has the form
$$w(x_1, x_2, x_3) = \sum \pm u' x_1 u'',$$
where the summation runs on some monomials u' and u'' of degree $q-1$ and $n-q$, respectively, and we define
$$\iota_{1q}(w(x_1, x_2, x_3)) = \sum \pm u' x_1^3 u'';$$
that is, $\iota_{1q}(w) = (w \cdot x_1^2)^\pi$, where
$$\pi^{-1} = \begin{pmatrix} 1 & \ldots & q & q+1 & q+2 & \ldots & n & n+1 & n+2 \\ 1 & \ldots & q & q+3 & q+4 & \ldots & n+2 & q+1 & q+2 \end{pmatrix}.$$
- Let $\tau = (2, 2)$ and let
$$\begin{aligned} w_{(2,2)}^{(2)}(x_1, x_2) &= \sum_{\sigma_1, \sigma_2 \in S_2} \text{sign}(\sigma_1)\text{sign}(\sigma_2) x_{\sigma_1(1)} x_{\sigma_2(1)} x_{\sigma_1(2)} x_{\sigma_2(2)} \\ &= x_1^2 x_2^2 - x_1 x_2^2 x_1 - x_2 x_1^2 x_2 + x_2^2 x_1^2 \end{aligned}$$
be the highest weight vector corresponding to the τ-tableau $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ (i.e. $w_{(2,2)}^{(2)} = w_\tau^\pi = ([x_1, x_2]^2)^\pi$ where π is the transposition $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$). Then we define
$$\iota_2(w(x_1, x_2, x_3)) = w(x_1, x_2, x_3) w_{(2,2)}^{(2)}(x_1, x_2).$$

- We define

$$\iota_3(w(x_1,x_2,x_3)) = w(x_1,x_2,x_3)s_3(x_1,x_2,x_3).$$

Lemma 3.5. (i) *Let* $\nu = (\nu_1,\nu_2,\nu_3) \vdash n$ *and let* $w(x_1,x_2,x_3) = w_\nu^\varrho(x_1,x_2,x_3)$ *be the highest weight vector corresponding to the permutation* $\varrho \in S_n$. *Then the polynomials* $\iota_{1q}(w(x_1,x_2,x_3)), \iota_2(w(x_1,x_2,x_3)), \iota_3(w(x_1,x_2,x_3))$ *are highest weight vectors of the form* w_μ^σ, *where* μ *is the partition* $(\nu_1+2,\nu_2,\nu_3), (\nu_1+2,\nu_2+2,\nu_3), (\nu_1+1,\nu_2+1,\nu_3+1)$, *respectively.*

(ii) *Let* $w^{(i)}(x_1,x_2,x_3) = w_\nu^{\varrho_i}(x_1,x_2,x_3)$, $\varrho_i \in S_n$, $i = 1,\dots,m$, *and let* $\{a_1,a_2,a_3\}$ *be the basis of* $so_3(K)$ *defined in the proof of Proposition 2.11. If the matrices* $w^{(i)}(a_1,a_2,a_3)$ *are linearly independent in* $M_3(K)$, *then the matrices of each set*

$$\{\iota_{1q_i}(w^{(i)}(a_1,a_2,a_3)) \mid i = 1,\dots,m\}, \quad \{\iota_2(w^{(i)}(a_1,a_2,a_3)) \mid i = 1,\dots,m\},$$

$$\{\iota_3(w^{(i)}(a_1,a_2,a_3)) \mid i = 1,\dots,m\}$$

are also linearly independent in $M_3(K)$.

Proof. (i) Applying ι_{1q} we insert x_1^2 between the q^{th} and $(q+1)^{\text{st}}$ positions of the monomials of $w(x_1,x_2,x_3)$. So $\iota_{1q}(w_\nu^\rho) = w_\mu^\psi$, where $\mu = (\nu_1+2,\nu_2,\nu_3)$ and the tableau $T_\mu(\psi)$ is obtained from the tableau $T_\nu(\rho)$ by adding 2 to each entry greater than q, and writing $q+1$, $q+2$ in the two new boxes at the end of the first row of the Young diagram of μ.

Hence $\iota_{1q}(w(x_1,x_2,x_3))$ is a highest weight vector corresponding to the partition (ν_1+2,ν_2,ν_3). Similarly, ι_2 multiplies $w(x_1,x_2,x_3)$ by a highest weight vector of weight $(2,2)$, thus $\iota_2(w(x_1,x_2,x_3))$ is a highest weight vector of weight (ν_1+2,ν_2+2,ν_3). Finally, ι_3 multiplies $w(x_1,x_2,x_3)$ by the standard polynomial $s_3(x_1,x_2,x_3)$ which is a highest weight vector of weight $(1,1,1)$, hence $\iota_3(w(x_1,x_2,x_3))$ is a highest weight vector with weight $(\nu_1+1,\nu_2+1,\nu_3+1)$. It is also clear that the resulting highest weight vectors are all of the form w_μ^σ for some partition μ and permutation σ.

(ii) Direct computations show that

$$a_1^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -a_1,$$

$$w_{(2,2)}^{(2)}(a_1,a_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad s_3(a_1,a_2,a_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

If the matrices $\iota_{1q_i}(w^{(i)}(a_1,a_2,a_3))$, $i = 1,\dots,m$, are linearly dependent, then the equality $\iota_{1q_i}(w^{(i)}(a_1,a_2,a_3)) = -w^{(i)}(a_1,a_2,a_3)$ implies the linear dependence for $w^{(i)}(a_1,a_2,a_3)$ which is a contradiction. Similarly, since the matrices $w_{(2,2)}^{(2)}(a_1,a_2)$ and $s_3(a_1,a_2,a_3)$ are invertible, the linear dependence of $\iota_2(w^{(i)}(a_1,a_2,a_3))$ and of $\iota_3(w^{(i)}(a_1,a_2,a_3))$, $i = 1,\dots,m$, gives the linear dependence of $w^{(i)}(a_1,a_2,a_3)$. □

Lemma 3.6. *For each of the following partitions* ν *the evaluations of the highest weight vectors* $w^{(i)}(a_1,a_2,a_3)$, $i = 1,\dots,m_\nu$, *are linearly independent if* $m_\nu > 1$ *and nonzero if* $m_\nu = 1$:

(i) *For* $\nu = (4,2)$ *and the* ν*-tableaux* $\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array}$, $\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6\\\cline{1-2}\end{array}$, *and* $\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5\\\cline{1-2}\end{array}$

$$w^{(1)} = [x_1, x_2]^2 x_1^2, \quad w^{(2)} = [x_1, x_2](x_1^3 x_2 - x_2 x_1^3),$$

$$w^{(3)} = (x_1^3 x_2^2 - x_1 x_2 x_1 x_2 x_1 - x_2 x_1^3 x_2 + x_2^2 x_1^3)x_1;$$

(ii) *For* $\nu = (2,2)$ *and the* ν*-tableaux* $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$ *and* $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$

$$w^{(1)} = [x_1, x_2]^2, \quad w^{(2)} = x_1^2 x_2^2 - x_1 x_2^2 x_1 - x_2 x_1^2 x_2 + x_2^2 x_1^2;$$

(iii) *For* $\nu = (3,1,1)$ *and the* ν*-tableaux* $\begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$ *and* $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1} 5\\\cline{1-1}\end{array}$

$$w^{(1)} = s_3(x_1, x_2, x_3)x_1^2, \quad w^{(2)} = \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_1^2 x_{\sigma(3)};$$

(iv) *For* $\nu = (0)$, $w^{(1)} = 1$*;*

(v) *For* $\nu = (3,1)$ *and the* ν*-tableaux* $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$ *and* $\begin{array}{|c|c|c|}\hline 3&1&2\\\hline 4\\\cline{1-1}\end{array}$

$$w^{(1)} = [x_1, x_2]x_1^2, \quad w^{(2)} = x_1^2[x_1, x_2];$$

(vi) *For* $\nu = (1,1)$ *and the* ν*-tableau* $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ $w^{(1)} = [x_1, x_2]$*;*

(vii) *For* $\nu = (2,1)$ *and the* ν*-tableaux* $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ *and* $\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}$

$$w^{(1)} = [x_1, x_2]x_1, \quad w^{(2)} = x_1[x_1, x_2];$$

(viii) *For* $\nu = (3,2)$ *and the* ν*-tableaux* $\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$ *and* $\begin{array}{|c|c|c|}\hline 2&4&1\\\hline 3&5\\\cline{1-2}\end{array}$

$$w^{(1)} = [x_1, x_2]^2 x_1, \quad w^{(2)} = x_1[x_1, x_2]^2;$$

(ix) *For* $\nu = (1)$ *and the* ν*-tableau* $\begin{array}{|c|}\hline 1\\\hline\end{array}$ $w^{(1)} = x_1$*.*

Proof. Direct computations show that:
(i) For $\nu = (4,2)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$w^{(3)}(a_1, a_2, a_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

(ii) For $\nu = (2,2)$

$$w^{(1)}(a_1, a_2, a_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad w^{(2)}(a_1, a_2, a_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

(iii) For $\nu=(3,1,1)$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}-2&0&0\\0&-2&0\\0&0&0\end{pmatrix},\quad w^{(2)}(a_1,a_2,a_3)=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-2\end{pmatrix};$$

(iv) For $\nu=(0)$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};$$

(v) For $\nu=(3,1)$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&-1&0\end{pmatrix},\quad w^{(2)}(a_1,a_2,a_3)=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix};$$

(vi) For $\nu=(1,1)$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix};$$

(vii) For $\nu=(2,1)$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}0&0&0\\0&0&0\\-1&0&0\end{pmatrix},\quad w^{(2)}(a_1,a_2,a_3)=\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix};$$

(viii) For $\nu=(3,2)$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad w^{(2)}(a_1,a_2,a_3)=\begin{pmatrix}0&-1&0\\0&0&0\\0&0&0\end{pmatrix};$$

(ix) For $\nu=(1)$ and the ν-tableau $\boxed{1}$

$$w^{(1)}(a_1,a_2,a_3)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

In all nine cases the matrices $w^{(i)}(a_1,a_2,a_3)$ are linearly independent if $m_\nu>1$ and nonzero if $m_\nu=1$. □

Now we state the main result of this section.

