DIFFERENTIAL INCLUSIONS INVOLVING OSCILLATORY TERMS

ALEXANDRU KRISTÁLY, ILDIKÓ I. MEZEI, KÁROLY SZILÁK

Abstract. Motivated by mechanical problems where external forces are non-smooth, we consider the differential inclusion problem

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in} \quad \Omega; \\ u \geq 0, & \text{in} \quad \Omega; \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \qquad (\mathcal{D}_\lambda)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain, and ∂F and ∂G stand for the generalized gradients of the locally Lipschitz functions F and G. In this paper we provide a quite complete picture on the number of solutions of $(\mathcal{D}_\lambda)$ whenever ∂F oscillates near the origin/infinity and ∂G is a generic perturbation of order $p > 0$ at the origin/infinity, respectively. Our results extend in several aspects those of Kristály and Moroşanu [*J. Math. Pures Appl.*, 2010].

1. Introduction

We consider the model Dirichlet problem

$$\begin{cases} -\Delta u(x) = f(u(x)) & \text{in} \quad \Omega; \\ u \geq 0, & \text{in} \quad \Omega; \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \qquad (P_0)$$

where Δ is the usual Laplace operator, $\Omega \subset \mathbb{R}^n$ is a bounded open domain ($n \geq 2$), and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying certain growth conditions at the origin and infinity. Usually, such a problem is studied on the Sobolev space $H_0^1(\Omega)$ and weak solutions of (P_0) become classical/strong solutions whenever f has further regularity. There are several approaches to treat problem (P_0), mainly depending on the behavior of the function f. When f is superlinear and subcritical at infinity (and superlinear at the origin), the seminal paper of Ambrosetti and Rabinowitz [2] guarantees the existence of at least a nontrivial solution of (P_0) by using variational methods. An important extension of (P_0) is its *perturbation*, i.e.,

$$\begin{cases} -\Delta u(x) = f(u(x)) + \lambda g(u(x)) & \text{in} \quad \Omega; \\ u \geq 0, & \text{in} \quad \Omega; \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \qquad (P_\lambda)$$

where $g : \mathbb{R} \to \mathbb{R}$ is another continuous function which is going to compete with the original function f. When both functions f and g are of *polynomial type* of sub- and super-unit degree, – the right hand side being called as a concave-convex nonlinearity – the existence of at least one or two nontrivial solutions of (P_λ) is guaranteed, depending on the range of $\lambda > 0$, see e.g. Ambrosetti, Brezis and Cerami [1], Autuori and Pucci [4], de Figueiredo, Gossez and Ubilla [8]. In these papers variational arguments, sub- and super-solution methods as well as fixed point arguments are employed.

2010 *Mathematics Subject Classification.* Primary 35R70, Secondary 35J61, 35A15.

Key words and phrases. Differential inclusions; competition; oscillation; Clarke's calculus.

A. Kristály was supported by the National Research, Development and Innovation Office, Hungary, K-18, project no. 127926.

Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th birthday.

Another important class of problems of the type (P_λ) is studied whenever f has a certain *oscillation* (near the origin or at infinity) and g is a *perturbation*. Although oscillatory functions seemingly call forth the existence of infinitely many solutions, it turns out that 'too classical' oscillatory functions do not have such a feature. Indeed, when $f(s) = c\sin s$ and $g = 0$, with $c > 0$ small enough, a simple use of the Poincaré inequality implies that problem (P_λ) has only the zero solution. However, when f *strongly* oscillates, problem (P_0) has indeed infinitely many different solutions; see e.g. Omari and Zanolin [19], Saint Raymond [21]. Furthermore, if $g(s) = s^p$ $(s > 0)$, a novel competition phenomena has been described for (P_λ) by Kristály and Moroşanu [12]. We notice that several extensions of [12] can be found in the literature, see e.g. Ambrosio, D'Onofrio and Molica Bisci [3] and Molica Bisci and Pizzimenti [16] for nonlocal fractional Laplacians; Molica Bisci, Rădulescu and Servadei [17] for general operators in divergence form; Mălin and Rădulescu [15] for difference equations. We emphasize that in the aforementioned papers the perturbations are either zero or have a (smooth) polynomial form.

