

A reverse Minkowski-type inequality

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Abstract

The famous Minkowski inequality provides a sharp lower bound for the mixed volume $V(K, M[n-1])$ of two convex bodies $K, M \subset \mathbb{R}^n$ in terms of powers of the volumes of the individual bodies K and M . The special case where K is the unit ball yields the isoperimetric inequality. In the plane, Betke and Weil (1991) found a sharp upper bound for the mixed area of K and M in terms of the perimeters of K and M . We extend this result to general dimensions by proving a sharp upper bound for the mixed volume $V(K, M[n-1])$ in terms of the mean width of K and the surface area of M . The equality case is completely characterized. In addition, we establish a stability improvement of this and related geometric inequalities of isoperimetric type.

Keywords. Geometric inequality, Brunn-Minkowski theory, Minkowski inequality, mean width, surface area, mixed volume, stability result

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1 Introduction

Mixed volumes of convex bodies in Euclidean space \mathbb{R}^n are fundamental functionals which encode geometric information about the involved convex bodies in a non-trivial way. Let \mathcal{K}^n denote the space of compact convex subsets of \mathbb{R}^n . For $K, M \in \mathcal{K}^n$ and $\alpha, \beta \geq 0$, the volume $V(\alpha K + \beta M)$ of the Minkowski sum $\alpha K + \beta M$ has the polynomial expansion

$$V(\alpha K + \beta M) = \sum_{i=0}^n \binom{n}{i} V(K[i], M[n-i]) \alpha^i \beta^{n-i}, \quad (1.1)$$

by which the coefficients $V(K[i], M[n-i])$ are uniquely determined. These are special mixed volumes involving i copies of K and $n-i$ copies of M , for $i \in \{0, \dots, n\}$. We refer to [8] for an introduction of more general mixed volumes and a thorough study of their basic properties. In the following, we simply write $V(K, M[n-1])$ or $V(K, M, \dots, M)$ if K appears with multiplicity one. In particular, the polynomial expansion (1.1) implies that

$$nV(K, M, \dots, M) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} (V(K + \varepsilon M) - V(K)).$$

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This explains why $nV(K, B^n, \dots, B^n)$ is the surface area $F(K)$ of K if $M = B^n$ is the Euclidean unit ball. The special choice $M = B^n$ leads to the intrinsic volumes

$$V_i(K) = \frac{1}{\kappa_{n-i}} \binom{n}{i} V(K[i], B^n[n-i]), \quad i \in \{0, \dots, n\},$$

where κ_m is the volume of B^m in \mathbb{R}^m . We note that $V_n = V$ is the volume functional, $V_1(K)$ is proportional to the mean width of K , and equal to the length of K if K is a segment. Furthermore, the intrinsic volume $V_{n-1}(K) = \frac{n}{2}V(K, B^n[n-1])$ is half of the surface area of K if $\text{int } K \neq \emptyset$, and $V_{n-1}(K) = \mathcal{H}^{n-1}(K)$ if $\dim K = n-1$. Here, we write \mathcal{H}^i for the i -dimensional Hausdorff-measure, which is normalized in such a way that it coincides with the Lebesgue measure on \mathbb{R}^i . In particular, in the Euclidean plane, $F(K) = 2V_1(K)$ is the perimeter of $K \in \mathcal{K}^2$.

One of the fundamental results for mixed volumes is Minkowski's inequality

$$V(K, M, \dots, M)^n \geq V(K)V(M)^{n-1} \quad \text{for } K, M \in \mathcal{K}^n. \quad (1.2)$$

If $\text{int } K, \text{int } M \neq \emptyset$, then equality holds if and only if K and M are homothetic, that is, $M = x + \lambda K$ for some $x \in \mathbb{R}^n$ and $\lambda > 0$ (we refer to [8] for notions and results in the Brunn-Minkowski theory which are used in the following without further explanation).

As a planar and reverse counterpart of the Minkowski inequality (1.2), Betke and Weil proved the following theorem (see [2, Theorem 1]) which yields a sharp upper bound for the mixed area of $K, M \in \mathcal{K}^2$ in terms of the perimeters of K and M .

Theorem 1.1 (Betke, Weil (1992)). *If $K, M \in \mathcal{K}^2$, then*

$$V(K, M) \leq \frac{1}{8} F(K) F(M)$$

with equality if and only if K and M are orthogonal (possibly degenerate) segments.

We extend this result to general dimensions and thus obtain the following reverse Minkowski-type inequality.

Theorem 1.2. *If $K, M \in \mathcal{K}^n$, then*

$$V(K, M[n-1]) \leq \frac{1}{n} V_1(K) V_{n-1}(M);$$

if $\dim(K) \geq 1$ and $\dim(M) \geq n-1$, then equality holds if and only if K is a segment and M is contained in a hyperplane orthogonal to K .

For Minkowski's inequality various stability versions have been found, the first is due to Minkowski himself. Here we cite only two such results. Groemer [6] proved that if $K, M \in \mathcal{K}^n$ with $\text{int } K, \text{int } M \neq \emptyset$ and $\varepsilon > 0$ is sufficiently small, then

$$V(K, M, \dots, M)^n \leq (1 + \varepsilon)V(K)V(M)^{n-1} \quad (1.3)$$

implies that there exist $y, z \in \mathbb{R}^n$ and $\lambda > 0$ such that

$$\lambda(K - z) \subset M - y \subset \left(1 + \gamma\varepsilon^{\frac{1}{n+1}}\right) \lambda(K - z)$$

where $\gamma > 0$ depends only on n .

In addition, Figalli, Maggi, Pratelli [3] showed that (1.3) implies that there is some $x \in \mathbb{R}^n$ such that

$$\mathcal{H}^n(M\Delta(x + \lambda K)) \leq \gamma \sqrt{\varepsilon} V(M)$$

where $\lambda = (V(M)/V(K))^{1/n}$, Δ stands for the symmetric difference and $\gamma > 0$ depends only on n .

These stability results improve Minkowski's first inequality provided some information about the deviation of the shapes of K and M (up to homothety) is available and at the same time they provide additional information on how close K and M are if almost equality holds in Minkowski's inequality.

We obtain the following stability version of the reverse Minkowski inequality given in Theorem 1.2. Here and in the following, we write $R(K)$ to denote the circumradius of $K \in \mathcal{K}^n$.

Theorem 1.3. *Let $K, M \in \mathcal{K}^n$ with $\dim(K) \geq 1$ and $\dim(M) \geq n - 1$. Suppose that*

$$V(K, M[n-1]) \geq (1 - \varepsilon) \frac{1}{n} V_1(K) V_{n-1}(M)$$

for some $\varepsilon \in (0, \varepsilon_0)$. Then there exist $e, f \in \mathbb{S}^{n-1}$ and a segment s of length $(2 - \gamma_1 \varepsilon)R(K)$ parallel to e such that $h_M(f) + h_M(-f) \leq \gamma_2 r \varepsilon^{\frac{1}{4}}$, $\langle e, f \rangle \geq 1 - \gamma_2 \sqrt{\varepsilon}$ and

$$s \subset K \subset s + \gamma_2 R(K) \sqrt{\varepsilon} B^n,$$

where r is the maximal radius of an $(n - 1)$ -ball in $M|_{e^\perp}$, and $\gamma_1, \gamma_2, \varepsilon_0 > 0$ are constants depending on n .

Note that the third condition ensures that M is contained in a slab of width at most $\gamma_2 r \varepsilon^{\frac{1}{4}}$ and the second condition implies that this slab is almost orthogonal (in a quantitative sense) to the segment s .

A key point in proving Theorem 1.2 and Theorem 1.3 is the following result, which is interesting in its own right.

