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Triangle areas in line arrangements

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ABSTRACT

A widely investigated subject in combinatorial geometry, originated from Erdős, is the following. Given a point set P of cardinality n in the plane, how can we describe the distribution of the determined distances? This has been generalized in many directions.

In this paper we propose the following variants. What is the maximum number of triangles of unit area, maximum area or minimum area, that can be determined by an arrangement of n lines in the plane?

We prove that the order of magnitude for the maximum occurrence of unit areas lies between $\Omega(n^2)$ and $O(n^{9/4+\varepsilon})$, for every $\varepsilon > 0$. This result is strongly connected to additive combinatorial results and Szemerédi–Trotter type incidence theorems. Next we show an almost tight bound for the maximum number of minimum area triangles. Finally, we present lower and upper bounds for the maximum area and distinct area problems by combining algebraic, geometric and combinatorial techniques.

1. Introduction

There are number of interesting question originating from Erdős, concerning distances in planar point sets [8]. He asked to determine the maximum number of equal distances that n planar points can form, the minimum number of distinct distances they can form and the maximum number of appearances of the largest/smallest distance. He also considered how large subset is guaranteed to exist in a point set such that the distances within that subset are distinct.

Erdős and Purdy also studied the related problem of the maximum number of occurrences of the same area among the triangles determined by n points in the plane [11]. Since then, several variants have been established and the former results of Erdős and Purdy have been settled for some cases, see e.g. [5,11,21].

In this paper we consider the following variants of the original problem, which can be considered as the dual setting. We are given n lines on the Euclidean plane and we are seeking for conditions on the distribution of the areas of triangles

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formed by the triples of lines. More precisely, we investigate the following four main problems and compare the results to the corresponding problems concerning triples of points.

Problem 1.1. Determine the largest possible number $f(n)$ of triangles of unit area formed by n lines in the Euclidean plane.

Problem 1.2. Determine the largest possible number $m(n)$ of triangles having minimum area formed by n lines in the Euclidean plane.

Problem 1.3. Determine the largest possible number $M(n)$ of triangles with maximum area formed by n lines in the Euclidean plane.

Problem 1.4. Determine the largest possible number $D(n)$ such that in any arrangement of n lines (satisfying some generality conditions) there are D(n) lines that form triangles of distinct areas.

Concerning these problems, we achieved the following results.

Theorem 1.5. *For the maximum number of triangles of unit area, we have*

$$f(n) = O\left(n^{\frac{9}{4}+\varepsilon}\right)$$

for every fixed $\varepsilon > 0$, while $f(n) = \Omega(n^2)$.

Theorem 1.6.

$$\left\lfloor \frac{n^2-n}{6} \right\rfloor \leq m(n) \leq \left\lfloor \frac{n^2-2n}{3} \right\rfloor$$

holds for the occurrences of the minimum area, if $n \geq 6$.

Theorem 1.7. *For the maximum number of triangles of maximum area, we have*

$$\frac{7}{5}n - O(1) < M(n) < \frac{2}{3}n(n-2).$$

Theorem 1.8. *For the largest subset of lines forming triangles of distinct areas, we have*

$$n^{\frac{1}{5}} < D(n),$$

provided that there are no six lines that are tangent to a common conic.

If one wishes to find a large subset of lines defining triangles that have distinct areas, it is necessary to make some additional assumptions about the set of lines we are considering. The most natural one is to assume that there are no parallel lines in the set, and no three of them through a common point. However, we were not able to obtain non-trivial bounds under these assumptions. Since 5 lines always have a common tangent conic (see e.g. in [1, p. 66]), another natural general position assumption is to require that no 6 of them do, as in the hypothesis of Theorem 1.8.

To put these results into perspective, let us recall a related problem, first asked by Oppenheim in 1967, which reads as follows: What is the maximum number of triangles of unit area that can be determined by n points in the plane? The first breakthrough after the investigation of Erdős and Purdy [11] was due to Pach and Sharir [18], who obtained an upper bound $O(n^{2+1/3})$ via a Szemerédi–Trotter type argument. This bound was improved in [2,7] and recently by Raz and Sharir to $O(n^{2+2/9})$ in [21]. Here the lower bound is a simple lattice construction from [11], yielding $\Omega(n^2 \log\log n)$. Our Theorem 1.5 also indicates that the straightforward application of some Szemerédi–Trotter type result can be improved. However, in the next Section we will point out that in some relaxation, it would provide the right order of magnitude.

As in the case of counting equal distances, the minimum and maximum area problems determined by point sets turned out to be easier, and they were asymptotically settled by Brass, Rote and Swanepoel [5]. The result on the minimum area problem was later refined in [7]. Concerning the occurrences of the maximum area, the upper bound happens to be exactly n. This is a rather common phenomenon in this field, we could mention the well-known theorem of Hopf and Pannwitz and similar results, see [4]. Surprisingly, Theorem 1.7 shows that this is not the case in our problem.

The problem of the largest subset of points with distinct pairwise distances was originally posed by Erdős [9] and generalized recently to distinct k-dimensional volumes in $\mathbb{R}^d$ by Conlon et al. [6]. For a point of comparison, they note that in the case $d = 2$, $\Omega(n^{1/5})$ points can be chosen from a set of n points in general position so that the triples in this subset determine triangles of distinct areas. The best upper bound so far is attained by choosing $\Omega(n)$ points in general position on the $n \times n$ grid. From Pick's Theorem [20] it is clear that twice the area of a lattice triangle is an integer. Therefore, lattice triangles on this grid define at most $O(n^2)$ areas, so the upper bound for the problem is $O(n^{2/3})$.

The paper is built up as follows. In Section 2 we discuss Problem 1.1 and prove Theorem 1.5. In order to do this, we consider first the maximum number of unit area triangles lying on a fixed line, and prove tight results up to a constant factor. Next we will apply a deep result of Pach and Zahl to complete the proof of our main theorem.

Section 3 is devoted to Problems 1.2 and 1.3, and we prove Theorems 1.6 and 1.7. Section 4 concerns Problem 1.4 and contains the proof of Theorem 1.8. Finally, we discuss some related problems and open questions in Section 5.