In mechanical applications, however, the perturbation may occur in a *discontinuous* manner as a non-regular external force, see e.g. the gluing force in von Kármán laminated plates, cf. Bocea, Panagiotopoulos and Rădulescu [5], Motreanu and Panagiotopoulos [18] and Panagiotopoulos [20]. In order to give a reasonable reformulation of problem (P_λ) in such a non-regular setting, the idea is to 'fill the gaps' of the discontinuities, considering instead of the discontinuous nonlinearity a *set-valued map* appearing as the generalized gradient of a locally Lipschitz function. In this way, we deal with an *elliptic differential inclusion* problem rather than an elliptic differential equation, see e.g. Chang [6], Gazzola and Rădulescu [9] and Kristály [10]; this problem can be formulated generically as

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in} \quad \Omega; \\ u \geq 0, & \text{in} \quad \Omega; \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathcal{D}_\lambda$}$$

where F and G are both nonsmooth, locally Lipschitz functions having various growths, while ∂F and ∂G stand for the generalized gradients of F and G, respectively.

The main purpose of the present paper is to extend the main results of Kristály and Moroşanu [12] in two directions:

(a) to allow the presence of nonsmooth nonlinear terms – reformulated into the inclusion $(\mathcal{D}_\lambda)$ – which are more suitable from mechanical point of view (mostly due to the perturbation term G, although we allow non-smoothness for the oscillatory term F as well);

(b) to consider a generic p-order perturbation ∂G at the origin/infinity, not necessarily of polynomial growth as in [12], $p > 0$.

In the present paper we study the inclusion $(\mathcal{D}_\lambda)$ in two different settings, i.e., we analyze the number of distinct solutions of $(\mathcal{D}_\lambda)$ whenever ∂F oscillates near the origin/infinity and ∂G is of order $p > 0$ near the origin/infinity. Roughly speaking, when ∂F *oscillates near the origin* and ∂G *is of order* $p > 0$ *at the origin*, we prove that the number of distinct, nontrivial solutions of $(\mathcal{D}_\lambda)$ is

- infinitely many whenever $p > 1$ ($\lambda \geq 0$ is arbitrary) or $p = 1$ and λ is small enough (see Theorem 2.1);
- at least (a prescribed number) $k \in \mathbb{N}$ whenever $0 < p < 1$ and λ is small enough (see Theorem 2.2).

As we can observe, in the first case, the term $\partial G(s) \sim s^p$ as $s \to 0^+$ with $p > 1$ has no effect on the number of solutions (i.e., the oscillatory term is the leading one), while in the second case, the situation changes dramatically, i.e., ∂G has a 'truth' competition with respect to the oscillatory term ∂F.

We can state a very similar result as above whenever ∂F *oscillates at infinity* and ∂G *is of order* $p > 0$ *at infinity* by proving that the number of distinct, nontrivial solutions of the differential inclusion $(\mathcal{D}_\lambda)$ is

- infinitely many whenever $p < 1$ ($\lambda \geq 0$ is arbitrary) or $p = 1$ and λ is small enough (see Theorem 2.3);
- at least (a prescribed number) $k \in \mathbb{N}$ whenever $p > 1$ and λ is small enough (see Theorem 2.4).

Contrary to the competition at the origin, in the first case the term $\partial G(s) \sim s^p$ as $s \to \infty$ with $p < 1$ has no effect on the number of solutions (i.e., the oscillatory term is the leading one), while in the second case, the perturbation term ∂G competes with the oscillator function ∂F.

We admit that the line of the proofs is conceptually similar to that of Kristály and Moroşanu [12]; however, the presence of the nonsmooth terms ∂F and ∂G requires a deep argumentation by fully exploring the nonsmooth calculus of locally Lipschitz functions in the sense of Clarke [7]. In addition, the presence of the generic p-order perturbation ∂G needs a special attention with respect to [12]; in particular, the p-order growth of ∂G is new even in smooth settings.

The organization of the present paper is the following. In Section 2 we state our main assumptions and results, providing also some examples of functions fulfilling the assumptions. Section 3 contains a generic localization theorem for differential inclusions, while Sections 4 and 5 are devoted to the proof of our main results. In Section 6 we formulate some concluding remarks, while in the Appendix (Section 7) we collect those notions and results on locally Lipschitz functions that are used throughout our arguments.