Theorem 1.4. *Let $K \in \mathcal{K}^n$ with $\text{diam}(K) \geq 1$.*

- (i) *Then $V_1(K) \geq 2R(K)$, with equality if and only if K is a segment.*
- (ii) *If $V_1(K) \leq (2 + \varepsilon)R(K)$ for some small $\varepsilon > 0$, then there exists a segment s of length $(2 - \gamma_1 \varepsilon)R(K)$ such that $s \subset K \subset s + \gamma_2 R(K) \sqrt{\varepsilon} B^n$, where $\gamma_1, \gamma_2 > 0$ are constants depending on n .*

The inequality between the circumradius and the first intrinsic volume (or the mean width) of a convex body, which is stated in Theorem 1.4 (i), is due to J. Linhart [5]. Our proof for part (i) follows Linhart's idea, but we introduce several modifications so as to simplify the discussion of the equality case and prepare for the proof of part (ii). The proof of the assertion in part (ii) provides a substantial strengthening and refinement of Linhart's argument.

The order of the error bound in Theorem 1.4 (ii) is $\sqrt{\varepsilon}$. This is the optimal order, as can be seen by considering isosceles triangles.

Geometric stability results have recently found applications in stochastic geometry, in particular in the study of shapes of large cells in certain random tessellations. The stability result stated in Theorem 1.4 now leads to the following probabilistic deviation result for stationary and isotropic Poisson hyperplane tessellations in \mathbb{R}^n . We refer to Section 5 for a brief introduction of the concepts used in the statement of Theorem 1.5. In particular, a suitable choice of a deviation functional ϑ is provided in (5.1).

Theorem 1.5. *Let Z_0 denote the zero cell of a stationary and isotropic Poisson hyperplane tessellation in \mathbb{R}^n with intensity $\lambda > 0$. Then there is a constant c_0 (depending on n) such that the following holds. If $\varepsilon > 0$ and $0 < a < b \leq \infty$, then*

$$\mathbb{P}(\vartheta(Z_0) \geq \varepsilon \mid R(Z_0) \in [a, b]) \leq c \exp\{c_0 \varepsilon a \lambda\},$$

where c is a constant which depends on n, ε .

We note that Betke and Weil [2] also proved that if $K \in \mathcal{K}^2$, then

$$V(K, -K) \leq \frac{\sqrt{3}}{18} F^2(K), \quad (1.4)$$

and under the additional assumption that K is a two-dimensional polygon they showed that equality holds in (1.4) if and only if K is an equilateral triangle.

Betke and Weil [2] suggested as a problem to characterize the equality cases of (1.4) among all planar compact convex sets $K \in \mathcal{K}^2$. This goal is achieved in the forthcoming manuscript [1].

The paper is structured as follows. Some basic notions which are used the following are introduced in Section 2. Then Theorem 1.4 is proved in Section 3. Our main results, Theorem 1.2 and its stability version Theorem 1.3, are established in Section 4. Finally, the application of Theorem 1.4 to stationary and isotropic Poisson hyperplane tessellation in \mathbb{R}^n is discussed in Section 5.

2 Preliminaries

For the basic notions and results from the Brunn-Minkowski theory which are used in this paper, we refer to the monograph [8]. We work in Euclidean space \mathbb{R}^n with scalar product $\langle \cdot, \cdot \rangle$ and induced Euclidean norm $\|\cdot\|$ in \mathbb{R}^n . The unit ball is denoted by B^n , its boundary is the unit sphere $\mathbb{S}^{n-1} = \partial B^n$. For a set A in a topological space we denote its closure by $\text{cl}(A)$. If $u \in \mathbb{S}^{n-1}$, then u^\perp denotes the linear $(n-1)$ -space orthogonal to u , and we write $X|u^\perp$ for the orthogonal projection of $X \subset \mathbb{R}^n$ into u^\perp . The support function of a convex body $K \in \mathcal{K}^n$ is $h_K(x) = \max_{y \in K} \langle x, y \rangle$ for $x \in \mathbb{R}^n$. On \mathcal{K}^n we use the Hausdorff metric

$$d_H(K, M) = \min\{r \geq 0 : K \subset M + rB^n \text{ and } M \subset K + rB^n\}$$

for $K, M \in \mathcal{K}^n$.

The surface area measure $S_{n-1}(K, \cdot)$ of $K \in \mathcal{K}^n$ is the (unique) finite Borel measure on \mathbb{S}^{n-1} such that if $M \in \mathcal{K}^n$, then

$$V(M, K[n-1]) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_M(u) S_{n-1}(K, du).$$

The surface area measure is weakly continuous on \mathcal{K}^n ; namely, if $K_m, K \in \mathcal{K}^n$ and $K_m \rightarrow K$ for $m \rightarrow \infty$ (with respect to the Hausdorff metric) and if $g : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ is continuous, then

$$\lim_{m \rightarrow \infty} \int_{\mathbb{S}^{n-1}} g(u) S_{n-1}(K_m, du) = \int_{\mathbb{S}^{n-1}} g(u) S_{n-1}(K, du).$$

We note that if $K \in \mathcal{K}^n$ and $e \in \mathbb{S}^{n-1}$, then

$$2 \mathcal{H}^{n-1}(K|e^\perp) = \int_{\mathbb{S}^{n-1}} |\langle e, u \rangle| S_{n-1}(K, du). \quad (2.1)$$

In fact, this holds even if K does not have interior points. We provide some additional information about the surface area measure for a convex body $K \in \mathcal{K}^n$. If $\dim K \leq n - 2$, then $S_{n-1}(K, \cdot)$ is the constant zero measure. If $\dim K = n - 1$ and the affine hull of K is parallel to u^\perp for $u \in \mathbb{S}^{n-1}$, then $S_{n-1}(K, \cdot)$ is the even measure concentrated on $\{\pm u\}$ with $S_{n-1}(K, \{u\}) = \mathcal{H}^{n-1}(\{u\})$. Now suppose that $\text{int } K \neq \emptyset$. For each $x \in \partial K$, there exists an exterior unit normal $u \in \mathbb{S}^{n-1}$ such that $h_K(u) = \langle x, u \rangle$. Moreover, for \mathcal{H}^{n-1} almost all $x \in \partial K$ the exterior unit normal of K at x is uniquely determined. In this case, x is called a regular boundary point and the exterior unit normal of K at x is denoted by $\nu_K(x)$. We write $\partial'K$ to denote the set of regular boundary points of K . In particular, if $g : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ is a bounded Borel function, then

$$\int_{\mathbb{S}^{n-1}} g(u) S_{n-1}(K, du) = \int_{\partial'K} g(\nu_K(x)) \mathcal{H}^{n-1}(dx).$$

If $K \in \mathcal{K}^n$ with $\text{int } K \neq \emptyset$ and $f \in \mathbb{S}^{n-1}$, then

$$S_{n-1}(K, \{u \in \mathbb{S}^{n-1} : \langle u, f \rangle > 0\}) > 0. \quad (2.2)$$

Since $2V_{n-1}(K)$ is the surface area $F(K)$ of K , we deduce from (2.1) and (2.2) that if $e \in \mathbb{S}^{n-1}$ and $K \in \mathcal{K}^n$ satisfies $\dim K \geq n - 1$, then

$$\mathcal{H}^{n-1}(K|e^\perp) \leq V_{n-1}(K), \quad (2.3)$$

with equality if and only if $\dim M = n - 1$ and e is normal to M . In addition, when projecting a convex body $K \in \mathcal{K}^n$ to e^\perp for some $e \in \mathbb{S}^{n-1}$, we have

$$\mathcal{H}^{n-1}(K|e^\perp) = \int_K |\langle e, u \rangle| \mathcal{H}^{n-1}(dx)$$

if $\dim K = n - 1$ and $u \in \mathbb{S}^{n-1}$ is normal to K , and

$$\mathcal{H}^{n-1}(K|e^\perp) = \frac{1}{2} \int_{\partial'K} |\langle e, \nu_K(x) \rangle| \mathcal{H}^{n-1}(dx)$$

if $\dim K = n$.