Theorem 3.7. *Let K be a field of characteristic 0 and let $I(M_3(K),so_3(K))$ be the ideal of the weak polynomial identities for the pair $(M_3(K),so_3(K))$. Then the cocharacter sequence of $I(M_3(K),so_3(K))$ is*

$$\chi_n(M_3(K),so_3(K))=\sum_{\nu\vdash n}m_\nu(M_3(K),so_3(K))\chi_\nu,\quad \nu=(\nu_1,\nu_2,\nu_3),$$

where the multiplicity $m_\nu(M_3(K),so_3(K))$ equals to m_ν from Lemma 3.3 for $\nu\notin\{(2k,0,0)\mid k=1,2,\dots\}$, whereas $m_{(2k,0,0)}(M_3(K),so_3(K))=1$.

Proof. The multiplicities of the cocharacter sequence $\chi_n(M_3(K), so_3(K))$ are determined by the structure of the relatively free algebra $F_3(M_3(K), so_3(K))$ as a $\mathrm{GL}_3(K)$-module and we shall work with the representations of $\mathrm{GL}_3(K)$.

The case $\nu = (n)$ is trivial because the multiplicity of $W_3(n)$ in the free associative algebra $K\langle X_3\rangle$ is equal to 1 and the generator x_1^n of $W_3(n)$ does not vanish in $(M_3(K), so_3(K))$. Hence we may assume that $\nu_2 > 0$. By Corollary 3.4 the multiplicities $m_\nu = m_\nu(M_3(K), so_3(K))$ are bounded from above by the multiplicities stated in the theorem. Hence it is sufficient to show that for a given ν in $F_3(M_3(K), so_3(K))$ there exist at least m_ν linearly independent highest weight vectors $w^{(i)}(x_1, x_2, x_3)$, $i = 1, \dots, m_\nu$.

Let $\nu_1 \equiv \nu_2 \equiv \nu_3 \pmod 2$ and $\nu_1 > \nu_2 > \nu_3$. Hence

$$\nu = (\nu_3 + 2r_2 + 2r_1, \nu_3 + 2r_2, \nu_3), \quad r_1, r_2 > 0.$$

By Lemma 3.6 for the partition $(4, 2)$ there exist three highest weight vectors $w^{(i)}(x_1, x_2, x_3) \in F_3(M_3(K), so_3(K))$, $i = 1, 2, 3$, with linearly independent evaluations $w^{(i)}(a_1, a_2, a_3)$, $i = 1, 2, 3$. Applying to them $r_1 - 1$ times suitable operations ι_{1q_i}, $r_2 - 1$ times the operation ι_2, and ν_3 times the operation ι_3, by Lemma 3.5 we obtain three highest weight vectors for the partition ν which are linearly independent in $F_3(M_3(K), so_3(K))$. Similarly, if $\nu_1 = \nu_2 > \nu_3$, i.e., $r_1 = 0$, $r_2 > 0$, we have two highest weight vectors corresponding to the partition $(2, 2)$ and apply to them $r_2 - 1$ times the operation ι_2 and ν_3 times the operation ι_3. All other cases are handled in a similar way. □

Theorem 3.7 gives also the cocharacter sequence of $I(M_3(K), \mathrm{ad}(sl_2(K)))$, thanks to the following:

Proposition 3.8. *The cocharacter sequence of the ideal of weak polynomial identities of the pair $(M_3(K), \mathrm{ad}(sl_2(K)))$ is the same as the cocharacter sequence of $I(M_3(K), so_3(K))$.*

Proof. Over the algebraic closure $\bar{K}$ of K, the pair $(M_3(\bar{K}), so_3(\bar{K}))$ is isomorphic to the pair $(M_3(\bar{K}), \mathrm{ad}(sl_2(\bar{K})))$, and as is well known, the cocharacter sequence does not change on extending the characteristic zero base field. □

Remark 3.9. The cocharacter sequence of the pair $(M_2(K), sl_2(K))$ was found by Procesi [P2], see also [Dr2, Exercise 12.6.12]:

$$\chi_n(M_2(K), sl_2(K)) = \sum_{\lambda \vdash n} \chi_\lambda, \quad \lambda = (\lambda_1, \lambda_2, \lambda_3),\ n = 0, 1, 2, \dots.$$

This is multiplicity free, unlike the cocharacter sequence of $I(M_3(\bar{K}), \mathrm{ad}(sl_2(\bar{K})))$ given in Theorem 3.7.

Problem 3.10. *Let $\varrho : sl_2(K) \to End_K(V_q) \cong M_q(K)$ be a q-dimensional irreducible representation of $sl_2(K)$, $q > 2$ (or $q = \infty$). Find the cocharacter sequence of the pair $(M_q(K), \varrho(sl_2(K)))$.*

4. The difference between $\mathcal{E}_p$ and $\mathcal{F}_p$

Denote by $\kappa : K\langle X_p\rangle \to \mathcal{F}_p$ the K-algebra surjection with $x_k \mapsto t_k$, $k = 1, \dots, p$. Clearly $\ker(\kappa) = I(M_3(K), so_3(K)) \cap K\langle X_p\rangle$. We can therefore reformulate Theorem 3.7 as follows:

Theorem 4.1. *For $p \geq 3$ we have the* $\mathrm{GL}_p(K)$*-module isomorphism*

$$\mathcal{F}_p \cong \bigoplus_{n=0}^{\infty} \bigoplus_{\nu=(\nu_1,\nu_2,\nu_3)\vdash n} m_\nu(M_3(K), so_3(K)) W_p(\nu),$$

where the multiplicities $m_\nu(M_3(K), so_3(K))$ are given in Theorem 3.7. For $p < 3$ the summands labeled by partitions ν with more than p non-zero parts have to be removed from the formula.

Theorem 4.2. (i) *The* $\mathrm{GL}_p(K)$*-module $\mathcal{E}_p$ decomposes as*

$$\mathcal{E}_p \cong \sum_{\nu=(\nu_1,\nu_2,\nu_3)} m_\nu W_p(\nu),$$

where the value of m_ν is the same as in Lemma 3.3 for $p \geq 3$; when $p < 3$, the summands labeled by partitions with more than p non-zero parts are removed.

(ii) *For all p we have*

$$\mathcal{E}_p = \mathcal{F}_p \oplus \bigoplus_{k=1}^{\infty} \langle \mathrm{tr}(t_1^{2k}) I \rangle_{\mathrm{GL}_p(K)}$$

(where I is the 3×3 identity matrix).

Proof. Statement (i) follows from Corollary 2.8, Proposition 2.7 (ii), Lemma 3.1 and Lemma 3.3. Combining statement (i) with Theorem 4.1 we get that the factor space $\mathcal{E}_p/\mathcal{F}_p$ decomposes as

$$\mathcal{E}_p/\mathcal{F}_p \cong \bigoplus_{k=1}^{\infty} W_p((2k)).$$

The only highest weight vector in $\mathcal{F}_p$ with weight $(2k)$ is t_1^{2k}. In $\mathcal{E}_p$ we have also the highest weight vector $\mathrm{tr}(t_1^{2k})I$ with weight $(2k)$, and these two highest weight vectors are linearly independent over K, as one can easily see by making the substitution $t_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. □

Remark 4.3. (i) The Cayley-Hamilton theorem gives that

$$t_1^3 - \frac{1}{2}\mathrm{tr}(t_1^2)t_1 = 0 \text{ and hence } t_1^4 = \frac{1}{2}\mathrm{tr}(t_1^2)t_1^2.$$

This easily implies that

$$\mathrm{tr}(t_1^{2k}) = \frac{1}{2^{k-1}}\mathrm{tr}^k(t_1^2) = 2(-1)^k((t_{12}^{(1)})^2 + (t_{13}^{(1)})^2 + (t_{23}^{(1)})^2)^k, \quad k \geq 1.$$

(ii) The $\mathrm{GL}_p(K)$-module structure of the algebra of $\mathrm{GL}_2(K)$-equivariant polynomial maps $sl_2(K)^{\oplus p} \to M_2(K)$, where $\mathrm{GL}_2(K)$ acts by (simultaneous) conjugation, is determined in [P2, Theorem 2.2]. This turns out to be multiplicity free, unlike our $\mathcal{E}_p$.

(iii) It follows from Theorem 4.2 (ii) that $\mathrm{tr}(t_1t_2)t_3$ can be expressed as a K-linear combination of monomials in t_1, t_2, t_3. Indeed, the explicit identity of this form is

$$\mathrm{tr}(t_1t_2) \cdot t_3 = t_1t_2t_3 - t_2t_3t_1 + t_3t_1t_2 + t_2t_1t_3 + t_3t_2t_1 - t_1t_3t_2.$$

We close this section with a description of the center $C(\mathcal{E}_p)$ and $C(\mathcal{F}_p)$ of the algebra $\mathcal{E}_p$ and $\mathcal{F}_p$.