2. The number of unit area triangles

2.1. Unit area triangles supported by a single line

A natural way to give an upper bound on $f(n)$ is to consider how many of the unit area triangles can be supported by a fixed line. Then $f(n)$ is at most $n/3$ times larger.

Problem 2.1. Let ℓ be a line and $\mathcal{L}$ be a set of n lines and consider the triangles formed by ℓ and two elements of $\mathcal{L}$. Determine the largest possible number $g(n)$ of triangles of unit area among these.

We determine the order of magnitude of $g(n)$ by turning the problem into an incidence problem for points and lines.

Theorem 2.2. *For the maximum number of triangles of unit area having a common supporting line ℓ, $g(n) = \Theta(n^{4/3})$ holds.*

We may assume that ℓ is horizontal, and the lines in $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$ are not horizontal lines. Let x_i denote the x-coordinate of the intersection of ℓ and ℓ_i and let $y_i = \cot \alpha_i$ where α_i denotes the (directed) angle determined by ℓ and ℓ_i. Let T_{ij} denote the triangle formed by ℓ, ℓ_i and ℓ_j. Notice that the parameters (x_i, y_i) $i = 1, \dots, n$ provide an exact description of any line not parallel to ℓ, while a parallel line $\ell' \parallel \ell$ would not contribute to the number of unit area triangles supported by ℓ. Let us denote by $e(x, y)$ the line described by parameters (x, y).

Lemma 2.3. *Assume that $x_i \neq x_j$ and $y_i \neq y_j$. The area of triangle T_{ij} is*

$$Area(T_{ij}) = \frac{(x_j - x_i)^2}{2|(y_i - y_j)|}.$$

Proof. The equations of the lines $e(x_i, y_i)$ and $e(x_j, y_j)$ are $y = \frac{x - x_i}{y_i}$ and $y = \frac{x - x_j}{y_j}$ respectively. Therefore, their intersection point is $(x, y) = \left(\frac{x_j y_i - x_i y_j}{y_i - y_j}, \frac{x_j - x_i}{y_i - y_j}\right)$.

$$\text{Area}(T_{ij}) = \frac{1}{2}|x_j - x_i| \left|\frac{x_j - x_i}{y_i - y_j}\right| = \frac{(x_j - x_i)^2}{2|(y_i - y_j)|} \quad \square$$

Proof of Theorem 2.2. We apply the lemma above. Supposing that $y_i > y_j$, T_{ij} is of unit area if and only if $2y_i - x_i^2 = -2x_i x_j + x_j^2 + 2y_j$. In other words T_{ij} is of unit area if and only if the point $(x_i, 2y_i - x_i^2)$ lies on the line $y = -2x_j x + 2y_j + x_j^2$.

By the Szemerédi–Trotter theorem, n lines and n points have $O(n^{4/3})$ incidences. Applying this to the lines $y = -2x_j x + 2y_j + x_j^2$ and the points $(x_i, 2y_i - x_i^2)$ we get $g(n) = O(n^{4/3})$.

On the other hand there exist $n/2$ lines and $n/2$ points that have $\Omega(n^{4/3})$ incidences. We can write these points in the form $(x_i, 2y_i - x_i^2)$ for some $(x_1, y_1), \dots, (x_{n/2}, y_{n/2})$. Similarly we can write the lines in the form $y = -2x_j x + 2y_j + x_j^2$ for some $(x_{n/2+1}, y_{n/2+1}), \dots, (x_n, y_n)$. The n lines given by the assignment $(x_i, y_i) \to e(x_i, y_i)$ determine $\Omega(n^{4/3})$ unit area triangles. Therefore, $g(n) = \Theta(n^{4/3})$. $\square$

Let us mention that the same upper bound is also implied by the powerful theorem of Pach and Sharir [19].

Theorem 2.4 ([19])**.** *Let P be a set of m points and let Γ be a set of n distinct irreducible algebraic curves of degree at most k, both in $\mathbb{R}^2$. If the incidence graph of $P \times \Gamma$ contains no copy of $K_{s,t}$, then the number of incidences is*

$$O(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n).$$

Indeed, the lines were described by their parameters $(x_1, y_1), \dots, (x_n, y_n)$, and consider the unit parabolas $2y = x^2 - 2xx_j + x_j^2 - 2y_j$ $(j = 1, \dots, n)$. The ith point lies on the jth parabola if and only if the triangle T_{ij} has unit area. A unit parabola is determined by two of its points, so the incidence graph does not contain $K_{2,2}$ and Theorem 2.4 can be applied for $s = t = 2$. Since we have n points and n curves, the number of unit area triangles having a common supporting line is $O(n^{2/3} n^{2/3}) = O(n^{4/3})$.

Corollary 2.5. *The bound above yields $f(n) = O(n^{7/3})$ for the maximum number of unit area triangles.*

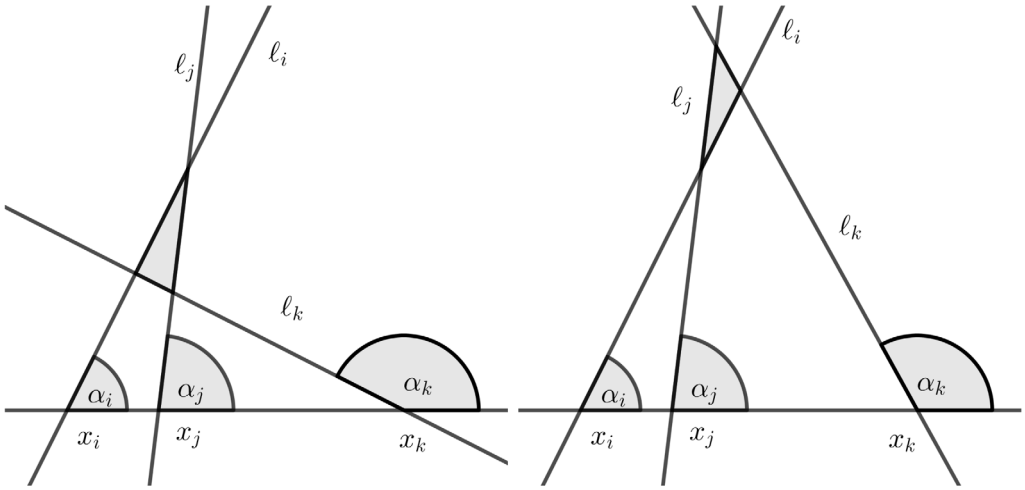


Fig. 1. The calculation of the triangle area, formed by the lines ℓ_i, ℓ_j and ℓ_k.