3 Proof of Theorem 1.4

For $z \in \mathbb{S}^{n-1}$ and $\alpha \in (0, \pi)$, let $B(z, \alpha) = \{x \in \mathbb{S}^{n-1} : \langle x, z \rangle \geq \cos \alpha\}$ be the spherical cap (geodesic ball) centered at z and of radius α . For a spherical set $X \subset \mathbb{S}^{n-1}$, we write $\text{int}_s X$ to denote the interior of X on \mathbb{S}^{n-1} and $\partial_s X$ for the boundary of X with respect to \mathbb{S}^{n-1} (and its topology induced by the geodesic metric, which is equal to the subspace topology of the ambient space). For a point $x \in \mathbb{S}^{n-1}$, the point $-x$ is the point of \mathbb{S}^{n-1} which is antipodal to x . We call $X \subset \mathbb{S}^{n-1}$ starshaped with respect to a point $x_0 \in \mathbb{S}^{n-1}$ if $x_0 \in X$, $-x_0 \notin X$, and for any $x \in X \setminus \{x_0\}$, the spherical geodesic arc connecting x and x_0 is contained in X .

The following observation is a key step in proving Theorem 3.3.

Lemma 3.1. *If $\alpha \in (0, \frac{\pi}{2}]$, $n \geq 2$, $z \in \mathbb{S}^{n-1}$ and $\Pi \subset B(z, \alpha)$ is compact and starshaped with respect to z , then*

$$\int_{\Pi} \langle z, u \rangle \mathcal{H}^{n-1}(du) \geq \frac{\int_{B(z, \alpha)} \langle z, u \rangle \mathcal{H}^{n-1}(du)}{\mathcal{H}^{n-1}(B(z, \alpha))} \cdot \mathcal{H}^{n-1}(\Pi).$$

Proof. For the proof, we can assume that $\mathcal{H}^{n-1}(\Pi) > 0$. For $u \in z^\perp \cap \mathbb{S}^{n-1}$, let $\varphi(u) \in [0, \alpha]$ be the ‘‘spherical radial function’’ of Π which is given by

$$\varphi(u) = \max\{t \in [0, \alpha] : z \cdot \cos t + u \cdot \sin t \in \Pi\}.$$

In addition, let

$$\Xi = \{u \in z^\perp \cap \mathbb{S}^{n-1} : \varphi(u) > 0\}.$$

An application of the transformation formula shows that

$$\begin{aligned} \int_{\Pi} \langle z, u \rangle \mathcal{H}^{n-1}(du) &= \int_{\Xi} \int_0^{\varphi(u)} (\cos s)(\sin s)^{n-2} ds \mathcal{H}^{n-2}(du), \\ \mathcal{H}^{n-1}(\Pi) &= \int_{\Xi} \int_0^{\varphi(u)} (\sin s)^{n-2} ds \mathcal{H}^{n-2}(du). \end{aligned}$$

To shorten the formulas, we set $\varrho(s) = (\sin s)^{n-2}$ for $s \in (0, \pi)$. Since $\cos s$ is decreasing in s , for any $u \in \Xi$ with $\varphi(u) < \alpha$ we have

$$\frac{\int_{\varphi(u)}^{\alpha} (\cos s) \varrho(s) ds}{\int_{\varphi(u)}^{\alpha} \varrho(s) ds} < \cos \varphi(u) < \frac{\int_0^{\varphi(u)} (\cos s) \varrho(s) ds}{\int_0^{\varphi(u)} \varrho(s) ds},$$

which in turn yields that

$$\begin{aligned} \frac{\int_0^{\alpha} (\cos s) \varrho(s) ds}{\int_0^{\alpha} \varrho(s) ds} &= \frac{\int_0^{\varphi(u)} \varrho(s) ds}{\int_0^{\alpha} \varrho(s) ds} \cdot \frac{\int_0^{\varphi(u)} (\cos s) \varrho(s) ds}{\int_0^{\varphi(u)} \varrho(s) ds} + \frac{\int_{\varphi(u)}^{\alpha} \varrho(s) ds}{\int_0^{\alpha} \varrho(s) ds} \cdot \frac{\int_{\varphi(u)}^{\alpha} (\cos s) \varrho(s) ds}{\int_{\varphi(u)}^{\alpha} \varrho(s) ds} \\ &\leq \frac{\int_0^{\varphi(u)} (\cos s) \varrho(s) ds}{\int_0^{\varphi(u)} \varrho(s) ds}. \end{aligned}$$

This holds in fact for any $u \in \Xi$. Therefore

$$\begin{aligned} \int_{\Pi} \langle z, u \rangle \mathcal{H}^{n-1}(du) &\geq \frac{\int_0^{\alpha} (\cos s) \varrho(s) ds}{\int_0^{\alpha} \varrho(s) ds} \int_{\Xi} \int_0^{\varphi(u)} \varrho(s) ds \mathcal{H}^{n-2}(du) \\ &= \frac{\int_{B(z, \alpha)} \langle z, u \rangle \mathcal{H}^{n-1}(du)}{\mathcal{H}^{n-1}(B(z, \alpha))} \cdot \mathcal{H}^{n-1}(\Pi), \end{aligned}$$

which proves the assertion. \square

We note that for any $z \in \mathbb{S}^{n-1}$, we have

$$\frac{\int_{B(z, \frac{\pi}{2})} \langle z, u \rangle \mathcal{H}^{n-1}(du)}{\mathcal{H}^{n-1}(B(z, \frac{\pi}{2}))} = \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}. \quad (3.1)$$

The following lemma shows how the left side increases when $B(z, \frac{\pi}{2})$ is replaced by $B(z, \alpha)$ and $0 < \alpha \leq \frac{\pi}{2} - \varepsilon$.

Lemma 3.2. *If $0 < \alpha \leq \frac{\pi}{2} - \varepsilon$, $\varepsilon \in [0, \frac{\pi}{6}]$, $n \geq 2$ and $z \in \mathbb{S}^{n-1}$, then*

$$\frac{\int_{B(z, \alpha)} \langle z, u \rangle \mathcal{H}^{n-1}(du)}{\mathcal{H}^{n-1}(B(z, \alpha))} \geq (1 + c_1 \varepsilon) \cdot \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})},$$

where $c_1 > 0$ depends on n .

Proof. For $\alpha \in (0, \frac{\pi}{2}]$, let

$$f(\alpha) = \frac{\int_{B(z,\alpha)} \langle z, u \rangle \mathcal{H}^{n-1}(du)}{\mathcal{H}^{n-1}(B(z,\alpha))} = \frac{\int_0^\alpha (\cos s) \varrho(s) ds}{\int_0^\alpha \varrho(s) ds},$$

and hence

$$f'(\alpha) = \frac{\varrho(\alpha)}{\left(\int_0^\alpha \varrho(s) ds\right)^2} \left(\cos \alpha \cdot \int_0^\alpha \varrho(s) ds - \int_0^\alpha (\cos s) \varrho(s) ds \right) < 0$$

as $\cos s > \cos \alpha$ for $0 < s < \alpha$.

Since f is monotone decreasing on $(0, \frac{\pi}{2}]$, it is sufficient to prove that $f'(\alpha) \leq -c_2$ for $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$, where $c_2 > 0$ is a constant depending on n . We observe that

$$\int_0^\alpha (\cos s) \varrho(s) ds \geq \cos \alpha \cdot \int_0^\alpha \varrho(s) ds + \cos \frac{\pi}{6} \cdot \int_0^{\frac{\pi}{6}} \varrho(s) ds,$$

and therefore

$$\cos \alpha \cdot \int_0^\alpha \varrho(s) ds - \int_0^\alpha (\cos s) \varrho(s) ds \leq \left(\cos \alpha - \cos \frac{\pi}{6} \right) \int_0^{\frac{\pi}{6}} \varrho(s) ds < 0.$$

Since $\alpha \geq \frac{\pi}{3} > \frac{\pi}{6}$, we conclude that

$$f'(\alpha) \leq \frac{(\sin \frac{\pi}{3})^{n-2}}{\left(\int_0^{\frac{\pi}{2}} \varrho(s) ds\right)^2} \cdot \left(\cos \frac{\pi}{3} - \cos \frac{\pi}{6} \right) \cdot \int_0^{\frac{\pi}{6}} \varrho(s) ds,$$

which proves Lemma 3.2. □

Recall that $R(K)$ denotes the circumradius of a convex body K , which is the radius of the (uniquely determined) smallest ball containing K . We slightly rephrase Theorem 1.4 from the introduction as follows.