Proposition 4.4. *For $p \geq 2$, the algebra $C(\mathcal{E}_p)$ is isomorphic to the generic trace algebra $K[T_p]^{SO_3(K)}$.*

Proof. Denote by $\mathbb{F}$ the field of fractions of $K[T_p]$. Let $a_1, a_3 \in so_3(K)$ be the matrices introduced in the proof of Proposition 2.11. Then I, a_1, a_3, a_1^2, a_3^2, a_1a_3, a_3a_1, $a_1^2a_3$, $a_3^2a_1$ are linearly independent over K. It follows that I, t_1, t_2, t_1^2, t_2^2, t_1t_2, t_2t_1, $t_1^2t_2$, $t_2^2t_1$ are linearly independent over $\mathbb{F}$ in $M_3(\mathbb{F})$; indeed, otherwise we could arrange the entries of the above 9 matrices into a 9×9 matrix whose determinant would be the zero element of $K[T_p]$, contrary to the fact that the substitution $t_1 \mapsto a_1$, $t_2 \mapsto a_3$ in this polynomial gives a non-zero value. So the above 9 monomials in t_1, t_2 constitute an $\mathbb{F}$-vector space basis of $M_3(\mathbb{F})$. Take any element $c \in C(\mathcal{E}_p)$. The centralizer of c in $M_3(\mathbb{F})$ contains t_1 and t_2, hence it contains the above $\mathbb{F}$-vector space basis of $M_3(\mathbb{F})$. Consequently, c is central in $M_3(\mathbb{F})$, and thus c is a scalar matrix. So $c = f \cdot I$ for some $f \in K[T_p]$. Taking into account that c gives an $\mathrm{SO}_3(K)$-equivariant map from $so_3(K)^{\oplus p} \to M_3(K)$, we get that f is an element of the generic trace algebra. □

Corollary 4.5. *(i) As a $\mathrm{GL}_p(K)$-module ($p \geq 2$), $C(\mathcal{E}_p)$ decomposes as*

$$C(\mathcal{E}_p) \cong \bigoplus W_p(\lambda), \tag{4}$$

where the summation runs on all $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_1 - \lambda_2 \equiv \lambda_2 - \lambda_3 \equiv 0$ (mod 2) for $p \geq 3$; for $p < 3$ the terms corresponding to partitions with more than p non-zero parts should be omitted.

(ii) For $p \geq 2$ the center $C(\mathcal{F}_p)$ of $\mathcal{F}_p$ is the direct sum of all $W_p(\lambda)$ from (4) such that $\lambda_2 > 0$.

Proof. Statement (i) follows from Proposition 4.4 and Corollary 2.16. Statement (ii) follows from (i) and Theorem 4.2 (ii). □

By analogy with the notion of a weak polynomial identity one may define *weak central polynomials for the pair (R, L)* as elements of the free algebra $K\langle X\rangle$ which take central values in R when evaluated on L. Denote by $\chi_n^{\mathrm{c}}(R, L)$ the S_n-character of the factor space $P_n^{\mathrm{c}}/(P_n^{\mathrm{c}} \cap I(R, L))$, where P_n^{c} is the space of multilinear weak central polynomials for the pair (R, L), and call $\chi_n^{\mathrm{c}}(R, L)$ *the central cocharacter sequence for the pair (R, L)*. We can restate the structure of $C(\mathcal{F}_p)$ as a $\mathrm{GL}_p(K)$-module in the language of central cocharacter sequence as follows:

Theorem 4.6.

$$\chi_n^c(M_3(K), so_3(K)) = \sum_{\lambda \vdash n} \chi_\lambda,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_2 > 0$, and $\lambda_1 - \lambda_2 \equiv \lambda_2 - \lambda_3 \equiv 0$ (mod 2). Moreover, writing λ in the form

$$\lambda = (2(\mu_1 + \mu_2) + \lambda_3, 2\mu_2 + \lambda_3, \lambda_3), \text{ where } \mu_2 + \lambda_3 > 0,$$

the corresponding highest weight vector is

$$s_3^{\lambda_3}(x_1, x_2, x_3)([x_1, x_2]^2 - \frac{1}{3}(x_1^2x_2^2 - x_1x_2^2x_1 - x_2x_1^2x_2 + x_2^2x_1^2))^{\mu_2 - 1} w'_{\mu_1}(x_1, x_2),$$

$$w'_{\mu_1}(x_1, x_2) = [x_1, x_2]^2 x_1^{2\mu_1} - [x_1, x_2](x_1^{2\mu_1+1}x_2 - x_2x_1^{2\mu_1+1})$$

$$+(x_1^3x_2^2 - x_1x_2x_1x_2x_1 - x_2x_1^3x_2 + x_2^2x_1^3)x_1^{2\mu_1-1},\ \textit{if } \mu_1,\mu_2>0;$$

$$s_3^{\lambda_3-1}(x_1,x_2,x_3)w''_{\mu_1}(x_1,x_2),$$

$$w''_{\mu_1}(x_1,x_2) = s_3(x_1,x_2,x_3)x_1^{2\mu_1}+2\sum_{\sigma\in S_3}\mathrm{sign}(\sigma)x_{\sigma(1)}x_{\sigma(2)}x_1^{2\mu_1}x_{\sigma(3)},\ \textit{if } \mu_1>0,\mu_2=0;$$

$$s_3^{\lambda_3}(x_1,x_2,x_3)([x_1,x_2]^2-\frac{1}{3}(x_1^2x_2^2-x_1x_2^2x_1-x_2x_1^2x_2+x_2^2x_1^2))^{\mu_2},\ \textit{if } \mu_1=0.$$

Proof. The statement on the central cocharacter sequence is a reformulation of Corollary 4.5 (ii). For each summand χ_λ of the central cocharacter the statement gives a multihomogeneous element of $K\langle X_3\rangle$ of $\mathbb{N}_0^3$-degree λ, moreover, this element is easily seen to be a highest weight vector (by the explanations in Section 2.3). It remains to show that they are weak central polynomials for the pair $(M_3(K),so_3(K))$. This holds for $s_3(x_1,x_2,x_3)$ by Theorem 2.2. One can check by direct computation (for example by substituting $x_i\mapsto t_i$) that $[x_1,x_2]^2-\frac{1}{3}(x_1^2x_2^2-x_1x_2^2x_1-x_2x_1^2x_2+x_2^2x_1^2)$ is a weak central polynomial for $(M_3(K),so_3(K))$, and that w'_1,w''_1 are weak central polynomials for $(M_3(K),so_3(K))$. Given that, we claim that for any $\mu_1>0$, $w'_{\mu_1}(b_1,b_2)$ is a scalar matrix for any $b_1,b_2\in so_3(K)$. It is sufficient to show this in the special case when $K=\mathbb{R}$, the field of real numbers. Moreover, the adjoint $\mathrm{SO}_3(\mathbb{R})$-orbit of b_1 contains a scalar multiple of the matrix a_1 (introduced in the proof of Proposition 2.11. For any $g\in\mathrm{SO}_3(\mathbb{R})$ we have $w'_{\mu_1}(gb_1g^{-1},gb_2g^{-1})=gw'_{\mu_1}(b_1,b_2)g^{-1}$. Thus (taking into account the homogeneity of w'_{μ_1} in x_1) we get that it is sufficient to show that $w'_{\mu_1}(a_1,b_2)$ is a scalar matrix. Inspection of the explicit form of $w'_{\mu_1}(x_1,x_2)$ shows that the equality $a_1^3=-a_1$ implies that for $\mu_1>0$ we have $w'_{\mu_1+1}(a_1,b_2)=-w'_{\mu_1}(a_1,b_2)$. Since $w'_1(a_1,b_2)$ is a scalar matrix, we conclude that $w'_{\mu_1}(a_1,b_2)$ is a scalar matrix for all $\mu_1>0$. Similar argument works for w''_{μ_1}. □

5. The module of covariants

We saw above (cf. Corollary 2.8) that to a large extent, the analysis of $\mathcal{E}_p$ for arbitrary p can be reduced to the special case $p=3$. Our aim is to describe $\mathcal{E}_3$ as a module over the ring $K[T_3]^{\mathrm{SO}_3(K)}$.

We set

$$C:=\iota(\mathcal{E}_3)=(K[T_3,Z]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)},\qquad \text{where } \iota \text{ is defined in Proposition 2.7.}$$

As a special case of the Clebsch-Gordan rules, the space of 3×3 matrices has the decomposition

$$M_3(K)=KI\oplus so_3(K)\oplus M_3(K)_0^+$$

as a direct sum of irreducible $\mathrm{SO}_3(K)$-invariant subspaces, where I is the identity matrix and $M_3(K)_0^+$ is the space of trace zero symmetric 3×3 matrices. Accordingly we have the decomposition

$$C=C_1\oplus C_2\oplus C_3 \tag{5}$$

into a direct sum of $K[so_3(K)^{\oplus 3}]^{\mathrm{SO}_3(K)}$-submodules (that are also $\mathrm{GL}_3(K)$-submodules) of C. Namely

$$C_1=K[T_3]^{\mathrm{SO}_3(K)}\cdot\mathrm{tr}(z),$$

$$C_2=(K[T_3,Z^-]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)},$$

and

$$C_3\cong(K[T_3,Z_0^+]^{\mathrm{SO}_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)},$$

where

$$Z^- = \{u_{ij} := \frac{1}{2}(z_{ij} - z_{ji}) \mid 1 \le i < j \le 3\}, \tag{6}$$

$$Z_0^+ = \{s_{ij} := \frac{1}{2}(z_{ij} + z_{ji}),\ s_{kk} := z_{kk} - \frac{1}{3}(z_{11} + z_{22} + z_{33}) \mid 1 \le i \le j \le 3,\ k = 1,2,3\},$$

and $(\mathbb{N}_0, \mathbb{N}_0, \mathbb{N}_0, 1)$ in the exponents above indicates that we take the component consisting of the polynomials that have total degree one in the variables belonging to Z.