2.2. Upper bound on the maximum number of unit area triangles

2.2.1. Reformulation in additive combinatorics

For any arrangement of n lines, an y-axis can be chosen such that the parameters of these lines as introduced above in Lemma 2.3, satisfy $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$. Let $H = \{(x_i, y_i) \mid 1 \leq i \leq n\} \subseteq \mathbb{R}^2$.

Proposition 2.6. *The number of unit-area triangles in the arrangement equals the number of solutions in $H \times H \times H$ to the (rational) equation*

$$\frac{(x_j - x_i)^2}{y_i - y_j} + \frac{(x_k - x_j)^2}{y_j - y_k} + \frac{(x_i - x_k)^2}{y_k - y_i} = 2. \tag{1}$$

Proof. We may assume by rotation that none of the n lines are horizontal, and consider a horizontal line ℓ located under all the intersections of the n lines. Taking ℓ as the x axis of a coordinate system, we let x_i to be the coordinate of the ith intersection of ℓ with another line (which we denote by ℓ_i). Let α_i denote the (directed) angle appearing between ℓ and ℓ_i, see Fig. 1. Let $y_i = \cot \alpha_i$. Since there are no intersections under or on ℓ, we have $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and therefore $y_1 \geq y_2 \geq \cdots \geq y_n$.

Since $y_i \geq y_j$, the area of the triangle T_{ij} determined by ℓ, ℓ_i and ℓ_j is $\frac{(x_j - x_i)^2}{|2(y_i - y_j)|} = \frac{(x_j - x_i)^2}{2(y_i - y_j)}$. (If $y_i = y_j$, then ℓ_i and ℓ_j are parallel, and they form no triangle.) The area of the triangle determined by the lines ℓ_i, ℓ_j and ℓ_k can be calculated as

$$Area(T_{ij}) + Area(T_{jk}) - Area(T_{ik}) = \frac{(x_j - x_i)^2}{2(y_i - y_j)} + \frac{(x_k - x_j)^2}{2(y_j - y_k)} - \frac{(x_k - x_i)^2}{2(y_i - y_k)} =$$

$$\frac{(x_j - x_i)^2}{2(y_i - y_j)} + \frac{(x_k - x_j)^2}{2(y_j - y_k)} + \frac{(x_i - x_k)^2}{2(y_k - y_i)}.$$

Therefore, the problem of finding an arrangement of lines determining $f(n)$ triangles of unit area is equivalent to finding some reals $x_1 < x_2 < \cdots < x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ such that (1) is satisfied for the maximal number of index triples. □

2.2.2. Improved upper bound for $f(n)$

We improve here the bound achieved by Corollary 2.5. To do this, we recall a recent result of Sharir and Zahl [22], which is a strengthening of Theorem 2.4.

Theorem 2.7 (*Incidences Between Points and Algebraic Curves, [22]*)**.** *Let P be a set of m points in the plane. Let $\mathcal{C}$ be a set of c algebraic plane curves of degree at most D, no two of which share a common irreducible component. Assume that we can parameterize these curves using s parameters. Then for any $\varepsilon > 0$, the number $I(P, \mathcal{C})$ of incidences between the points of P and the curves of $\mathcal{C}$ satisfies*

$$I(P, \mathcal{C}) = O\left(m^{\frac{2s}{5s-4}} c^{\frac{5s-6}{5s-4}+\varepsilon} + m^{2/3} c^{2/3} + m + c\right).$$

Now we are ready to prove our main result.

Theorem 2.8. *For every fixed $\varepsilon > 0$, the maximum number of triangles of unit area satisfies*

$$f(n) = O(n^{\frac{9}{4}+\varepsilon}).$$

Proof. Consider the additive combinatorial equivalent form of the problem in Eq. (1), take the solution set with maximum number of solutions and denote it by H. For every ordered pair $(x_i, y_i), (x_j, y_j)$ where $x_i < x_j$, the solutions of Eq. (1) are points (x_k, y_k) of a bounded degree rational curve defined by (1), with the condition that $x_j < x_k$ must hold. Hence we obtain $m = n$ points on at most $c = \binom{n}{2}$ plane curves belonging to an $s = 4$-dimensional family, as the family depends on the real values $\{x_i, y_i, x_j, y_j\}$. One can verify also rather easily that no two of these curves share a common irreducible component. This can be done either directly, by deducing from Eq. (1) an equivalent reformulation, a degree 2 polynomial in variables x_k, y_k where the coefficients are (polynomial) functions of x_i, y_i, x_j, y_j, or by referring to Lemma 4.1 which provides a description of the locus of the points (x_k, y_k) satisfying (1) in terms of $\{x_i, y_i, x_j, y_j\}$. Hence, applying the result of Sharir and Zahl (Theorem 2.7), we get the desired bound. □

The lower bound for $f(n)$ follows from the results in the next section, by scaling the triangles of minimum area to have area 1, which have a quadratic cardinality in n.

3. Number of maximum and minimum area triangles, bounds on $m(n)$ and $M(n)$

3.1. Minimum area triangles

In this subsection we prove Theorem 1.6 by determining the maximal possible number of triangles of minimal area constituted by n lines, up to a factor of 2. This will follow from the results on the lower and upper bound below.

Proposition 3.1. *$m(n) \leq \lfloor n(n-2)/3 \rfloor$ for every n and $m(n) \leq \lfloor n(n-2)/3 \rfloor - 1$ if $n \equiv 0, 2 \pmod 6$.*

Proof. Observe that if a triangle is of minimal area, then none of the lines can intersect its sides. Hence the maximal number of triangles of minimal area is at most the number of triangular faces $K(n)$ in an arrangement of n lines. The latter problem became famous as the so-called Tokyo puzzle or the problem of Kobon triangles. The best bound is by Bader and Clément [3], who showed that $K(n) \leq \lfloor n(n-2)/3 \rfloor$ for every n and $K(n) \leq \lfloor n(n-2)/3 \rfloor - 1$ if $n \equiv 0, 2 \pmod 6$. □

The bound on $K(n)$ is almost sharp since Füredi and Palásti constructed a general arrangement to prove $K(n) \geq \lfloor n(n-3)/3 \rfloor$ [14]. See also the construction of Forge and Ramírez-Alfonsín [13].