Theorem 3.3. *Let $K \in \mathcal{K}^n$.*

- (i) *Then $V_1(K) \geq 2R(K)$, with equality if and only if K is a segment*
- (ii) *If $V_1(K) \leq (2 + \varepsilon)R(K)$ for some small $\varepsilon > 0$, then there exists a vector $c \in \mathbb{R}^n$ and a segment s of length $2 - c_3\varepsilon$ such that $R(K)s \subset K - c \subset R(K)(s + c_4\sqrt{\varepsilon}B^n)$, where $c_3, c_4 > 0$ are constants depending on n .*

Note also that Theorem 1.4 remains true if $R(K) = 0$ (as stated above). In this case K is a point and all assertions hold trivially. If $R(K) > 0$, then the explicit use of the vector c can be avoided by considering a translation of the segment s .

Proof of Theorem 3.3. For the proof, we can assume that $R(K) > 0$. By homogeneity and translation invariance, we can then assume that B^n is the circumball of K , and hence $R(K) = 1$. It follows that the origin o is contained in the convex hull of $\mathbb{S}^{n-1} \cap K$. Let k be the minimal number of points of $\mathbb{S}^{n-1} \cap K$ whose convex hull contains o , and hence $2 \leq k \leq n + 1$ by Carathéodory's theorem. Let $x_1, \dots, x_k \in \mathbb{S}^{n-1} \cap K$ and $\lambda_1, \dots, \lambda_k > 0$

with $\lambda_1 + \dots + \lambda_k = 1$ be such that $\lambda_1 x_1 + \dots + \lambda_k x_k = o$. For $i = 1, \dots, k$, we define the Dirichlet-Voronoi cell

$$D_i = \{x \in \mathbb{S}^{n-1} : \langle x, x_i \rangle \geq \langle x, x_j \rangle \text{ for } j = 1, \dots, k\},$$

and hence D_i is starshaped with respect to x_i and $\sum_{i=1}^k \mathcal{H}^{n-1}(D_i) = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. In fact, since $\lambda_1 x_1 + \dots + \lambda_k x_k = o$ implies that for any $x \in \mathbb{S}^{n-1}$, there exists $i \in \{1, \dots, k\}$ such that $\langle x, x_i \rangle \geq 0$, it follows that

$$D_i \subset B\left(x_i, \frac{\pi}{2}\right), \quad i = 1, \dots, k.$$

For $x \in D_i$, we have $h([x_1, \dots, x_k], x) = \langle x, x_i \rangle$. Thence, we deduce from Lemma 3.1 and (3.1) that

$$\begin{aligned} V_1(K) &\geq V_1([x_1, \dots, x_k]) = \frac{n}{\kappa_{n-1}} V([x_1, \dots, x_k], B^n[n-1]) \\ &= \frac{1}{\kappa_{n-1}} \sum_{i=1}^k \int_{D_i} \langle x, x_i \rangle \mathcal{H}^{n-1}(dx) \end{aligned} \quad (3.2)$$

$$\geq \frac{1}{\kappa_{n-1}} \sum_{i=1}^k \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(D_i) = 2. \quad (3.3)$$

If $V_1(K) = 2$, then equality in (3.2) yields that $K = [x_1, \dots, x_k]$. Moreover, by the monotonicity shown in the first part of the proof of Lemma 3.2, it follows that strict inequality holds in (3.3) if $D_i \neq B(x_i, \pi/2)$ for some $i \in \{1, \dots, k\}$. Hence, if equality holds, we must have $k = 2$ and $K = [x_1, x_2]$.

To prove the stability statement (ii), we again assume that $R(K) = 1$. First, we show that there exists a constant $\eta_0 > 0$, depending only on n , such that if $\text{diam } K \leq 2 - \eta$ for some $\eta \in [0, \eta_0]$, then

$$V_1(K) \geq 2 + c_5 \sqrt{\eta} \quad \text{for a constant } c_5 > 0 \text{ depending on } n. \quad (3.4)$$

For $\eta = 0$, the assertion holds by (i). Since $\sum_{i=1}^k \mathcal{H}^{n-1}(D_i) = n\kappa_n$, we may assume that $\mathcal{H}^{n-1}(D_1) \geq n\kappa_n/k$. For $i = 2, \dots, k$, we consider the set $F_i = \{x \in D_1 : \langle x, x_1 \rangle = \langle x, x_i \rangle\}$, which is contained in the hyper-sphere $\{x \in \mathbb{S}^{n-1} : \langle x, x_1 - x_i \rangle = 0\}$ and compact. In addition, let C_i be the union of all spherical geodesic arcs connecting x_1 to the points of F_i , and hence each C_i is compact and starshaped with respect x_1 and $\sum_{i=2}^k \mathcal{H}^{n-1}(C_i) = \mathcal{H}^{n-1}(D_1)$. In particular, since $2 \leq k \leq n+1$ we may assume that

$$\mathcal{H}^{n-1}(C_2) \geq \frac{n\kappa_n}{k(k-1)} \geq \frac{\kappa_n}{n+1}.$$

Let $u \in x_1^\perp \cap \mathbb{S}^{n-1} \cap \text{lin}\{x_1, x_2\}$ be the vector such that $\langle u, x_2 \rangle > 0$, and hence writing β for the angle enclosed by x_1 and x_2 , we have $x_2 = x_1 \cos \beta + u \sin \beta$. We deduce from $F_2 \subset B(x_1, \frac{\pi}{2}) \cap (x_1 - x_2)^\perp$ that $F_2 \subset B(x_1, \frac{\pi}{2}) \cap B(u, \frac{\pi}{2})$, and thus

$$C_2 \subset B(x_1, \frac{\pi}{2}) \cap B(u, \frac{\pi}{2}) \cap B\left(\frac{x_1 - x_2}{\|x_1 - x_2\|}, \frac{\pi}{2}\right).$$

Let

$$\Xi' := \{x \in x_1^\perp \cap \mathbb{S}^{n-1} : 0 \leq \langle x, u \rangle \leq \tau\},$$

where $\tau = \tau(n)$, depending only on n , is chosen such that

$$\mathcal{H}^{n-2}(\Xi') = \frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n(n+1)}$$

Then we connect each point of Ξ' to x_1 by a geodesic arc and take the union of all such arcs to get a compact subset $\Xi \subset B(x_1, \frac{\pi}{2})$ which is starshaped with respect to x_1 and satisfies

$$\mathcal{H}^{n-1}(\Xi) = \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2n(n+1)} = \frac{\kappa_n}{2(n+1)}.$$

Now we show that if $0 \leq \eta \leq \tau^{-2} =: \eta_0$ and $\gamma_1 := \frac{1}{2}\tau$, then

$$\langle x, x_1 \rangle \geq \gamma_1 \sqrt{\eta} \quad \text{for } x \in C_2 \setminus \Xi. \quad (3.5)$$

For the proof, we write x in the form $x = x_1 \cos s + x_0 \sin s$, where $x_0 \in x_1^\perp$ and $s \in [0, \frac{\pi}{2}]$. Since $x \in C_2 \setminus \Xi$, we conclude further that $s \in (0, \frac{\pi}{2}]$ and $\langle x_0, u \rangle > \tau > 0$.