Denote by P the subalgebra of $K[T_3]^{\mathrm{SO}_3(K)}$ generated by $\mathrm{tr}(t_1^2)$, $\mathrm{tr}(t_2^2)$, $\mathrm{tr}(t_3^2)$, $\mathrm{tr}(t_1t_2)$, $\mathrm{tr}(t_1t_3)$, $\mathrm{tr}(t_2t_3)$. Note that the six generators are algebraically independent over K. The algebra $K[T_3]^{\mathrm{SO}_3(K)}$ is a free P-module of rank two, generated by 1 and $\mathrm{tr}(t_1t_2t_3)$ (see Corollary 2.12 (ii)). It follows that the $\mathbb{N}_0^3$-graded Hilbert series (or in other words, the formal $\mathrm{GL}_3(K)$-character of C_1) is

$$H(C_1; \tau_1, \tau_2, \tau_3) = \frac{1 + \tau_1\tau_2\tau_3}{\prod_{1\le i\le j\le 3}(1 - \tau_i\tau_j)}. \tag{7}$$

Proposition 5.1. *The $\mathbb{N}_0^3$-graded Hilbert series of C_2 and C_3 are the following:*

$$H(C_2; \tau_1, \tau_2, \tau_3) = \frac{(S_{(1)} + S_{(1,1)})(\tau_1, \tau_2, \tau_3)}{\prod_{1\le i\le j\le 3}(1 - \tau_i\tau_j)} \tag{8}$$

$$H(C_3; \tau_1, \tau_2, \tau_3) = \frac{(S_{(2)} + S_{(2,1)} - S_{(2,2,1)} - S_{(2,2,2)})(\tau_1, \tau_2, \tau_3)}{\prod_{1\le i\le j\le 3}(1 - \tau_i\tau_j)} \tag{9}$$

where

$$S_{(1)}(\tau_1, \tau_2, \tau_3) = \tau_1 + \tau_2 + \tau_3,$$
$$S_{(2)}(\tau_1, \tau_2, \tau_3) = \sum_{1\le i\le j\le 3} \tau_i\tau_j,$$
$$S_{(1,1)}(\tau_1, \tau_2, \tau_3) = \sum_{1\le i<j\le 3} \tau_i\tau_j,$$
$$S_{(2,1)}(\tau_1, \tau_2, \tau_3) = \sum_{i\ne j} \tau_i^2\tau_j + 2\tau_1\tau_2\tau_3,$$
$$S_{(2,2,1)}(\tau_1, \tau_2, \tau_3) = \tau_1\tau_2\tau_3 S_{(1,1)}(\tau_1, \tau_2, \tau_3),$$
$$S_{(2,2,2)}(\tau_1, \tau_2, \tau_3) = S_{(1,1,1)}(\tau_1, \tau_2, \tau_3)^2 = (\tau_1\tau_2\tau_3)^2$$

are Schur polynomials (the formal characters of the $\mathrm{GL}_3(K)$*-modules* $W_3(1)$*,* $W_3(2)$*,* $W_3(1,1)$*,* $W_3(2,1)$*,* $W_3(2,2,1)$*,* $W_3(2,2,2)$*).*

Proof. It is well known that the Hilbert series in question are independent of the characteristic zero base field K. Therefore we may assume $K = \mathbb{C}$, the field of complex numbers. View the $SO_3(\mathbb{C})$-module $\mathbb{C}[T_3, Z_0^+]$ as an $SL_2(\mathbb{C})$-module via the natural surjection $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$ with kernel consisting of the 2×2 identity matrix and its negative. The maximal compact subgroup $SU_2(\mathbb{C})$ (the special unitary group) of $SL_2(\mathbb{C})$ has the same subspace of invariants in $\mathbb{C}[T_3, Z_0^+]$ as $SL_2(\mathbb{C})$. We compute the Hilbert series of $C_3 = (\mathbb{C}[T_3, Z_0^+]^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)})^{SU_2(\mathbb{C})}$ using standard methods. Namely, it can be expressed by the Molien-Weyl formula and the Weyl integration formula as an integral over a maximal torus of $SU_2(\mathbb{C})$ as follows. Consider the maximal torus $\mathbb{T} = \{\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \mid |\rho| = 1\}$ in $SU_2(\mathbb{C})$. The character of the multihomogenous components of $\mathbb{C}[T_3, Z_0^+]^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)}$ as a $\mathbb{T}$-module is given by the series

$$\frac{\rho^4+\rho^2+1+\rho^{-2}+\rho^{-4}}{\prod_{j=1}^3(1-\rho^2\tau_j)(1-\tau_j)(1-\rho^{-2}\tau_j)}.$$

The roots of $SU_2(\mathbb{C})$ are ρ^2 and ρ^{-2}, and the order of the Weyl group is 2. Therefore the Molien-Weyl formula combined with the Weyl integration formula yields

$$\begin{aligned}H(C_3;\tau_1,\tau_2,\tau_3) &= \frac{1}{2}\int_{|\rho|=1}\frac{(\rho^4+\rho^2+1+\rho^{-2}+\rho^{-4})(1-\rho^2)(1-\rho^{-2})}{\prod_{j=1}^3(1-\rho^2\tau_j)(1-\tau_j)(1-\rho^{-2}\tau_j)}\frac{\mathrm{d}\rho}{2\pi\mathrm{i}\rho}\\ &= \frac{1}{2}\cdot\frac{1}{2\pi\mathrm{i}}\int_{|\rho|=1}\frac{-\rho^{12}+\rho^{10}+\rho^2-1}{\rho\prod_{j=1}^3(1-\rho^2\tau_j)(1-\tau_j)(\rho^2-\tau_j)}\mathrm{d}\rho.\end{aligned}$$

The above integral can be evaluated by residue calculus. Suppose that τ_1,τ_2,τ_3 are non-zero complex numbers of absolute value less than 1. Then the integrand has poles inside the unit circle at $\rho=\pm\sqrt{\tau_k}$, $k=1,2,3$, and at $\rho=0$. The residue at $\pm\sqrt{\tau_k}$ is

$$\frac{-\tau_k^6+\tau_k^5+\tau_k-1}{2\tau_k(1-\tau_k^2)(1-\tau_k)\prod_{j\in\{1,2,3\}\setminus\{k\}}(1-\tau_k\tau_j)(1-\tau_j)(\tau_k-\tau_j)},$$

whereas the residue of the integrand at $\rho=0$ is

$$\frac{1}{\prod_{j=1}^3(1-\tau_j)\tau_j}.$$

It follows that

$$H(C_3;\tau_1,\tau_2,\tau_3)=\frac{1}{2}\left(\frac{1}{\prod_{j=1}^3(1-\tau_j)\tau_j}+ 2\sum_{k=1}^3\frac{-\tau_k^6+\tau_k^5+\tau_k-1}{2\tau_k(1-\tau_k^2)(1-\tau_k)\prod_{j\in\{1,2,3\}\setminus\{k\}}(1-\tau_k\tau_j)(1-\tau_j)(\tau_k-\tau_j)}\right).$$

Bringing to common denominator the summands on the right hand side and after some cancellations we obtain (9).

Similarly,

$$\begin{aligned}H(C_2;\tau_1,\tau_2,\tau_3) &= \frac{1}{2}\int_{|\rho|=1}\frac{(\rho^2+1+\rho^{-2})(1-\rho^2)(1-\rho^{-2})}{\prod_{j=1}^3(1-\rho^2\tau_j)(1-\tau_j)(1-\rho^{-2}\tau_j)}\frac{\mathrm{d}\rho}{2\pi\mathrm{i}\rho}\\ &= \frac{1}{2}\cdot\frac{1}{2\pi\mathrm{i}}\int_{|\rho|=1}\frac{-\rho^9+\rho^7+\rho^3-\rho}{\prod_{j=1}^3(1-\rho^2\tau_j)(1-\tau_j)(\rho^2-\tau_j)}\mathrm{d}\rho\\ &= \frac{1}{2}\sum_{k=1}^3\frac{-\tau_k^4+\tau_k^3+\tau_k-1}{(1-\tau_k^2)(1-\tau_k)\prod_{j\in\{1,2,3\}\setminus\{k\}}(1-\tau_k\tau_j)(1-\tau_j)(\tau_k-\tau_j)},\end{aligned}$$

from which one gets (8) after bringing the summands to common denominator and cancelling certain factors. □

Remark 5.2. It would be possible to derive Theorem 4.2 (i) (giving the multiplicities of the irreducible $\mathrm{GL}_p(K)$-module summands in $\mathcal{E}_p$) using Proposition 5.1 and Corollary 2.8.