Proposition 3.2. *Assume that $n \geq 3$. Then*

$$m(n) \geq \begin{cases} 6l^2 & \text{if } n = 6l, \\ 6l^2 + 2jl + j - 2 & \text{if } n = 6l + j,\ 1 \leq j \leq 5. \end{cases}$$

Proof. Take the grid depicted in Fig. 2. Choose n lines such that they are as close to the center of a hexagonal face as possible. If there are 2, 3 or 4 lines in the outermost layer, pick these to be in consecutive clockwise position.

Assume that $n = 6l$. Add the lines to the diagram layer by layer, starting from the center. Adding the six lines of the ith layer will create $6(2i-1)$ new triangular faces ($6i$ outside the hexagon formed by those lines and $6(i-1)$ inside it). Therefore, the total number of triangular faces is $\sum_{i=1}^{l} 6(2i-1) = 6l^2$. If $n = 6l + j$ the result follows similarly with elementary counting on the outermost layer. □

Conjecture 3.3. *The lower bound of Proposition 3.2 is sharp if n is large enough.*

Note that these lower bounds are not met if n is small. C. T. Zamfirescu [23] recently proved that even the number of facial *congruent* triangles exceeds this bound if $n \leq 12$, see Table 1. On the other hand, the construction described in Proposition 3.2 provides a general lower bound as well for the number of facial *congruent* triangles in terms of the number of lines, which exceeds the bound of C.T. Zamfirescu if n is large.

We can obtain the same order of magnitude in an essentially different way as well.

Proposition 3.4. *Assume that $n \geq 3$. Then*

$$m(n) \geq \begin{cases} 6l^2 + 2jl - 2 & \text{if } n = 6l + j,\ j \in \{0, \pm 1, \pm 2\}, \\ 6l^2 + 6l & \text{if } n = 6l + 3. \end{cases}$$

Table 1
Comparison of the constructions for the number of congruent or minimal area triangles in small cases.

# of lines, n	3	4	5	6	7	8	9	10	11	12
# of congruent facial triangles, lower bound	1	2	5	6	≥ 9	≥ 12	≥ 15	≥ 20	≥ 23	≥ 26
# of congruent triangles, lower bound via Proposition 3.2	1	2	3	6	7	10	13	16	19	24
# of congruent triangles, lower bound via Proposition 3.4	0	1	2	4	6	8	12	14	18	22

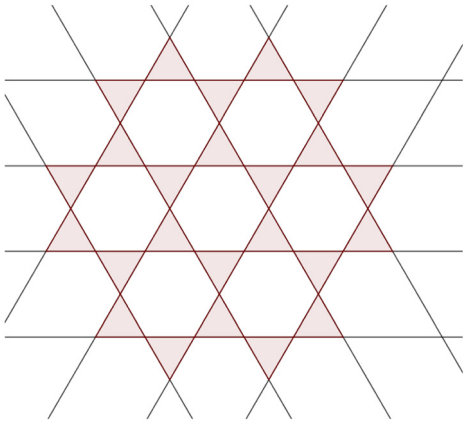


Fig. 2. A hexagonal grid formed by 12 lines.

Proof. Take a triangular grid. If $n \equiv 3 \pmod 6$, choose those n lines of the grid, which are the closest to a fixed point on the grid. If $n \not\equiv 3 \pmod 6$, choose those n lines of the grid, which are the closest to a fixed point, which is the center of a triangle in the grid. A simple inductive argument similar to the one in Proposition 3.2 shows that the number of constructed facial triangles equals the desired quantity, see Fig. 3. □

The above constructions differ in several aspects. Firstly, the former one does not contain concurrent triples of lines. Secondly, note that in the upper bound on $K(n)$ of Bader and Clément [3] a key observation was that every line segment between consecutive intersections on a line belongs to at most one triangular region. This property appears only in the former construction. Thus if Conjecture 3.3 holds, it would imply that the arrangements that almost attain the extremum may have significantly different structure.

3.2. Maximum area triangles

We start with a construction to prove the lower bound of Theorem 1.7 on the number of maximum area triangles. The main idea is the following. Suppose you have a construction with some number of maximal area triangles. Then we can add a new line that does not create large triangles, i.e. the maximal area does not increase. We can slide this line until it creates an extra triangle of maximal area. This way we can create a new maximal area triangle per line. To improve this we will show that we can add five lines together to get seven new maximal area triangles. Five of the new maximal triangles will appear between these five new lines and then by sliding the five lines together we will get two extra ones.

The precise construction requires a couple of lemmas first.

Proposition 3.5. *Let ABC be one of the maximal area triangles in the arrangement and let ℓ be one of the lines of the arrangement. Then either ℓ intersects the interior of ABC or it is parallel to one of the sides of ABC.*

Proof. If ℓ is not parallel to one of the three sides, then it intersects each of the three lines. Suppose ℓ avoids the interior of the triangle. By symmetry we may assume that ℓ runs as in Fig. 4(a). Hence $A'BC'$ is a triangle of larger area which contradicts the maximality of ABC. □

Proposition 3.6. *Assume that there are no parallel lines in the arrangement and that the triangle Δ, formed by lines (ℓ_1, ℓ_2, ℓ_3), is a maximal area triangle. Then all the maximal area triangles that are supported by ℓ_1 lie on the same side of ℓ_1.*

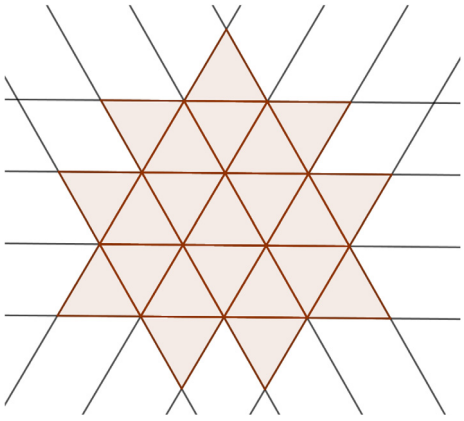


Fig. 3. A triangular grid formed by 12 lines.