We first observe that

$$\frac{x_1 - x_2}{\|x_1 - x_2\|} = x_1 \sin \frac{\beta}{2} - u \cos \frac{\beta}{2} \quad (3.6)$$

and

$$\sin \frac{\beta}{2} = \frac{1}{2} \|x_1 - x_2\| \leq 1 - \frac{\eta}{2},$$

since $x_1, x_2 \in K$ and $\text{diam } K \leq 2 - \eta$. In addition, we have

$$\left(\tan \frac{\beta}{2} \right)^2 \leq \frac{(1 - \frac{\eta}{2})^2}{1 - (1 - \frac{\eta}{2})^2} \leq \frac{4}{3\eta}.$$

Since $x \in C_2$, we have $x \in B(\frac{x_1 - x_2}{\|x_1 - x_2\|}, \frac{\pi}{2})$, and hence by (3.6) it follows that

$$\langle x, u \rangle \leq \langle x, x_1 \rangle \tan \frac{\beta}{2} \leq \langle x, x_1 \rangle \frac{2}{\sqrt{3}} \frac{1}{\sqrt{\eta}}.$$

If $s \in [\frac{\pi}{3}, \frac{\pi}{2}]$, then

$$\langle x, u \rangle = \langle x_0, u \rangle \sin s \geq \tau \sin s \geq \frac{\sqrt{3}}{2} \tau,$$

and hence

$$\langle x, x_1 \rangle \geq \frac{3}{4} \tau \sqrt{\eta} \geq \frac{1}{2} \tau \sqrt{\eta}.$$

If $s \in (0, \frac{\pi}{3}]$, then again

$$\langle x, x_1 \rangle = \cos s \geq \frac{1}{2} \geq \frac{1}{2} \tau \sqrt{\eta},$$

since $0 < \eta \leq \tau^{-2}$. This proves the claim.

It follows from the construction of C_2 and Ξ that

$$\mathcal{H}^{n-1}(\text{cl}(C_2 \setminus \Xi)) = \mathcal{H}^{n-1}(C_2 \setminus \Xi) \geq \frac{\kappa_n}{2(n+1)} = \frac{1}{2n(n+1)} \cdot \mathcal{H}^{n-1}(\mathbb{S}^{n-1}). \quad (3.7)$$

We define $\alpha \in (0, \frac{\pi}{2})$ by $\cos \alpha = \gamma_1 \sqrt{\eta} \in (0, \frac{1}{2}]$, and hence $\alpha \geq \frac{\pi}{3}$. Then (3.5) implies that for $x \in \text{cl}(C_2 \setminus \Xi)$ we have $x \in B(x_1, \alpha)$ and

$$\gamma_1 \sqrt{\eta} = \cos \alpha = \sin \left(\frac{\pi}{2} - \alpha \right) \leq \frac{\pi}{2} - \alpha,$$

so that $0 < \alpha \leq \frac{\pi}{2} - \gamma_1 \sqrt{\eta}$ and $\gamma_1 \sqrt{\eta} \leq \frac{\pi}{6}$. Therefore, we can apply Lemma 3.1 to the topological closure of $C_2 \setminus \Xi$, which is starshaped with respect to x_1 , and also use Lemma 3.2 and (3.7) to get

$$\begin{aligned} \int_{C_2 \setminus \Xi} \langle x_1, x \rangle \mathcal{H}^{n-1}(dx) &\geq \frac{\int_{B(x_1, \alpha)} \langle x_1, x \rangle \mathcal{H}^{n-1}(dx)}{\mathcal{H}^{n-1}(B(x_1, \alpha))} \cdot \mathcal{H}^{n-1}(C_2 \setminus \Xi) \\ &\geq (1 + c_1 \gamma_1 \sqrt{\eta}) \cdot \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(C_2 \setminus \Xi) \\ &\geq \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(C_2 \setminus \Xi) + \frac{\kappa_{n-1} c_1 \gamma_1}{n(n+1)} \cdot \sqrt{\eta}. \end{aligned}$$

In addition, using again Lemma 3.1 we also have

$$\begin{aligned} \int_{C_2 \cap \Xi} \langle x_1, x \rangle \mathcal{H}^{n-1}(dx) &\geq \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(C_2 \cap \Xi), \\ \int_{C_j} \langle x_1, x \rangle \mathcal{H}^{n-1}(dx) &\geq \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(C_j) \quad \text{for } j \geq 3, \\ \int_{D_i} \langle x_i, x \rangle \mathcal{H}^{n-1}(dx) &\geq \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(D_i) \quad \text{for } i \geq 2. \end{aligned}$$

Summing up the individual contributions from the subsets, we get

$$\begin{aligned} V_1(K) &\geq V_1([x_1, \dots, x_k]) = \frac{1}{\kappa_{n-1}} \sum_{i=1}^k \int_{D_i} \langle x, x_i \rangle \mathcal{H}^{n-1}(dx) \\ &\geq \frac{\kappa_{n-1} c_1 \gamma_1}{n(n+1)} \cdot \sqrt{\eta} + \frac{1}{\kappa_{n-1}} \sum_{i=1}^k \frac{2\kappa_{n-1}}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \cdot \mathcal{H}^{n-1}(D_i) \\ &= 2 + \frac{\kappa_{n-1} c_1 \gamma_1}{n(n+1)} \cdot \sqrt{\eta}, \end{aligned}$$

which completes the proof of (3.4).

Let us assume that $V_1(K) \leq 2 + \varepsilon$. If $[y_1, y_2] \subset K$ is a longest segment in K , then $\|y_1 - y_2\| \geq 2 - c_6 \varepsilon^2$ for $c_6 = c_5^{-2}$ by (3.4). For any $y \in K$, writing t for the distance of y from $[y_1, y_2]$, we have

$$\begin{aligned} 2 + \varepsilon &\geq V_1(K) \geq V_1([y_1, y_2, y]) \geq \frac{\|y_1 - y_2\|}{2} + \sqrt{\frac{\|y_1 - y_2\|^2}{4} + t^2} \\ &\geq 1 - \frac{c_6 \varepsilon^2}{2} + \sqrt{\left(1 - \frac{c_6 \varepsilon^2}{2}\right)^2 + t^2}. \end{aligned}$$

Assuming that $(\frac{c_6 \varepsilon}{2} + 1)\varepsilon \leq 1$ and using that $(1 + s)^2 \leq 1 + 3s$ for $s \in [0, 1]$, we deduce that

$$t^2 \leq \left(\frac{5c_6}{2} + 3\right) \varepsilon,$$

which in turn yields that

$$K \subset [y_1, y_2] + \sqrt{(3 + 3c_6)\varepsilon} B^n.$$

Finally we note that (ii) again implies the equality condition in (i). \square

4 Proofs of Theorem 1.2 and Theorem 1.3

Part (i) of Theorem 3.3 is the main ingredient for the proof of Theorem 1.2.

Proof of Theorem 1.2. We can assume that the circumball of K has its centre at the origin. Then

$$\begin{aligned} V(K, M[n-1]) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) S_{n-1}(M, du) \leq \frac{1}{n} R(K) F(M) \\ &\leq \frac{1}{n} \frac{1}{2} V_1(K) 2V_{n-1}(M) = \frac{1}{n} V_1(K) V_{n-1}(M), \end{aligned} \quad (4.1)$$

where we used Theorem 3.3 for the second inequality. If equality holds, then equality holds in Theorem 3.3, since $V_{n-1}(M) > 0$, and therefore $K = [-Re, Re]$ with $R = R(K)$ and for some $e \in \mathbb{S}^{n-1}$. Moreover, we then also have equality in the first inequality, which yields

$$\int_{\mathbb{S}^{n-1}} |\langle u, e \rangle| S_{n-1}(M, du) = S_{n-1}(M, \mathbb{S}^{n-1}).$$

This implies that the area measure of M is concentrated in $\{-e, e\}$, hence M is contained in a hyperplane orthogonal to e . \square

We now start to build the argument leading to the stability version Theorem 1.3 of Theorem 1.2. Recall that if M is an at least $(n-1)$ -dimensional compact convex set in \mathbb{R}^n and $e \in \mathbb{S}^{n-1}$, then

$$\begin{aligned} 2\mathcal{H}^{n-1}(M|e^\perp) &= \int_{\mathbb{S}^{n-1}} |\langle e, u \rangle| S_{n-1}(M, du) \\ &= \int_{\partial' M} |\langle e, \nu_M(x) \rangle| \mathcal{H}^{n-1}(dx). \end{aligned}$$

Moreover,

$$\int_{\mathbb{S}^{n-1}} |\langle e, u \rangle| S_{n-1}(M, du) \leq 2V_{n-1}(M), \quad (4.2)$$

with equality if and only if $M \subset e^\perp$.