Proposition 5.3. *Write s for the symmetric trace zero matrix whose entries above and in the diagonal are s_{ij}, $1 \leq i \leq j \leq 3$, and write u for the skew-symmetric matrix whose entries above the diagonal are u_{ij}, $1 \leq i < j \leq 3$ (where s_{ij}, u_{ij} were introduced in* (6)*).* (i) $\mathrm{tr}(t_1u)$ *is a highest weight vector in the* $\mathrm{GL}_3(K)$*-module* C_2 *generating a* $\mathrm{GL}_3(K)$*-submodule isomorphic to* $W_3(1)$*.*

(ii) $\mathrm{tr}(t_1t_2u)$ *is a highest weight vector in the* $\mathrm{GL}_3(K)$*-module* C_2 *generating a* $\mathrm{GL}_3(K)$*-submodule isomorphic to* $W_3(1,1)$*.*

(iii) $\mathrm{tr}(t_1^2s)$ *is a highest weight vector in the* $\mathrm{GL}_3(K)$*-module* C_3 *generating a* $\mathrm{GL}_3(K)$*-submodule isomorphic to* $W_3(2)$*.*

(iv) $\mathrm{tr}([t_1^2,t_2]s)$ *is a highest weight vector in the* $\mathrm{GL}_3(K)$*-module* C_3 *generating a* $\mathrm{GL}_3(K)$*-submodule isomorphic to* $W_3(2,1)$*.*

Proof. The map $K\langle X_3\rangle \to K[T_3, Z_0^+]$, $f(x_1,x_2,x_3) \mapsto \mathrm{tr}(f(t_1,t_2,t_3)s)$ is a $\mathrm{GL}_3(K)$-module homomorphism. As explained in Section 2.3, x_1^2 is a highest weight vector in $K\langle X_3\rangle^{(2)}$ generating a $\mathrm{GL}_3(K)$-submodule isomorphic to $W_3(2)$, whereas $[x_1^2,x_2]$ is a highest weight vector in $K\langle X_3\rangle^{(3)}$ generating a $\mathrm{GL}_3(K)$-submodule isomorphic to $W_3(2,1)$. Since the images of these highest weight vectors in C_3 are non-zero, they are also highest weight vectors as required. Similarly, the map $K\langle X_3\rangle \to K[T_3, Z^-]$, $f(x_1,x_2,x_3) \mapsto \mathrm{tr}(f(t_1,t_2,t_3)u)$ is a $\mathrm{GL}_3(K)$-module homomorphism. Now $x_1 \in K\langle X_3\rangle^{(1)}$ is a highest weight vector generating a $\mathrm{GL}_3(K)$-module isomorphic to $W_3(1)$. Also $\frac{1}{2}(x_1x_2 - x_2x_1) \in K\langle X_3\rangle^{(2)}$ is a highest weight vector generating a $\mathrm{GL}_3(K)$-module isomorphic to $W_3(1,1)$, and its image in C_2 is $\mathrm{tr}(t_1t_2u)$, since $t_1t_2 + t_2t_1$ is a symmetric matrix, hence $\mathrm{tr}((t_1t_2+t_2t_1)u) = 0$. □

The $\mathrm{GL}_3(K)$-submodule $\langle \mathrm{tr}(t_1u)\rangle_{\mathrm{GL}_3(K)}$ generated by $\mathrm{tr}(t_1u)$ has the K-vector space basis $\{\mathrm{tr}(t_1u), \mathrm{tr}(t_2u), \mathrm{tr}(t_3u)\}$, and the $\mathrm{GL}_3(K)$-submodule $\langle \mathrm{tr}(t_1t_2u)\rangle_{\mathrm{GL}_3(K)}$ generated by $\mathrm{tr}(t_1t_2u)$ has the K-vector space basis $\{\mathrm{tr}(t_1t_2u), \mathrm{tr}(t_1t_3u), \mathrm{tr}(t_2t_3u)\}$.

Proposition 5.4. C_2 *is a rank* 6 *free* P*-module generated by* $\mathrm{tr}(t_1u)$, $\mathrm{tr}(t_2u)$, $\mathrm{tr}(t_3u)$, $\mathrm{tr}(t_1t_2u)$, $\mathrm{tr}(t_1t_3u)$, $\mathrm{tr}(t_2t_3u)$*.*

Proof. The fact that the above 6 elements generate C_2 as a $K[T_3]^{\mathrm{SO}_3(K)}$-module is an immediate consequence of Corollary 2.12. The following two relations hold by Theorem 2.10 and the proof of Proposition 2.11. They (together with relations obtained by permuting t_1,t_2,t_3) show that the 6 elements in the statement in fact generate C_2 as a P-module:

$$\mathrm{tr}(t_1t_2t_3)\mathrm{tr}(t_1u) = \mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3u) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3u) + \mathrm{tr}(t_1t_3)\mathrm{tr}(t_1t_2u) \tag{10}$$

$$\begin{aligned} \mathrm{tr}(t_1t_2t_3)\mathrm{tr}(t_1t_2u) &= \frac{1}{8}\left(\mathrm{tr}(t_1t_3)\mathrm{tr}(t_2^2) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_2t_3)\right)\mathrm{tr}(t_1u) \\ &+ \frac{1}{8}\left(\mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3)\right)\mathrm{tr}(t_2u) \\ &+ \frac{1}{8}\left(\mathrm{tr}(t_1t_2)^2 - \mathrm{tr}(t_1^2)\mathrm{tr}(t_2^2)\right)\mathrm{tr}(t_3u) \end{aligned} \tag{11}$$

Therefore denoting by $e_1,\dots,e_6$ the standard generators of the free P-module $P^{\oplus 6}$, we have a P-module surjection

$$\begin{aligned} \mu : P^{\oplus 6} \to C_2,\ & e_1 \mapsto \mathrm{tr}(t_1u),\ e_2 \mapsto \mathrm{tr}(t_2u),\ e_3 \mapsto \mathrm{tr}(t_3u), \\ & e_4 \mapsto \mathrm{tr}(t_1t_2u),\ e_5 \mapsto \mathrm{tr}(t_1t_3u),\ e_6 \mapsto \mathrm{tr}(t_2t_3u). \end{aligned}$$

This is a homomorphism of graded P-modules, where we endow $P^{\oplus 6}$ with the grading given by $\deg(e_1) = \deg(e_2) = \deg(e_3) = 1$ and $\deg(e_4) = \deg(e_5) = \deg(e_6) = 2$, and C_2 is endowed with the standard grading coming from the action of the subgroup of scalar matrices in $\mathrm{GL}_3(K)$. The Hilbert series of $P^{\oplus 6}$ is $\frac{3\tau+3\tau^2}{(1-\tau^2)^6}$, and by Proposition 5.1 this agrees with the Hilbert series of C_2. It follows that μ is an isomorphism. $\square$

The $\mathrm{GL}_3(K)$-submodule $\langle \mathrm{tr}(t_1^2 s)\rangle_{\mathrm{GL}_3(K)}$ generated by $\mathrm{tr}(t_1^2 s)$ has the basis

$$e_{ij} := \mathrm{tr}(t_i t_j s),\ 1 \leq i \leq j \leq 3. \tag{12}$$

The $\mathrm{GL}_3(K)$-submodule $\langle \mathrm{tr}([t_1^2, t_2]s)\rangle_{\mathrm{GL}_3(K)}$ generated by $\mathrm{tr}([t_1^2, t_2]s)$ has the basis

$$\begin{gathered} e_{iij} := \mathrm{tr}([t_i^2, t_j]s),\ i \neq j \in \{1,2,3\}, \\ e_{132} := \mathrm{tr}([t_1t_3 + t_3t_1, t_2]s),\ e_{123} := \mathrm{tr}([t_1t_2 + t_2t_1, t_3]s). \end{gathered} \tag{13}$$

Theorem 5.5. (i) *As a P-module, C_3 is generated by*

$$\{e_{iij}, e_{132}, e_{123}, e_{kl} \mid i \neq j \in \{1,2,3\},\ 1 \leq k \leq l \leq 3\}.$$

Moreover, it has the direct sum decomposition

$$C_3 = C_3^{(0)} \oplus C_3^{(1)}, \quad \text{where} \quad C_3^{(0)} = P \cdot \langle e_{11}\rangle_{\mathrm{GL}_3(K)},\ C_3^{(1)} = P \cdot \langle e_{112}\rangle_{\mathrm{GL}_3(K)}.$$

(ii) *The P-module $C_3^{(0)}$ has the free resolution*

$$0 \longrightarrow P \xrightarrow{\psi^{(0)}} P^{\oplus 6} \xrightarrow{\varphi^{(0)}} C_3^{(0)} \longrightarrow 0$$

where denoting by $e_1, e_2, e_3, e_4, e_5, e_6$ the standard generators of P^6, $\varphi^{(0)}$ is the P-module homomorphism given by

$$\varphi^{(0)} : e_1 \mapsto e_{11}, e_2 \mapsto e_{12}, e_3 \mapsto e_{13}, e_4 \mapsto e_{22}, e_5 \mapsto e_{23}, e_6 \mapsto e_{33},$$

and $\psi^{(0)}$ maps the generator of the rank one P-module P to

$$\begin{pmatrix} \frac{1}{2}(\mathrm{tr}(t_2^2)\mathrm{tr}(t_3^2) - \mathrm{tr}(t_2t_3)^2) \\ \mathrm{tr}(t_1t_3)\mathrm{tr}(t_2t_3) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_3^2) \\ \mathrm{tr}(t_1t_2)\mathrm{tr}(t_2t_3) - \mathrm{tr}(t_1t_3)\mathrm{tr}(t_2^2) \\ \frac{1}{2}(\mathrm{tr}(t_1^2)\mathrm{tr}(t_3^2) - \mathrm{tr}(t_1t_3)^2) \\ \mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3) - \mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3) \\ \frac{1}{2}(\mathrm{tr}(t_1^2)\mathrm{tr}(t_2^2) - \mathrm{tr}(t_1t_2)^2) \end{pmatrix} \in P^{\oplus 6}. \tag{14}$$