2. Main theorems

Let $F, G : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz functions and as usual, let us denote by ∂F and ∂G their generalized gradients in the sense of Clarke (see the Appendix). Hereafter, $\mathbb{R}_+ = [0, \infty)$. Let $p > 0$, $\lambda \geq 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open domain, and consider the elliptic differential inclusion problem

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in} \quad \Omega; \\ u \geq 0 & \text{in} \quad \Omega; \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{$\mathcal{D}_\lambda$}$$

We distinguish the cases when ∂F oscillates near the *origin* or at *infinity*.

2.1. Oscillation near the origin.

We assume:

(F_0^0) $F(0) = 0$;

(F_1^0) $-\infty < \liminf_{s \to 0^+} \frac{F(s)}{s^2}$; $\limsup_{s \to 0^+} \frac{F(s)}{s^2} = +\infty$;

(F_2^0) $l_0 := \liminf_{s \to 0^+} \frac{\max\{\xi : \xi \in \partial F(s)\}}{s} < 0$.

(G_0^0) $G(0) = 0$;

(G_1^0) There exist $p > 0$ and $\underline{c}, \overline{c} \in \mathbb{R}$ such that

$$\underline{c} = \liminf_{s \to 0^+} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \leq \limsup_{s \to 0^+} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \overline{c}.$$

Remark 2.1. Hypotheses (F_1^0) and (F_2^0) imply a strong oscillatory behavior of ∂F near the origin. Moreover, it turns out that $0 \in \partial F(0)$; indeed, if we assume the contrary, by the upper semicontinuity of ∂F we also have that $0 \notin \partial F(s)$ for every small $s > 0$. Thus, by (F_2^0) we have that $\partial F(s) \subset (-\infty, 0]$ for these values of $s > 0$. By using (F_0^0) and Lebourg's mean value theorem (see Proposition 7.3 in the Appendix), it follows that $F(s) = F(s) - F(0) = \xi s \leq 0$ for some $\xi \in \partial F(\theta s) \subset (-\infty, 0]$ with $\theta \in (0,1)$. The latter inequality contradicts the second assumption from (F_1^0). Similarly, one obtains that $0 \in \partial G(0)$ by exploring (G_0^0) and (G_1^0), respectively.

In conclusion, since $0 \in \partial F(0)$ and $0 \in \partial G(0)$, it turns out that $0 \in H_0^1(\Omega)$ is a solution of the differential inclusion $(\mathcal{D}_\lambda)$. Clearly, we are interested in nonzero solutions of $(\mathcal{D}_\lambda)$.

Example 2.1. Let us consider $F_0(s) = \displaystyle\int_0^s f_0(t)$, $s \geq 0$, where $f_0(t) = \sqrt{t}(\frac{1}{2} + \sin t^{-1})$, $t > 0$ and $f_0(0) = 0$, or some of its jumping variants. One can prove that $\partial F_0 = f_0$ verifies the assumptions $(F_0^0) - (F_2^0)$. For a fixed $p > 0$, let $G_0(s) = \ln(1 + s^{p+2}) \max\{0, \cos s^{-1}\}$, $s > 0$ and $G_0(0) = 0$. It is clear that G_0 is not of class C^1 and verifies (G_1^0) with $\underline{c} = -1$ and $\overline{c} = 1$, respectively; see Figure 1 representing both f_0 and G_0 (for $p = 2$).