In the following proposition, we explore what can be said about M if the integral on the left side of (4.2) is ε -close to the upper bound.

Proposition 4.1. *Let $\varepsilon \in (0, \frac{1}{2} (\frac{1}{2n})^n)$ and $e \in \mathbb{S}^{n-1}$. Suppose that M is an at least $(n-1)$ -dimensional compact convex set in \mathbb{R}^n such that*

$$\int_{\mathbb{S}^{n-1}} |\langle e, u \rangle| S_{n-1}(M, du) \geq (1 - \varepsilon) 2V_{n-1}(M). \quad (4.3)$$

Then there is some $f \in \mathbb{S}^{n-1}$ such that $h_M(f) + h_M(-f) \leq c_7 r \sqrt{\varepsilon}$ and $\langle e, f \rangle \geq 1 - c_8 \varepsilon$, where $c_7 \leq 48n^2 \sqrt{6}^n$, $c_8 \leq (10n)^4 (2n)^n$ and r is the maximal radius of an $(n-1)$ -ball in $M|e^\perp$.

Remark The lemma is essentially optimal for $n \geq 3$, in the sense that one cannot conclude in general that $h_M(e) + h_M(-e) \leq c_0 V_{n-1}(M)^{\frac{1}{n-1}} \sqrt{\varepsilon}$. To show this, let $f \in \mathbb{S}^{n-1}$ with $\langle e, f \rangle = 1 - \frac{\varepsilon}{2}$, and let f_1, \dots, f_n be an orthonormal basis such that $f_1 = f$ and $e \in \text{lin}\{f_1, f_2\}$. For large λ , we define

$$M = [\pm\sqrt{\varepsilon}f_1, \pm\lambda f_2, \pm n f_3, \dots, \pm n f_n],$$

which satisfies $\mathcal{H}^{n-1}(M|e^\perp) \geq (1 - \varepsilon) V_{n-1}(M)$ if $\lambda > 0$ is large and $\varepsilon > 0$ is small enough.

Proof of Proposition 4.1. For the proof, we can assume that M is n -dimensional. This follows from an approximation argument (which will require adjustments of M , ε and r).

The main idea is as follows. Let us consider some n -dimensional convex body C with $V_{n-1}(C) \leq V_{n-1}(M)$ and $\mathcal{H}^{n-1}(C|e^\perp) = \mathcal{H}^{n-1}(M|e^\perp)$, and let

$$\begin{aligned}\partial_+ C &= \{y \in \partial' C : \langle e, \nu_C(y) \rangle > 0\}, \\ \partial_- C &= \{y \in \partial' C : \langle e, \nu_C(y) \rangle < 0\}.\end{aligned}$$

Claim Suppose there exist $\eta, \gamma > 0$ and a compact convex set $X \subset C|e^\perp$ with $\mathcal{H}^{n-1}(X) = \gamma \mathcal{H}^{n-1}(C|e^\perp)$ such that any $y \in \partial_+ C$ with $y|e^\perp \in X$ satisfies $\tan \angle(e, \nu_C(y)) \geq \eta$. Then

$$\eta \leq \frac{4\sqrt{\varepsilon}}{\sqrt{\gamma}} \quad \text{provided } \varepsilon < \frac{\gamma}{4}. \quad (4.4)$$

To prove the Claim, let Y denote the set of all $y \in \partial_+ C$ with $y|e^\perp \in X$. For any $y \in Y$, we have

$$0 < \langle e, \nu_C(y) \rangle = \cos \angle(e, \nu_C(y)) = \sqrt{\frac{1}{1 + (\tan \angle(e, \nu_C(y)))^2}} \leq \sqrt{\frac{1}{1 + \eta^2}}.$$

It follows that

$$\gamma \mathcal{H}^{n-1}(C|e^\perp) = \mathcal{H}^{n-1}(X) = \int_Y |\langle e, \nu_C(y) \rangle| \mathcal{H}^{n-1}(dy) \leq \sqrt{\frac{1}{1 + \eta^2}} \cdot \mathcal{H}^{n-1}(Y). \quad (4.5)$$

Furthermore, we have

$$\begin{aligned}(2 - \gamma) \mathcal{H}^{n-1}(C|e^\perp) &= \mathcal{H}^{n-1}(C|e^\perp) + \mathcal{H}^{n-1}((C|e^\perp) \setminus X) \\ &= \int_{(\partial' C) \setminus Y} |\langle e, \nu_C(y) \rangle| \mathcal{H}^{n-1}(dy) \\ &\leq \mathcal{H}^{n-1}((\partial C) \setminus Y).\end{aligned} \quad (4.6)$$

From (4.3), (4.5) and (4.6), we deduce that

$$\begin{aligned}(1 + 2\varepsilon) \cdot 2\mathcal{H}^{n-1}(C|e^\perp) &= (1 + 2\varepsilon) \cdot 2\mathcal{H}^{n-1}(M|e^\perp) \\ &\geq (1 + 2\varepsilon)(1 - \varepsilon) \cdot \mathcal{H}^{n-1}(\partial M) \\ &\geq \mathcal{H}^{n-1}(\partial M) \geq \mathcal{H}^{n-1}(\partial C) \\ &= \mathcal{H}^{n-1}((\partial C) \setminus Y) + \mathcal{H}^{n-1}(Y) \\ &\geq \left(2 - \gamma + \gamma\sqrt{1 + \eta^2}\right) \mathcal{H}^{n-1}(C|e^\perp),\end{aligned}$$

and hence $4\varepsilon \geq \gamma(\sqrt{1 + \eta^2} - 1)$. Since $(1 + s)^2 \leq 1 + 4s$ for $s \in (0, 1)$, we conclude (4.4), which proves the Claim.

We set

$$t = \max \{ \mathcal{H}^1((x + \mathbb{R}e) \cap M) : x \in M|e^\perp \}$$

and write r to denote the maximal radius of $(n-1)$ -balls in $M|e^\perp$. Possibly after a translation of M , there exists an origin symmetric ellipsoid E such that

$$E \subset M \subset nE \quad (4.7)$$

according to John's theorem. By changing the orientation of $e \in \mathbb{S}^{n-1}$, if necessary, there is a positive τ such that $\tau e \in \partial E$ and $\tau \leq \frac{t}{2} \leq n\tau$, and hence $\tau \in [\frac{t}{2n}, \frac{t}{2}]$. Let $f \in \mathbb{S}^{n-1}$ be the exterior unit normal at τe to E . It follows that

$$\begin{aligned} h_M(f) + h_M(-f) &\leq h_{nE}(f) + h_{nE}(-f) = 2nh_E(f) \\ &= 2n\langle \tau e, f \rangle = 2n\tau \langle e, f \rangle \leq 2n\tau, \end{aligned}$$

and thus

$$h_M(f) + h_M(-f) \leq nt. \quad (4.8)$$

We prove Lemma 4.1 in two steps. First, we bound t from above, then we establish a lower bound for $|\langle e, f \rangle|$.