(iii) *The P-module $C_3^{(1)}$ has the free resolution*

$$0 \longrightarrow P^{\oplus 3} \xrightarrow{\psi^{(1)}} P^{\oplus 8} \xrightarrow{\varphi^{(1)}} C_3^{(1)} \longrightarrow 0$$

where denoting by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ the standard generators of P^8, $\varphi^{(1)}$ is the P-module homomorphism given by

$$\begin{aligned} \varphi^{(1)} : \quad & e_1 \mapsto e_{112}, e_2 \mapsto e_{221}, e_3 \mapsto e_{113}, e_4 \mapsto e_{331}, \\ & e_5 \mapsto e_{223}, e_6 \mapsto e_{332}, e_7 \mapsto e_{132}, e_8 \mapsto e_{123} \end{aligned}$$

and $\psi^{(1)} : P^{\oplus 3} \to P^{\oplus 8}$ *is given by the matrix below:*

$$\begin{pmatrix} \mathrm{tr}(t_2t_3) & 0 & -\mathrm{tr}(t_3^2) \\ \mathrm{tr}(t_1t_3) & -\mathrm{tr}(t_3^2) & 0 \\ -\mathrm{tr}(t_2^2) & 0 & \mathrm{tr}(t_2t_3) \\ 0 & -\mathrm{tr}(t_2^2) & \mathrm{tr}(t_1t_2) \\ -\mathrm{tr}(t_1^2) & \mathrm{tr}(t_1t_3) & 0 \\ 0 & \mathrm{tr}(t_1t_2) & -\mathrm{tr}(t_1^2) \\ 0 & -\mathrm{tr}(t_2t_3) & \mathrm{tr}(t_1t_3) \\ \mathrm{tr}(t_1t_2) & -\mathrm{tr}(t_2t_3) & 0 \end{pmatrix} \in P^{8\times 3} \tag{15}$$

Proof. (i) C_3 is spanned as a K-vector space by products

$$\mathrm{tr}(t_{i_1}\cdots t_{i_k})\cdots\mathrm{tr}(t_{j_1}\cdots t_{j_l})\mathrm{tr}(t_{a_1}\cdots t_{a_m}s)$$

by Theorem 2.3 and Theorem 2.4. For $k \geq 4$, $\mathrm{tr}(t_{i_1}\cdots t_{i_k})$ can be expressed as a polynomial in $\mathrm{tr}(t_it_j)$ and $\mathrm{tr}(t_it_jt_k)$ by Corollary 2.12 (ii). Moreover, $\mathrm{tr}(t_it_jt_k)$ is non-zero only if i,j,k are distinct.

Claim: for $k \geq 4$, $\mathrm{tr}(t_{i_1}\cdots t_{i_k}s)$ can be expressed by products of traces of shorter products.

Indeed, one can easily verify the identity

$$\mathrm{tr}(t_1t_2t_3t_4s) = \frac{1}{2}(\mathrm{tr}(t_1t_2)\mathrm{tr}(t_3t_4s) + \mathrm{tr}(t_3t_4)\mathrm{tr}(t_1t_2s) - \mathrm{tr}(t_1t_4)\mathrm{tr}(t_2t_3s)), \tag{16}$$

implying our claim for $k = 4$. Apply next the fundamental trace identity (see for example [DrF, p. 63, Theorem 5.2.4]) for the four 3×3 matrices t_1t_2, t_3t_4, t_5, s, and take into acount that $0 = \mathrm{tr}(t_i) = \mathrm{tr}(s) = \mathrm{tr}(t_is)$ to get

$$\begin{aligned} 0 \quad = \quad & \mathrm{tr}(t_1t_2t_3t_4t_5s) + \mathrm{tr}(t_3t_4t_5t_1t_2s) + \mathrm{tr}(t_5t_1t_2t_3t_4s) \\ & +\mathrm{tr}(t_3t_4t_1t_2t_5s) + \mathrm{tr}(t_1t_2t_5t_3t_4s) + \mathrm{tr}(t_5t_3t_4t_1t_2s) \\ & -\mathrm{tr}(t_1t_2)\mathrm{tr}(t_3t_4t_5s) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_5t_3t_4s) - \mathrm{tr}(t_3t_4)\mathrm{tr}(t_1t_2t_5s) \\ & -\mathrm{tr}(t_3t_4)\mathrm{tr}(t_5t_1t_2s) - \mathrm{tr}(t_1t_2t_5)\mathrm{tr}(t_3t_4s) - \mathrm{tr}(t_3t_4t_5)(\mathrm{tr}(t_1t_2s). \end{aligned} \tag{17}$$

For $f, g \in C_3$ write $f \equiv g$ if $f - g \in K[T_3]_+^{\mathrm{SO}_3(K)}C_3$, where $K[T_3]_+^{\mathrm{SO}_3(K)}$ stands for the sum of the positive degree homogeneous components of $K[T_3]^{\mathrm{SO}_3(K)}$. Since $[t_i, t_j]$ is a skew-symmetric matrix, the identity (16) implies that

$$\mathrm{tr}(t_1t_2t_3t_4t_5s) \equiv \mathrm{tr}(t_{\pi(1)}t_{\pi(2)}t_{\pi(3)}t_{\pi(4)}t_{\pi(5)}s) \text{ for any permutation } \pi \in S_5.$$

Therefore (17) implies $6\mathrm{tr}(t_1t_2t_3t_4t_5s) \equiv 0$. This settles our claim for $k = 5$. Finally, for $k \geq 6$, recall that $\mathrm{tr}(z_1z_2z_3z_4z_5z_6z_7)$ can be expressed by traces of shorter products where $z_1, \ldots, z_7$ are arbitrary (not necessarily skew-symmetric or symmetric) 3×3 matrices (see for example [DrF, p. 78, Theorem 6.1.6 and p. 79]), so our Claim holds for $k \geq 6$ as well.

Thus we proved that C_3 is generated as a $K[T_3]^{\mathrm{SO}_3(K)}$-module by

$$V := \mathrm{Span}_K\{\mathrm{tr}(t_it_js),\ \mathrm{tr}(t_it_jt_ks) \mid i,j,k \in \{1,2,3\}\}.$$

This is a $\mathrm{GL}_3(K)$-submodule of C_3. Consider the surjective $\mathrm{GL}_3(K)$-module homomorphism $\rho : K\langle X_3\rangle^{(2)} \oplus K\langle X_3\rangle^{(3)} \to V$ given by $\rho(f(x_1,x_2,x_3)) = \mathrm{tr}(f(t_1,t_2,t_3)s)$. As a $\mathrm{GL}_3(K)$-module, $K\langle X_3\rangle^{(2)}$ is generated by x_1^2 and $[x_1,x_2]$, whereas $K\langle X_3\rangle^{(3)}$ is generated by x_1^3, $[x_1^2,x_2]$, $[x_1,[x_1,x_2]]$, $s_3(x_1,x_2,x_3) = \sum_{\pi\in S_3}\mathrm{sign}(\pi)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$.

Now $\rho([x_1,x_2])$, $\rho(x_1^3)$, $\rho([x_1,[x_1,x_2]])$, $\rho(s_3(x_1,x_2,x_3))$ are all zero. Hence we conclude

$$V = \langle e_{11}\rangle_{\mathrm{GL}_3(K)} \oplus \langle e_{112}\rangle_{\mathrm{GL}_3(K)}.$$

Recall that $K[T_3]^{\mathrm{SO}_3(K)}$ is a rank two free P-module generated by 1 and $\mathrm{tr}(t_1t_2t_3)$ by Theorem 2.9, Theorem 2.10 and Proposition 2.11. Thus by $C_3 = K[T_3]^{\mathrm{SO}_3(K)}\cdot V$ we conclude that C_3 is generated as a P-module by $V + \mathrm{tr}(t_1t_2t_3)V$. Next we show that

$$\mathrm{tr}(t_1t_2t_3)V \subseteq PV. \tag{18}$$

Indeed, observe that $\mathrm{tr}(t_1t_2t_3)$ spans a 1-dimensional $\mathrm{GL}_3(K)$-invariant subspace in $K[T_3, Z_0^+]$. Therefore to prove (18), it suffices to show that the $\mathrm{GL}_3(K)$-module generators e_{11} and e_{112} are multiplied by $\mathrm{tr}(t_1t_2t_3)$ into PV. This follows from the following two equalities:

$$\mathrm{tr}(t_1t_2t_3)e_{11} = \frac{1}{4}\mathrm{tr}(t_1t_3)e_{112} - \frac{1}{4}\mathrm{tr}(t_1t_2)e_{113} - \frac{1}{12}\mathrm{tr}(t_1^2)e_{132} + \frac{1}{12}\mathrm{tr}(t_1^2)e_{123} \tag{19}$$

$$\begin{aligned}\mathrm{tr}(t_1t_2t_3)e_{112} = {} & \frac{1}{2}\left(\mathrm{tr}(t_1t_3)\mathrm{tr}(t_2^2) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_2t_3)\right)e_{11} \\ & + \frac{1}{2}\left(\mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3)\right)e_{12} \\ & + \frac{1}{2}\left(\mathrm{tr}(t_1t_2)^2 - \mathrm{tr}(t_1^2)\mathrm{tr}(t_2^2)\right)e_{13}\end{aligned} \tag{20}$$

So we proved

$$C_3 = P\langle e_{11}\rangle_{\mathrm{GL}_3(K)} + P\langle e_{112}\rangle_{\mathrm{GL}_3(K)}.$$

The above sum is necessarily direct, as the polynomials in the first summand have odd total degree, whereas the polynomials in the second summand have even total degree. This finishes the proof of (i).