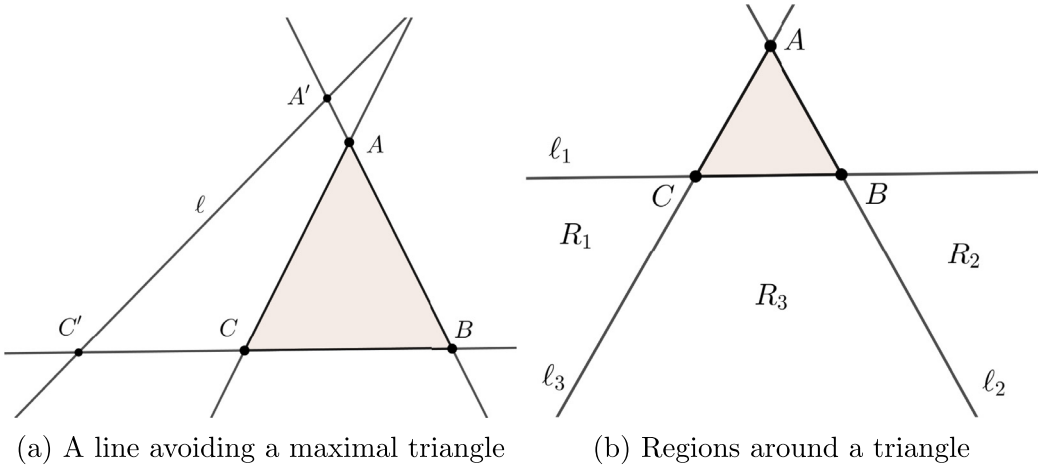


Fig. 4. Line positions with respect to a maximum area triangle.

Proof. Suppose that a triangle Δ', formed by the lines (ℓ_1, ℓ_4, ℓ_5), is also a maximal area triangle and it lies on the opposite side of ℓ_1. Let $P = \ell_4 \cap \ell_5$ and consider the possible positions of P. We will denote the three regions by R_1, R_2 and R_3 as seen in Fig. 4(b).

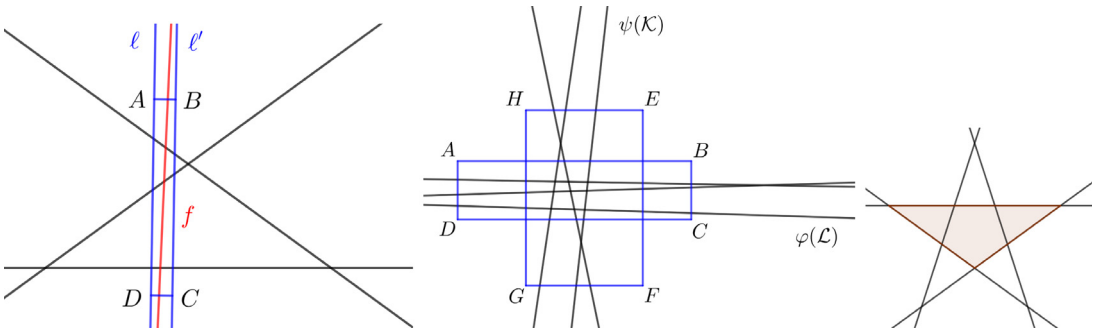
Suppose that P lies in the interior of $R_1 \cup R_3$. By Proposition 3.5 we know that ℓ_4 and ℓ_5 must intersect the interior of the triangle ABC, therefore they intersect ℓ_1 on the interior of the $\overrightarrow{BC}$ ray. But then the line ℓ_2 avoids the maximal triangle Δ', contradicting Proposition 3.5. Similarly P cannot lie in the interior of $R_2 \cup R_3$. □

Proposition 3.7. *If there are no parallel lines in an arrangement, then we can add a new line ℓ to the arrangement such that it supports no maximal area triangle in the new arrangement.*

Proof. Pick an arbitrary direction that is not parallel to any of the lines of the arrangement. Choose ℓ to be the line that has the chosen direction and for which the largest new triangle area created is the smallest possible. Suppose this area is q. Then ℓ must support two triangles on opposite sides that have area q. Otherwise we could translate ℓ slightly in one direction to decrease all the new areas below q. By Proposition 3.6 this implies that the q cannot be the maximal area in the whole arrangement. □

Proposition 3.8. *If there are no parallel lines in an arrangement, then we can find a rectangle ABCD such that if we add any line to the arrangement that intersects both AB and CD we create no new maximal area triangles.*

Proof. By Proposition 3.2 we can find a line ℓ that creates no new maximal area triangles. Let ℓ' be a line parallel to ℓ which also does not create a new maximal triangle and lies so close to ℓ that no two line of the arrangement intersects each other between ℓ and ℓ'. Then any line f that intersects all lines of the arrangement between ℓ and ℓ' does not create



(a) No new maximal triangles. (b) Combining two constructions (c) Pentagon

Fig. 5. Ingredients for the recursive construction.

a new maximal area triangle. This follows from the fact that if f supports a triangle, then either ℓ or ℓ' avoids that triangle, so by Proposition 3.5 the triangle cannot be maximal. Next we can choose points A, D on ℓ and B, C on ℓ' appropriately, see Fig. 5(a). □

For an arrangement $\mathcal{L}$ let $T(\mathcal{L})$ denote the number of maximal area triangles. For example $T(\mathcal{L}) = 5$ if $\mathcal{L}$ consists of five lines forming a regular pentagon. For an affine transformation φ let $\varphi(\mathcal{L})$ denote the image of $\mathcal{L}$.

Proposition 3.9. *If $\mathcal{L}$ and $\mathcal{K}$ are arrangements of lines that contain no parallel lines, then there exist affine transformations φ and ψ such that $T(\varphi(\mathcal{L}) \cup \psi(\mathcal{K})) \geq T(\mathcal{L}) + T(\mathcal{K}) + 2$.*

Proof. We may assume that the maximal area triangles have the same area in $\mathcal{L}$ and $\mathcal{K}$. Using Proposition 3.8 we can define rectangle *ABCD* for $\mathcal{L}$ and rectangle *EFGH* for $\mathcal{K}$. Then, applying an area preserving affine transformation, we can place the two construction such that the two rectangles cross each other (see Fig. 5(b)). Now every line of $\varphi(\mathcal{L})$ crosses *EF* and *GH* and every line of $\psi(\mathcal{K})$ crosses *AB* and *CD*. By Proposition 3.8 this means that in the new construction the maximal area triangles are the same as they are in $\mathcal{K}$ and $\mathcal{L}$. So we have exactly $T(\mathcal{L}) + T(\mathcal{K})$ maximal triangles.