Step 1 We show that $t \leq c_9 r \sqrt{\varepsilon}$ with a constant $c_9 \leq 48n\sqrt{6}^{n-1}$.

Let $w \in \mathbb{S}^{n-1} \cap e^\perp$ be such that $h_E(w) = \min\{h_E(u) : u \in \mathbb{S}^{n-1} \cap e^\perp\}$, which equals the inradius of $E|e^\perp$, and hence $h_E(w) \leq r$. In turn, for $y \in M$ we deduce from (4.7) that

$$|\langle y, w \rangle| \leq \max\{h_M(w), h_M(-w)\} \leq h_{nE}(w) = nh_E(w) \leq nr,$$

that is,

$$|\langle y, w \rangle| \leq nr \quad \text{for } y \in M. \quad (4.9)$$

We may choose an orthonormal basis e_1, \dots, e_n of \mathbb{R}^n such that $e_1 = e$ and $e_2 = w$, and let M' be the convex body resulting from M via successive Steiner symmetrizations with respect to $e_1^\perp, \dots, e_n^\perp$. Then M' satisfies

$$|\langle y, w \rangle| \leq nr \quad \text{for } y \in M', \quad \pm \frac{1}{2}te \in \partial M', \quad (rB^n \cap e^\perp) \subset M',$$

M' is symmetric with respect to the coordinate subspaces $e_1^\perp, \dots, e_n^\perp$, and hence in particular M' is centrally symmetric, $\mathcal{H}^{n-1}(M'|e^\perp) = \mathcal{H}^{n-1}(M|e^\perp)$, $M'|e^\perp = M' \cap e^\perp$ and $V_{n-1}(M') \leq V_{n-1}(M)$. Finally, we consider the double cone

$$\widetilde{M} = \text{conv} \left\{ \frac{1}{2}te, -\frac{1}{2}te, M' \cap e^\perp \right\},$$

which satisfies

$$|\langle y, w \rangle| \leq nr \quad \text{for } y \in \widetilde{M}, \quad \pm \frac{1}{2}te \in \partial \widetilde{M}, \quad (rB^n \cap e^\perp) \subset \widetilde{M},$$

\widetilde{M} is symmetric with respect to the coordinate subspaces $e_1^\perp, \dots, e_n^\perp$ (and hence centrally symmetric), $\mathcal{H}^{n-1}(\widetilde{M}|e^\perp) = \mathcal{H}^{n-1}(M|e^\perp)$ and $V_{n-1}(\widetilde{M}) \leq V_{n-1}(M)$.

For $\varrho = h_{\widetilde{M}}(w)$, we have $\varrho w \in \partial \widetilde{M}$, and hence $\varrho \leq nr$. To prepare an application of (4.4), we consider

$$X = \frac{5}{6}\varrho w + \frac{1}{6} \left(\widetilde{M}|e^\perp \right) \subset \widetilde{M}|e^\perp.$$

Let $L = e^\perp \cap w^\perp$. Let Y denote the set of all $y \in \partial_+ \widetilde{M}$ such that $y|e^\perp \in X$. For $y \in Y$, we write α for the angle of $\nu_{\widetilde{M}}(y)$ and e , thus $y = se + pw + v_1$ and $\nu_{\widetilde{M}}(y) = e \cos \alpha + qw + v_2$, where

$$v_1 \in \frac{1}{6} (\widetilde{M} \cap L), \quad v_2 \in L, \quad s \geq 0, \quad \frac{2}{3} \varrho \leq p \leq \varrho, \quad q \leq \sin \alpha.$$

Since \widetilde{M} is a double cone, there exists $z \in e^\perp \cap \partial \widetilde{M}$ such that $y \in [z, \frac{1}{2} te]$ and $y|e^\perp \in [z, o]$. We deduce that

$$\frac{s}{\frac{1}{2}t} = \frac{\|z - (y|e^\perp)\|}{\|z\|} = \frac{\langle z - (y|e^\perp), w \rangle}{\langle z, w \rangle} \leq \frac{1}{3},$$

thus $s \leq \frac{1}{6}t$. As $\frac{1}{2}te \in \widetilde{M}$, we have

$$0 \leq \langle y - \frac{1}{2}te, \nu_{\widetilde{M}}(y) \rangle = (s - \frac{1}{2}t) \cos \alpha + pq + \langle v_1, v_2 \rangle,$$

which yields

$$-\langle v_1, v_2 \rangle \leq (s - \frac{1}{2}t) \cos \alpha + pq.$$

Therefore, since $2v_1 \in \widetilde{M}$ we obtain

$$\begin{aligned} 0 &\leq \langle y - 2v_1, \nu_{\widetilde{M}}(y) \rangle = s \cdot \cos \alpha + pq - \langle v_1, v_2 \rangle \leq (2s - \frac{1}{2}t) \cos \alpha + 2pq \\ &\leq -\frac{1}{6}t \cos \alpha + 2\varrho \sin \alpha \leq -\frac{1}{6}t \cos \alpha + 2nr \sin \alpha, \end{aligned}$$

which implies that $\tan \alpha \geq \frac{t}{12nr}$. Now an application of (4.4) proves the estimate of Step 1.

Step 2 We show that $\langle e, f \rangle \geq 1 - c_8 \varepsilon$.

Let $\beta = \angle(f, e) \in [0, \frac{\pi}{2})$, and let $\tilde{w} \in \mathbb{S}^{n-1} \cap e^\perp$ be such that $f = e \cos \beta + \tilde{w} \sin \beta$. Since the shadow boundary of E in direction e lies in a hyperplane and by [7, Theorem 1], it follows from the definition of f that $E|e^\perp = (E \cap f^\perp)|e^\perp$. Hence (4.7) yields that

$$(E \cap f^\perp)|e^\perp \subset \pm M|e^\perp \subset n(E \cap f^\perp)|e^\perp. \quad (4.10)$$

Since there is some $a \in e^\perp$ such that $(rB^n \cap e^\perp) + a \subset M|e^\perp$ it follows that

$$\frac{r}{n}(B^n \cap e^\perp) \pm \frac{a}{n} \subset \pm \frac{1}{n}(M|e^\perp) \subset E|e^\perp,$$

which shows that

$$\frac{r}{n}(B^n \cap e^\perp) \subset (E \cap f^\perp)|e^\perp,$$

and we deduce that

$$\varrho = \max \{ \xi > 0 : \xi \tilde{w} \in (E \cap f^\perp)|e^\perp \} \geq \frac{r}{n}, \quad (4.11)$$

In order to apply (4.4), we choose $C = M$ and the set $X \subset M|e^\perp$ is chosen as

$$X = \frac{1}{2}(E|e^\perp) = \frac{1}{2}(E \cap f^\perp)|e^\perp \subset M|e^\perp,$$

which satisfies

$$\mathcal{H}^{n-1}(X) = \tilde{\gamma} \mathcal{H}^{n-1}(M|e^\perp), \quad \tilde{\gamma} \geq (2n)^{-(n-1)},$$

according to (4.10).

As before, we define Y as the set of all $y \in \partial_+ M$ with $y|e^\perp \in X$. Then, for $y \in Y$ we have

$$x = y|e^\perp \in \frac{1}{2}(E \cap f^\perp)|e^\perp.$$

It follows that

$$x - \frac{\varrho}{2}\tilde{w} \in (E \cap f^\perp)|e^\perp,$$

and hence $x - \frac{\varrho}{2}\tilde{w} = z|e^\perp$ for a $z \in E \cap f^\perp \subset M$. In addition, the definition of t implies the existence of an $s \in [0, t]$ such that $y - se \in E \cap f^\perp$. Since $f = e \cos \beta + \tilde{w} \sin \beta$, we deduce that

$$z - (y - se) = -\frac{1}{2}\varrho\tilde{w} + e \cdot \frac{1}{2}\varrho \tan \beta,$$

thus

$$y - z = \frac{1}{2}\varrho\tilde{w} + e \cdot \left(s - \frac{1}{2}\varrho \tan \beta \right).$$

We set $\alpha = \angle(e, \nu_M(y)) \in [0, \frac{\pi}{2})$, and hence

$$\nu_M(y) = e \cos \alpha + \tilde{p}\tilde{w} + \tilde{v},$$

where $\tilde{v} \in (\text{lin}\{\tilde{w}, e\})^\perp$ and $\tilde{p}^2 + \|\tilde{v}\|^2 = (\sin \alpha)^2$.