(ii) We proved above that $\varphi^{(0)}$ is surjective onto $C_3^{(0)}$. Using [CoCoA] we found the following relation:

$$\begin{aligned} 0 \; = \; & \frac{1}{2}(\mathrm{tr}(t_2^2)\mathrm{tr}(t_3^2) - \mathrm{tr}(t_2t_3)^2)e_{11} + (\mathrm{tr}(t_1t_3)\mathrm{tr}(t_2t_3) - \mathrm{tr}(t_1t_2)\mathrm{tr}(t_3^2))e_{12} \\ & (\mathrm{tr}(t_1t_2)\mathrm{tr}(t_2t_3) - \mathrm{tr}(t_1t_3)\mathrm{tr}(t_2^2))e_{13} + \frac{1}{2}(\mathrm{tr}(t_1^2)\mathrm{tr}(t_3^2) - \mathrm{tr}(t_1t_3)^2)e_{22} \\ & (\mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3) - \mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3))e_{23} + \frac{1}{2}(\mathrm{tr}(t_1^2)\mathrm{tr}(t_2^2) - \mathrm{tr}(t_1t_2)^2)e_{33}\end{aligned}$$

Hence we have established $\psi^{(0)}(P) \subseteq \ker(\varphi^{(0)})$. Taking into account the Hilbert series of $C_3^{(0)}$ we may conclude the equality $\psi^{(0)}(P) = \ker(\varphi^{(0)})$. Indeed, by Proposition 5.1 we have that the univariate Hilbert series of C_3 with the standard $\mathbb{N}_0$-grading (coming from the action of the subgroup of scalar matrices in $\mathrm{GL}_3(K)$) is

$$\frac{6\tau^2 - \tau^6}{(1-\tau^2)^6}.$$

The Hilbert series of the free module $P^{\oplus 6}$ (endowed with the appropriate grading respected by $\varphi^{(0)}$ is $\frac{6\tau^2}{(1-\tau^2)^6}$. It follows that the Hilbert series of $\ker(\varphi^{(0)})$ is $\frac{\tau^6}{(1-\tau^2)^6}$, which obviously agrees with the Hilbert series of the rank one free P-submodule $\psi^{(0)}(P)$ generated by a single element of degree 6.

(iii) In the proof of (i) we saw already that $\varphi^{(1)}$ is surjective onto $C_3^{(1)}$. Using [CoCoA] we found the relation

$$0 = \mathrm{tr}(t_2t_3)e_{112} + \mathrm{tr}(t_1t_3)e_{221} - \mathrm{tr}(t_1^2)e_{223} - \mathrm{tr}(t_2^2)e_{113} + \mathrm{tr}(t_1t_2)e_{123}. \quad (21)$$

This means that the first column of the 8×3 matrix in the statement (iii) belongs to $\ker(\varphi^{(1)})$. Permuting cyclically the matrix variables t_1, t_2, t_3 in (21) we get two other relations, meaning that the second and third columns of the 8×3 matrix in (15) belong to $\ker(\varphi^{(1)})$. So we have $\psi^{(1)}(P^{\oplus 3}) \subseteq \ker(\varphi^{(1)})$. As the upper 3×3 minor of the 8×3 matrix in (15) has non-zero determinant, we get that $\psi^{(1)}$ is injective, and consequently the univariate Hilbert series of $\psi^{(1)}(P^{\oplus 3})$ agrees with $\frac{3\tau^5}{(1-\tau^2)^6}$, the Hilbert series of $P^{\oplus 3}$ (graded appropriately). On the other hand, by Proposition 5.1 we know that the Hilbert series of $C_3^{(1)}$ is $\frac{8\tau^3 - 3\tau^5}{(1-\tau^2)^6}$. the Hilbert series of $P^{\oplus 8}$ (with the suitable grading) is $\frac{8\tau^3}{(1-\tau^2)^6}$, implying that the Hilbert series of $\ker(\varphi^{(1)})$ is $\frac{3\tau^5}{(1-\tau^2)^6}$, the same as the Hilbert series of $\psi^{(1)}(P^{\oplus 3})$. This proves the equality $\mathrm{im}(\psi^{(1)}) = \ker(\varphi^{(1)})$. □

Theorem 5.6. (i) *The P-module $\mathcal{E}_3$ has the direct sum decomposition*

$$\mathcal{E}_3 = \mathcal{E}_{3,1} \oplus \mathcal{E}_{3,2}^{(1)} \oplus \mathcal{E}_{3,2}^{(0)} \oplus \mathcal{E}_{3,3}^{(0)} \oplus \mathcal{E}_{3,3}^{(1)},$$

where

$$\begin{aligned}
\mathcal{E}_{3,1} &= P \cdot I \oplus P \cdot \mathrm{tr}(t_1t_2t_3)I = K[T_3]^{SO_3(K)} \cdot I \subset \mathcal{E}_3 \\
\mathcal{E}_{3,2}^{(1)} &= P \cdot \langle t_1 \rangle_{GL_3(K)} \\
\mathcal{E}_{3,2}^{(0)} &= P \cdot \langle [t_1, t_2] \rangle_{GL_3(K)} \\
\mathcal{E}_{3,3}^{(0)} &= P \cdot \langle t_1^2 - \frac{1}{3}\mathrm{tr}(t_1^2)I \rangle_{GL_3(K)} \\
\mathcal{E}_{3,3}^{(1)} &= P \cdot \langle [t_1^2, t_2] \rangle_{GL_3(K)}.
\end{aligned}$$

(ii) *Both $\mathcal{E}_{3,2}^{(1)}$ and $\mathcal{E}_{3,2}^{(0)}$ are free P-modules of rank* 3*:*

$$\mathcal{E}_{3,2}^{(1)} = P \cdot t_1 \oplus P \cdot t_2 \oplus P \cdot t_3 \text{ and } \mathcal{E}_{3,2}^{(0)} = P \cdot [t_1, t_2] \oplus P \cdot [t_1, t_3] \oplus P \cdot [t_2, t_3].$$

(iii) *The K-vector space $\langle t_1^2 - \frac{1}{3}\mathrm{tr}(t_1^2)I \rangle_{GL_3(K)}$ has the basis*

$$\{f_{ij} = \frac{1}{2}(t_it_j + t_jt_i) - \frac{1}{3}\mathrm{tr}(t_it_j)I \mid 1 \le i \le j \le 3\},$$

and the P-module $\mathcal{E}_{3,3}^{(0)}$ has the free resolution

$$0 \longrightarrow P \xrightarrow{\mu^{(0)}} P^{\oplus 6} \xrightarrow{\eta^{(0)}} C_3^{(0)} \longrightarrow 0,$$

where denoting by $e_1, e_2, e_3, e_4, e_5, e_6$ the standard generators of P^6, $\eta^{(0)}$ is the P-module surjection given by

$$\eta^{(0)} : e_1 \mapsto f_{11}, e_2 \mapsto f_{12}, e_3 \mapsto f_{13}, e_4 \mapsto f_{22}, e_5 \mapsto f_{23}, e_6 \mapsto f_{33},$$

and $\mu^{(0)}$ maps the generator of the rank one P-module P to the element of $P^{\oplus 6}$ given in (14) *in Theorem 5.5 (ii).*

(iv) *The K-vector space $\langle [t_1^2, t_2] \rangle_{GL_3(K)}$ has the basis*

$$\{f_{iij} = [t_i^2, t_j],\ f_{132} = [t_1t_3 + t_3t_1, t_2],\ f_{123} = [t_1t_2 + t_2t_1, t_3] \mid i \ne j \in \{1, 2, 3\}\},$$

and the P-module $C_3^{(1)}$ has the free resolution

$$0 \longrightarrow P^{\oplus 3} \xrightarrow{\mu^{(1)}} P^{\oplus 8} \xrightarrow{\eta^{(1)}} C_3^{(1)} \longrightarrow 0$$

where denoting by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ the standard generators of P^8, $\eta^{(1)}$ is the P-module surjection given by

$$\begin{aligned} \eta^{(1)}: \quad & e_1 \mapsto f_{112}, e_2 \mapsto f_{221}, e_3 \mapsto f_{113}, e_4 \mapsto f_{331}, \\ & e_5 \mapsto f_{223}, e_6 \mapsto f_{332}, e_7 \mapsto f_{132}, e_8 \mapsto f_{123} \end{aligned}$$

and $\mu^{(1)} : P^{\oplus 3} \to P^{\oplus 8}$ is given by the matrix in (15) *in Theorem 5.5 (iii).*