Finally, we increase this number by two in two steps. Translate first the lines of $\varphi(\mathcal{L})$ together in an arbitrary direction until a new maximal area triangle appears, formed by lines both from the translates of $\varphi(\mathcal{L})$ and $\psi(\mathcal{K})$. We may assume that only one such triangle Δ^* is formed, and it has exactly one supporting line ℓ^* in $\psi(\mathcal{K})$. Now if we translate again the lines of $\varphi(\mathcal{L})$, this time along the line ℓ^*, then obviously neither the area of triangles formed by the lines from $\varphi(\mathcal{L})$ or $\psi(\mathcal{K})$, nor the area of Δ^* will change. However, some translated lines of $\varphi(\mathcal{L})$ will eventually form yet another triangle of maximal area together with some lines from $\psi(\mathcal{K})$. □

It is easy to see that the lower bound of Theorem 1.7 follows. We start with five lines forming a regular star pentagon (see Fig. 5(c)). Then we use Proposition 3.9 repeatedly, always using the previous construction as $\mathcal{L}$ and five lines forming a regular pentagon as $\mathcal{K}$.

Theorem 3.10. $M(n) \leq \frac{2}{3}n(n-2)$.

Proof. We will show that in an arrangement of n lines, any fixed line ℓ supports at most $2(n-2)$ triangles of maximal area. This immediately implies the statement of the theorem.

Let ℓ be a fixed line in the arrangement. We may assume that all other lines intersect it as otherwise they would not form any triangle together. Consider ℓ as the x axis of a coordinate system, and let x_i denote the x coordinate of the intersection of ℓ and ℓ_i for all $i = 1, 2, \ldots, n-1$. We also use the notation y_i for the cotangent of the (directed) angle determined by ℓ and ℓ_i. By Lemma 2.3, the area of the triangle T_{ij} determined by the lines ℓ, ℓ_i and ℓ_j is $\text{Area}(T_{i,j}) = \frac{(x_i - x_j)^2}{2|y_i - y_j|}$. (If $x_i = x_j$ or $y_i = y_j$, then there is no triangle to speak of.) If the sign of $x_i - x_j$ and $y_i - y_j$ is the same, then the triangle is located under ℓ, otherwise it is located over it.

Without loss of generality, we may assume that the maximal triangle area is $1/2$. Then $(x_i - x_j)^2 \leq |y_i - y_j|$ applies to all pairs (i, j), with equality if and only if $\text{Area}(T_{i,j})$ is maximal.

Let us define the graph G_ℓ^+, and resp. G_ℓ^- on the vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$ and connect v_i to v_j if $(x_i - x_j)^2 = |y_i - y_j|$ and the sign of $x_i - x_j$ and $y_i - y_j$ is the same or respectively, the opposite. We will show that there is no cycle in G_ℓ^+, therefore $|E(G_\ell^+)| \leq n - 2$ holds for the cardinality of the edge set. The same argument applies to G_ℓ^- as well, yielding $|E(G_\ell^-)| \leq n-2$. Therefore, the total number of edges, which is equal to the number of triangles of maximal area supported by ℓ, is at most $2(n-2)$.

Suppose that there is a cycle $v_{i_1} v_{i_2} \ldots v_{i_k}$ in G_ℓ^+. We will get a contradiction using two simple propositions. In the following two statements we consider the indexing of the vertices modulo k.

Proposition 3.11. *The signs of $x_{i_t} - x_{i_{t+1}}$ and $x_{i_{t+1}} - x_{i_{t+2}}$ are the opposite.*

Proof. Suppose that the signs are the same. Then

$$|y_{i_{t+2}} - y_{i_t}| = |y_{i_{t+2}} - y_{i_{t+1}}| + |y_{i_{t+1}} - y_{i_t}| = (x_{i_{t+1}} - x_{i_{t+2}})^2 + (x_{i_t} - x_{i_{t+1}})^2 < (x_{i_t} - x_{i_{t+2}})^2$$

would hold, a contradiction. □

Proposition 3.12. *There are no four vertices v_a, v_b, v_c and v_d in G_ℓ^+ such that $x_a < x_b < x_c < x_d$ and $v_a v_c$, $v_b v_c$, $v_b v_d \in E(G_\ell^+)$.*

Proof. Suppose that there are four such vertices. Then

$$(x_d - x_a)^2 \leq y_d - y_a = (y_d - y_b) + (y_c - y_a) - (y_c - y_b) = (x_d - x_b)^2 + (x_c - x_a)^2 - (x_c - x_b)^2$$

After rearranging, we get

$$x_a x_c + x_b x_d \leq x_a x_d + x_b x_c,$$

which can be written as $(x_a - x_b)(x_c - x_d) \leq 0$, a contradiction. □

Returning to the cycle $v_{i_1} v_{i_2} \dots v_{i_k}$, Proposition 3.11 implies that k is even. We may assume that $|x_{i_1} - x_{i_2}| > |x_{i_2} - x_{i_3}|$ after shifting the indexing of the vertices if necessary. This means that x_{i_3} is between x_{i_1} and x_{i_2}.

Proposition 3.11 tells us that x_{i_4} must be in the same direction from x_{i_3} as x_{i_2}. However Proposition 3.12 implies that it cannot be past x_{i_2}. Note that $x_{i_2} = x_{i_4}$ is also impossible since this would imply $y_{i_2} = y_{i_4}$ and $\ell_{i_2} = \ell_{i_4}$. Therefore, x_{i_4} must be between x_{i_2} and x_{i_3}.