We deduce from Step 1, $0 \leq s \leq t$ and $\varrho \geq r/n$ (compare (4.11)) that

$$\begin{aligned} 0 &\leq \langle y - z, \nu_M(y) \rangle = \left\langle \frac{1}{2}\varrho\tilde{w} + e \cdot \left(s - \frac{1}{2}\varrho \tan \beta \right), e \cos \alpha + \tilde{p}\tilde{w} + \tilde{v} \right\rangle \\ &= \left(s - \frac{1}{2}\varrho \tan \beta \right) \cos \alpha + \frac{\tilde{p}}{2} \cdot \varrho \\ &\leq \left(c_1 n \varrho \sqrt{\varepsilon} - \frac{1}{2}\varrho \tan \beta \right) \cos \alpha + \frac{\sin \alpha}{2} \cdot \varrho, \end{aligned}$$

thus $\tan \alpha \geq \tan \beta - 2c_9 n \sqrt{\varepsilon}$. If $\tan \beta - 2c_9 n \sqrt{\varepsilon} > 0$, we conclude from (4.4) that

$$\tan \beta - 2c_9 n \sqrt{\varepsilon} \leq 4(2n)^{\frac{n-1}{2}} \sqrt{\varepsilon},$$

which in turn yields $\beta \leq \tan \beta \leq c_{10} \sqrt{\varepsilon}$ with $c_{10} \leq 96n^2 \sqrt{6}^{n-1} + 4\sqrt{2n}^{n-1} \leq (10n)^2 \sqrt{2n}^n$. If $\tan \beta \leq 2c_9 n \sqrt{\varepsilon}$, we directly arrive at the same conclusion. Therefore, in any case $\langle e, f \rangle = \cos \beta \geq 1 - \frac{1}{2}c_{10}^2 \varepsilon$, completing Step 2. \square

Finally, we combine the stability estimates above to derive (Theorem 1.3 in the following form.

Theorem 4.2. *Let $K, M \in \mathcal{K}^n$ with $\dim(K) \geq 1$ and $\dim(M) \geq n - 1$. Suppose that*

$$V(K, M[n-1]) \geq (1 - \varepsilon) \frac{1}{n} V_1(K) V_{n-1}(M) \quad (4.12)$$

for some sufficiently small $\varepsilon \in (0, 1/2)$. Then there is a segment s of length $(2 - c_3\varepsilon)$ and direction $e \in \mathbb{S}^{n-1}$ such that $R(K)s \subset K \subset R(K)(s + c_4\sqrt{\varepsilon}B^m)$, and there is a vector $f \in \mathbb{S}^{n-1}$ such that $\langle e, f \rangle \geq 1 - c_{11}\sqrt{\varepsilon}$ and $h_M(f) + h_M(-f) \leq c_{12}r\varepsilon^{\frac{1}{4}}$, where r is the maximal radius of $(n-1)$ -balls in $M|e^\perp$ and c_3, c_4, c_{11}, c_{12} are constants depending on n .

Proof. Suppose that (4.12) is satisfied for some $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \in (0, \frac{1}{2})$ is sufficiently small. A combination of (4.1) and (4.12) yields

$$\frac{1}{n}R(K)F(M) \geq V(K, M[n-1]) \geq (1-\varepsilon)\frac{1}{n}V_1(K)V_{n-1}(M),$$

and hence

$$V_1(K) \leq \frac{1}{1-\varepsilon}2R(K) \leq (1+2\varepsilon)2R(K) = (2+4\varepsilon)R(K).$$

By Theorem 3.3, there is a segment s of length $2 - c_3\varepsilon$ such that $R(K)s \subset K \subset R(K)(s + c_4\sqrt{\varepsilon}B^n)$. In particular,

$$V_1(K) \geq V_1(s)R(K) \geq (2 - c_3\varepsilon)R(K).$$

If s has direction $e \in \mathbb{S}^{n-1}$, then again by the assumption we obtain

$$\begin{aligned} & (1-\varepsilon)(2 - c_3\varepsilon)\frac{1}{n}R(K)V_{n-1}(M) \\ & \leq (1-\varepsilon)\frac{1}{n}V_1(K)V_{n-1}(M) \\ & \leq V(K, M[n-1]) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) S_{n-1}(M, du) \\ & \leq \frac{1}{n} \int_{\mathbb{S}^{n-1}} (h_s(u) + c_4\sqrt{\varepsilon}) R(K) S_{n-1}(M, du) \\ & = \frac{R(K)}{n} \frac{1}{2} V_1(s) \int_{\mathbb{S}^{n-1}} |\langle u, e \rangle| S_{n-1}(M, du) + c_4\sqrt{\varepsilon} \frac{R(K)}{n} 2V_{n-1}(M), \end{aligned}$$

and hence

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\langle u, e \rangle| S_{n-1}(M, du) & \geq (1-\varepsilon)2V_{n-1}(M) - c_4 \frac{c_2\sqrt{\varepsilon}}{2 - c_3\varepsilon} 2V_{n-1}(M) \\ & \geq 2V_{n-1}(M)(1 - c_{13}\sqrt{\varepsilon}). \end{aligned}$$

An application of Proposition 4.1 completes the proof. \square

5 An application

Let X be a stationary and isotropic Poisson hyperplane process in \mathbb{R}^n with intensity measure

$$\Theta = \lambda \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \in \cdot\} dt \sigma_0(du),$$

where $\lambda > 0$ is the intensity, σ_0 is the spherical Lebesgue measure normalized as a probability measure and $H(u, t) = \{x \in \mathbb{R}^n : \langle x, u \rangle = t\}$. We refer to [9] for an introduction to geometric Poisson processes (of hyperplanes) and random tessellations. Kendall's problem concerns the shape of large cells in random tessellations which are driven by a Poisson process. A very general view and treatment of problems of this kind, for Poisson hyperplane and Poisson Voronoi tessellations, for different types of cells and various size assumptions (and under weaker probabilistic invariance assumptions) has been developed in [4]. Here

we briefly explain how the current stability result can be used to supplement a remark at the end of Section 5 in [4, Added in Proof].

Specifically, in the general framework developed in [4], we choose as the size functional the circumradius, that is, $\Sigma = R$ (which is homogeneous of degree $k = 1$). Moreover, we have the hitting functional

$$\Phi(K) = \int_{\mathbb{S}^{n-1}} h_K(u) \sigma_0(du) = \frac{\kappa_{n-1}}{n\kappa_n} V_1(K).$$

Then

$$\Phi(K) = \frac{\kappa_{n-1}}{n\kappa_n} V_1(K) \geq \frac{2\kappa_{n-1}}{n\kappa_n} R(K) = \tau R(K),$$

which defines the isoperimetric constant τ , with equality if and only if K is a segment. Moreover, we define the deviation functional

$$\vartheta(K) = \min\{r \geq 0 : s \text{ is a segment, } s \subset R(K)^{-1}K \subset s + \sqrt{r}B^n, V_1(s) \geq 2 - r\}. \quad (5.1)$$

Then Theorem 1.4 shows that if $\vartheta(K) \geq \varepsilon$, then

$$\Phi(K) \geq (1 + c_{14}\varepsilon) \frac{2\kappa_{n-1}}{n\kappa_n} R(K),$$

where c_{14} is a positive constant which depends on n . Thus an application of [4, Theorem 1] yields Theorem 1.5.

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