Proof. Consider the $\mathrm{GL}_3(K)$-module isomorphism

$$\iota : \mathcal{E}_3 \to (K[T_3, Z]^{SO_3(K)})^{(\mathbb{N}_0,\mathbb{N}_0,\mathbb{N}_0,1)}, \quad f \mapsto \mathrm{tr}(fz)$$

from Proposition 2.7 (ii). Write the generic matrix z as the sum

$$z = \frac{1}{3}\mathrm{tr}(z)I + s + u, \text{ with } s, u \text{ as in Proposition 5.3.}$$

We have the equalities

$$\begin{aligned} 0 &= \mathrm{tr}(t_i) = \mathrm{tr}([t_i, t_j]) = \mathrm{tr}(t_i^2 - \frac{1}{3}\mathrm{tr}(t_i^2)I) = \mathrm{tr}([t_i^2, t_j]) \\ 0 &= \mathrm{tr}(s) = \mathrm{tr}(t_i s) = \mathrm{tr}([t_i, t_j]s) \\ 0 &= \mathrm{tr}(u) = \mathrm{tr}(t_i^2 u) = \mathrm{tr}([t_i^2, t_j]u) = \mathrm{tr}((t_i t_j + t_j t_i)u). \end{aligned}$$

These equalities show that

$$\begin{aligned} &\iota(t_i) = \mathrm{tr}(t_i u) \ (i = 1, 2, 3), \\ &\iota([t_i, t_j]) = \mathrm{tr}([t_i, t_j]u) = 2\mathrm{tr}(t_i t_j u) \ (1 \le i < j \le 3\} \\ &\iota(\frac{1}{2}(t_i t_j + t_j t_i)) = \mathrm{tr}(t_i t_j s) + \mathrm{tr}(t_i t_j)\mathrm{tr}(z), \ (1 \le i \le j \le 3). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \iota(f_{ij}) &= e_{ij} \ (1 \le i \le j \le 3), \\ \iota(f_{iij}) &= e_{iij} \ (i \ne j \in \{1, 2, 3\}), \\ \iota(f_{132}) &= e_{132}, \ \iota(f_{123}) = e_{123} \end{aligned}$$

(where e_{ij}, e_{iij}, e_{132}, e_{123} were defined in (12), (13)). Since ι is a P-module homomorphism, it follows that ι restricts to isomorphisms $\mathcal{E}_{3,1} \xrightarrow{\cong} C_1$, $\mathcal{E}_{3,2}^{(1)} + \mathcal{E}_{3,2}^{(0)} \xrightarrow{\cong} C_2$, $\mathcal{E}_{3,3}^{(0)} \xrightarrow{\cong} C_3^{(0)}$, and $\mathcal{E}_{3,3}^{(1)} \xrightarrow{\cong} C_3^{(1)}$. Thus our statements immediately follow from (5), Corollary 2.12 (ii), Proposition 5.4, and Theorem 5.5. □

We record a few relations in $\mathcal{E}_3$ that follow from (10), (11), (19), (20) by the proof of Theorem 5.6; these relations show the effect of multiplication by $\mathrm{tr}(t_1 t_2 t_3)$ on the P-module $\mathcal{E}_3$ written in the form as in Theorem 5.6:

Proposition 5.7. *We have the following equalities:* (i)

$$\mathrm{tr}(t_1 t_2 t_3) \cdot t_1 = \frac{1}{2}\left(\mathrm{tr}(t_1^2) \cdot [t_2, t_3] - \mathrm{tr}(t_1 t_2) \cdot [t_1, t_3] + \mathrm{tr}(t_1 t_3) \cdot [t_1, t_2]\right)$$

(ii)

$$\begin{aligned}\mathrm{tr}(t_1t_2t_3)\cdot[t_1,t_2] = \frac{1}{4}\left(\mathrm{tr}(t_1t_3)\mathrm{tr}(t_2^2)-\mathrm{tr}(t_1t_2)\mathrm{tr}(t_2t_3)\right)\cdot t_1\\ +\frac{1}{4}\left(\mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3)-\mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3)\right)\cdot t_2\\ +\frac{1}{4}\left(\mathrm{tr}(t_1t_2)^2-\mathrm{tr}(t_1^2)\mathrm{tr}(t_2^2)\right)\cdot t_3\end{aligned}$$

(iii)

$$\mathrm{tr}(t_1t_2t_3)f_{11} = \frac{1}{4}\mathrm{tr}(t_1t_3)f_{112}-\frac{1}{4}\mathrm{tr}(t_1t_2)f_{113}-\frac{1}{12}\mathrm{tr}(t_1^2)f_{132}+\frac{1}{12}\mathrm{tr}(t_1^2)f_{123}$$

(iv)

$$\begin{aligned}\mathrm{tr}(t_1t_2t_3)f_{112} = \frac{1}{2}\left(\mathrm{tr}(t_1t_3)\mathrm{tr}(t_2^2)-\mathrm{tr}(t_1t_2)\mathrm{tr}(t_2t_3)\right)f_{11}\\ +\frac{1}{2}\left(\mathrm{tr}(t_1^2)\mathrm{tr}(t_2t_3)-\mathrm{tr}(t_1t_2)\mathrm{tr}(t_1t_3)\right)f_{12}\\ +\frac{1}{2}\left(\mathrm{tr}(t_1t_2)^2-\mathrm{tr}(t_1^2)\mathrm{tr}(t_2^2)\right)f_{13}\end{aligned}$$

For an arbitrary $p \geq 3$, denote by A_p the subalgebra of $K[T_p]^{\mathrm{SO}_p(K)}$ generated by $\mathrm{tr}(t_it_j)$, $1 \leq i \leq j \leq p$ (note that for $p \geq 4$, A_p is not a polynomial algebra). The algebra $\mathcal{E}_p$ is naturally an A_p-module. Now Theorem 5.6 and Corollary 2.8 imply the following:

Proposition 5.8. *For any $p \geq 3$, the A_p-module $\mathcal{E}_p$ decomposes as*

$$\begin{aligned}\mathcal{E}_p = A_p\cdot I \oplus A_p\cdot \mathrm{tr}(t_1t_2t_3)I \oplus A_p\cdot\langle t_1\rangle_{GL_p(K)} \oplus A_p\cdot\langle [t_1,t_2]\rangle_{GL_p(K)}\\ \oplus A_p\cdot\langle t_1^2-\frac{1}{3}\mathrm{tr}(t_1^2)I\rangle_{GL_p(K)} \oplus A_p\cdot\langle [t_1^2,t_2]\rangle_{GL_p(K)}.\end{aligned}$$

In particular, as an A_p-module, $\mathcal{E}_p$ is generated by

$$I,\ \mathrm{tr}(t_1t_2t_3)I,\ t_i,\ t_it_j,\ t_it_jt_k \quad 1 \leq i,j,k \leq p.$$

Proposition 5.8 implies that for $m \geq 4$, any product $t_{i_1}t_{i_2}\cdots t_{i_m}$ is contained in $A_p^+\cdot\mathcal{E}_p$, where A_p^+ stands for the ideal in A_p generated by $\mathrm{tr}(t_it_j)$, $1 \leq i \leq j \leq p$. A more direct explanation of this fact is given by the following identity:

Proposition 5.9. *We have the equality*

$$\begin{aligned}t_1t_2t_3t_4 = \frac{1}{4}\left(\mathrm{tr}(t_1t_4)\mathrm{tr}(t_2t_3)-\mathrm{tr}(t_1t_2)\mathrm{tr}(t_3t_4)\right)I\\ +\frac{1}{2}\left(\mathrm{tr}(t_1t_2)t_3t_4-\mathrm{tr}(t_3t_4)t_1t_2-\mathrm{tr}(t_1t_4)t_3t_2\right).\end{aligned}$$

Proof. Proposition 5.8 implies that $t_1t_2t_3t_4$ must be a K-linear combination of $\mathrm{tr}(t_{\pi(1)}t_{\pi(2)})\mathrm{tr}(t_{\pi(3)}t_{\pi(4)})I$, $\mathrm{tr}(t_{\pi(1)}t_{\pi(2)})[t_{\pi(3)},t_{\pi(4)}]$, $\mathrm{tr}(t_{\pi(1)}t_{\pi(2)})f_{\pi(3)\pi(4)}$, $\pi \in S_4$. The actual coefficients were found using [CoCoA]:

$$\begin{aligned}t_1t_2t_3t_4 = \frac{1}{12}\left(\mathrm{tr}(t_1t_2)\mathrm{tr}(t_3t_4)+\mathrm{tr}(t_1t_4)\mathrm{tr}(t_2t_3)\right)I\\ +\frac{1}{4}\left(\mathrm{tr}(t_3t_4)[t_1,t_2]+\mathrm{tr}(t_1t_4)[t_2,t_3]+\mathrm{tr}(t_1t_2)[t_3,t_4]\right)\\ +\frac{1}{2}\left(\mathrm{tr}(t_3t_4)f_{12}-\mathrm{tr}(t_1t_4)f_{23}+\mathrm{tr}(t_1t_2)f_{34}\right).\end{aligned} \tag{22}$$

Plugging in the explicit expressions for f_{12}, f_{23}, f_{34} on the right hand side of the above formula, we obtain the desired statement. □

Remark 5.10. Based on Theorem 5.6 and its proof, it is possible to give a normal form for the elements in $\mathcal{E}_3$. With an iterated use of (22) it is possible to rewrite the product of any two P-module generators of $\mathcal{E}_3$ in normal form. This way one obtains a normal form plus a rewriting algorithm for products of elements given in normal form. The result is complicated and technical, so we leave out the details.

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