Following this argument, we find that $x_{i_{t+2}}$ must be between x_{i_t} and $x_{i_{t+1}}$ for all $t = 1, 2, \dots, k-2$. Then the vertices v_1, v_{k-1}, v_k, v_{k-2} violate Proposition 3.12, a contradiction. □

Remark 3.13. Theorem 3.10 can be even strengthened, as $M(n) \leq \frac{1}{3}n(n-1)$ also holds. Indeed, one can verify that Proposition 3.6 is true in a more general form, namely if there are parallel lines in the line arrangement, then there may exist maximal area triangles on both sides of a fixed line ℓ, but on one of the sides there is no more than one maximal area triangle. This result yields $|E(G_\ell^-)| + |E(G_\ell^+)| \leq n-1$ in the proof above, implying our stated improvement. The details are left to the interested reader.

4. Lines defining triangles with distinct areas

In this section we assume that the lines in the original arrangement are in general position. More specifically, we will require that no six of them are tangent to a common quadratic curve in the plane.

To prove Theorem 1.8, we begin with the following result.

Lemma 4.1. *Let r_1 and r_2 be two rays from a point O and $\lambda \in \mathbb{R}^+$ fixed. Then those lines that form a triangle with r_1 and r_2 of area λ are all tangent to a fixed hyperbola. The two rays belong to the asymptotes of this hyperbola.*

Proof. Affine transformations preserve lines, conics and ratios of areas. Therefore, we may assume that r_1 and r_2 are perpendicular and correspond to the positive parts of the x and y axis, respectively.

Now, for a positive real number c consider the hyperbola $xy = c$ and any point (x_1, y_1) on it. The tangent t at (x_1, y_1) is given by the equation $xy_1 + yx_1 = 2c$. Let $P_1 = (2c/y_1, 0)$ and $P_2 = (0, 2c/y_2)$ be the intersections of t and the x and y axis, respectively. Then the area of the triangle OP_1P_2 is $\frac{4c^2}{2x_1y_1} = 2c$.

Any line that intersects the positive parts of the x and y axis must be tangent to exactly one of these hyperbolas, and as seen above the area of the triangle it defines depends completely and injectively on c. Therefore, triangles with the same area must all be tangent to a fixed one of these hyperbolas. □

From here, we deduce the following result.

Corollary 4.2. *Let ℓ_1 and ℓ_2 be two intersecting lines. Then for any fixed value $\lambda \geq 0$, there can be at most 20 lines in general position such that each of them forms a triangle with ℓ_1 and ℓ_2 of area λ.*

Proof. Note that ℓ_1 and ℓ_2 define four quadrants. If $\lambda > 0$, each line defines a triangle of positive area with ℓ_1 and ℓ_2, so it intersects both rays of one of the quadrants. By Lemma 4.1, we can have at most 5 lines per quadrant, so we obtain at most 20 lines.

If $\lambda = 0$, then each line has to go through the point of intersection of ℓ_1 and ℓ_2. There can be at most 5 of these lines, as otherwise the intersection point would be a common degenerate conic tangent to 6 lines. □

The second ingredient that we use is a rainbow Ramsey result. We apply the following particular version of a result proven by Conlon et al. [6] and independently by Martínez-Sandoval, Raggi and Roldán-Pensado [17]. It has been used before to obtain similar results in combinatorial geometry in which a large structure with distinct substructures is desired.

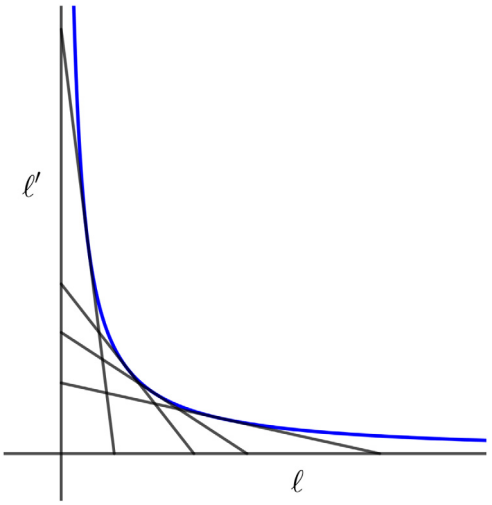


Fig. 6. Maximum area triangles lying on ℓ, formed by ℓ, ℓ', and a third line from the tangent line set.

Theorem 4.3. *Let H be an m-uniform hypergraph and k a positive integer. Assume that the hyperedges of H are colored in such a way that no two vertices lie in k edges of the same color.*

Then there exists a set of

$$\Omega_k(n^{1/(2m-1)})$$

vertices for which all the hyperedges have distinct colors.

We are ready to prove the main result of this section.

Proof of Theorem 1.8. Note that there cannot be six or more lines with the same slope, as otherwise a seventh line would be a degenerate conic tangent to all of them at infinity. Therefore, the lines define at least $\frac{n}{5}$ distinct slopes, and taking at most one for each slope we may extract a subset L' of size at least $\frac{n}{5}$ so that no two lines of L' are parallel.

Consider the complete 3-uniform hypergraph H whose vertex set is L'. Since no two lines of L' are parallel, we may provide a coloring of the 3-edges of H by assigning to each triple the area of the triangle it defines. By Corollary 4.2, no pair of vertices belongs to 21 or more triples of the same color. Therefore, by Theorem 4.3 we obtain a set of $\Omega((n/5)^{1/5}) = \Omega(n^{1/5})$ lines such that the triangles that they define have all distinct areas. □

5. Discussion and open problems

One could also raise here an analogue question to the well known problem due to Erdős, Purdy and Strauss, which is formulated as

Problem 5.1 (*Erdős, Purdy, Straus, [12]*)**.** Let S be a set of n points in $\mathbb{R}^d$ not all in one hyperplane. What is the minimal number of distinct volumes of non-degenerate simplices with vertices in S?

Concerning the case $d = 2$, we refer to e.g. [7] and its reference list. Note that to obtain reasonable results on the cardinality of *distinct areas*, one has to prescribe certain restrictions to avoid huge classes of parallel lines hence obtaining only few triangles. However, having assumed e.g. that no pair of parallel lines appear, the distribution of the areas can change significantly. We conjecture that the number of unit area triangles drops to $O(n^2)$ in that case, and in fact we could not even find evidence of that the order of magnitude is $\Omega(n^2)$.

The proof of the upper bound on the number of maximum area triangles was relying on an argument about maximum area triangles sharing a common line ℓ that provides a linear upper bound. Although it is easy to see that a linear lower bound is realizable by a set of $n-2$ tangent and two asymptotes of a hyperbole branch (see Fig. 6), we conjecture that this will not provide the right (quadratic) order of magnitude for $M(n)$. Note that this phenomenon appeared concerning the unit area triangles as well when we compared $g(n)$ and $f(n)$, see Section 2.

In fact, we believe that the following holds.

Conjecture 5.2. *The order of magnitude of $M(n)$, largest possible number of triangles with maximum area in arrangements of n lines in the plane is $O(n^{1+\varepsilon})$ for every $\varepsilon > 0$.*

In general we have seen that in these types of combinatorial geometry problems, small (or minimum) distances (or areas) may occur much more frequently than large (or maximum) distances (areas).

Supposing that this assertion holds, it raises yet another interesting inverse research problem from a statistical point of view.

Problem 5.3. Suppose that for each n, a set of n lines is given in the Euclidean plane. Assume that the number of triangles having unit area, determined by triples of lines, is $\phi(n)$, where $\phi(n)/n \to \infty$. Prove a lower bound (in terms of $\phi(n)$) on the number of triangles having area greater than 1.

The analogue of Problem 5.3 for the original Erdős–Purdy problem on distances in a planar point set seems also widely open. Some results were obtained by Erdős, Lovász and Vesztergombi [10].

We also note that the problem may be investigated in a finite field setting as well, similarly to [15].

In Theorem 1.8, we assume that no six lines are tangent to a common conic. This implies, in particular, that no six lines are pairwise parallel. A more natural condition would be to simply require that no two lines are parallel. We were not able to obtain any non-trivial bounds under this hypothesis.

Problem 5.4. What is the maximum number $D'(n)$ such that in any arrangement of n lines on the plane, no two of them parallel or three through a common point, there are $D'(n)$ lines that form triangles of distinct areas?

The bound in Theorem 1.8 can be improved by a logarithmic factor, as mentioned in [6]. The problem could also be generalized to higher dimensions as follows.

Problem 5.5. A set of n hyperplanes in general position are given in $\mathbb{R}^d$. What is the maximum number $D_d(n)$ such that we can always find a subset of these hyperplanes of this size for which all the simplices that they define have distinct d-dimensional volumes?

We finish by mentioning that there are very few geometric problems with this combinatorial flavor in which the bounds are asymptotically tight. A related question concerning circumradii is discussed in [16].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] A.V. Akopyan, A.A. Zaslavsky, Geometry of conics, Amer. Math. Soc. (2007).
[2] R. Apfelbaum, M. Sharir, An improved bound on the number of unit area triangles, Discrete Comput. Geom. 44 (4) (2010) 753–761.
[3] J. Bader, G. Clement, Tighter upper bound for the number of Kobon triangles, in: J. Knowles, D. Corne, K. Deb (Eds.), Multiobjective Problem Solving from Nature: From Concepts to Applications, Springer, 2007, pp. 177–200.
[4] P. Brass, W.O.J. Moser, J. Pach, Research Problems in Discrete Geometry, Springer, 2005.
[5] P. Brass, G. Rote, K.J. Swanepoel, Triangles of extremal area or perimeter in a finite planar point set, Discrete Comput. Geom. 26 (1) (2001) 51–58.
[6] D. Conlon, J. Fox, W. Gasarch, D.G. Harris, D. Ulrich, S. Zbarsky, Distinct volume subsets, SIAM J. Discrete Math. 29 (1) (2015) 472–480.
[7] A. Dumitrescu, M. Sharir, C.D. Tóth, Extremal problems on triangle areas in two and three dimensions, J. Combin. Theory Ser. A 116 (7) (2009) 1177–1198.
[8] P. Erdős, On sets of distances of n points, Amer. Math. Monthly 53 (5) (1946) 248–250.
[9] P. Erdős, Néhány geometriai problémáról , Math. Lapok 8 (1957) 86–92 (On some geometrical problems, in Hungarian).
[10] P. Erdős, L. Lovász, K. Vesztergombi, On the graph of large distances, Discrete Comput. Geom. 4 (6) (1989) 541–549.
[11] P. Erdős, G. Purdy, Some extremal problems in geometry, J. Combin. Theory Ser. A 10 (1971) 246–252.
[12] P. Erdös, G. Purdy, E.G. Straus, On a problem in combinatorial geometry, Discrete Math. 40 (1) (1982) 45–52.
[13] D. Forge, J.R. Alfonsin, Straight line arrangements in the real projective plane, Discrete Comput. Geom. 20 (2) (1998) 155–161.
[14] Z. Füredi, I. Palásti, Arrangements of lines with a large number of triangles, Proc. Amer. Math. Soc. 92 (1984) 561–566.
[15] A. Iosevich, M. Rudnev, Y. Zhai, Areas of triangles and Beck's theorem in planes over finite fields, Combinatorica 35 (3) (2015) 295–308.
[16] L. Martínez-Sandoval, Roldán-Pensado E., Points defining triangles with distinct circumradii, Acta Math. Hungar. 145 (1) (2015) 136–141.
[17] L. Martínez-Sandoval, M. Raggi, E. Roldán-Pensado, A sunflower anti-Ramsey theorem and its applications, 2015, ArXiv e-prints ArXiv: 1505.05170.
[18] J. Pach, M. Sharir, Repeated angles in the plane and related problems, J. Combin. Theory Ser. A 59 (1992) 12–22.
[19] J. Pach, M. Sharir, On the number of incidences between points and curves, Combin. Probab. Comput. 7 (1998) 121–127.
[20] G. Pick, Geometrisches zur Zahlenlehre, Sitzungsberichte des deutschen naturwissenschaftlich-medicinischen Vereines für Böhmen Lotos in Prag. (Neue Folge) 19 (1899) 311–319.
[21] O.E. Raz, M. Sharir, The number of unit-area triangles in the plane: Theme and variation, Combinatorica 37 (6) (2017) 1221–1240.
[22] M. Sharir, J. Zahl, Cutting algebraic curves into pseudo-segments and applications, J. Combin. Theory Ser. A 150 (2017) 1–35.
[23] C.T. Zamfirescu, Congruent triangles in arrangements of lines, Ars Math. Contemp. 14 (2) (2017) 359–373.