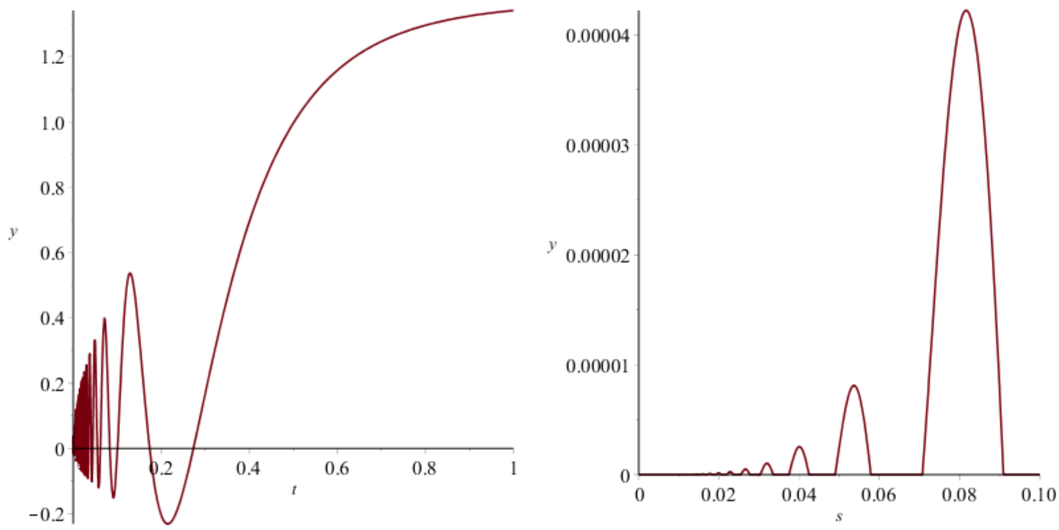

FIGURE 1. Graphs of f_0 and G_0 around the origin, respectively.

In the sequel, we provide a quite complete picture about the competition concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we are going to show that when $p \geq 1$ then the 'leading' term is the oscillatory function ∂F; roughly speaking, one can say that the effect of $s \mapsto \partial G(s)$ is negligible in this competition. More precisely, we prove the following result.

Theorem 2.1. (Case $p \geq 1$) *Assume that $p \geq 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^0) - (F_2^0)$ and $(G_0^0) - (G_1^0)$. If*

(i) *either $p = 1$ and $\lambda\overline{c} < -l_0$ (with $\lambda \geq 0$),*

(ii) *or $p > 1$ and $\lambda \geq 0$ is arbitrary,*

then the differential inclusion problem $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i\to\infty} \|u_i\|_{H_0^1} = \lim_{i\to\infty} \|u_i\|_{L^\infty} = 0. \tag{2.1}$$

In the case when $p<1$, the perturbation term ∂G may compete with the oscillatory function ∂F; namely, we have:

Theorem 2.2. (Case $0<p<1$) *Assume $0<p<1$ and that the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^0)-(F_2^0)$ and $(G_0^0)-(G_1^0)$. Then, for every $k\in\mathbb{N}$, there exists $\lambda_k>0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda},...,u_{k,\lambda}\}\subset H_0^1(\Omega)$ whenever $\lambda\in[0,\lambda_k]$. Moreover,*

$$\|u_{i,\lambda}\|_{H_0^1} < i^{-1} \ \ \textit{and} \ \ \|u_{i,\lambda}\|_{L^\infty} < i^{-1} \ \ \textit{for any } i=\overline{1,k};\ \lambda\in[0,\lambda_k]. \tag{2.2}$$

2.2. **Oscillation at infinity.** Let assume:

(F_0^∞) $F(0)=0$;
(F_1^∞) $-\infty<\liminf_{s\to\infty}\frac{F(s)}{s^2}$; $\limsup_{s\to\infty}\frac{F(s)}{s^2}=+\infty$;
(F_2^∞) $l_\infty:=\liminf_{s\to\infty}\frac{\max\{\xi:\xi\in\partial F(s)\}}{s}<0$.
(G_0^∞) $G(0)=0$;
(G_1^∞) There exist $p>0$ and $\underline{c},\overline{c}\in\mathbb{R}$ such that

$$\underline{c}=\liminf_{s\to\infty}\frac{\min\{\xi:\xi\in\partial G(s)\}}{s^p}\le\limsup_{s\to\infty}\frac{\max\{\xi:\xi\in\partial G(s)\}}{s^p}=\overline{c}.$$

Remark 2.2. Hypotheses (F_1^∞) and (F_2^∞) imply a strong oscillatory behavior of the set-valued map ∂F at infinity.

Example 2.2. We consider $F_\infty(s)=\displaystyle\int_0^s f_\infty(t)$, $s\ge0$, where $f_\infty(t)=\sqrt{t}(\frac{1}{2}+\sin t)$, $t\ge0$, or some of its jumping variants; one has that F_∞ verifies the assumptions $(F_0^\infty)-(F_2^\infty)$. For a fixed $p>0$, let $G_\infty(s)=s^p\max\{0,\sin s\}$, $s\ge0$; it is clear that G_∞ is a typically locally Lipschitz function on $[0,\infty)$ (not being of class C^1) and verifies (G_1^∞) with $\underline{c}=-1$ and $\overline{c}=1$; see Figure 2 representing both f_∞ and G_∞ (for $p=2$), respectively.

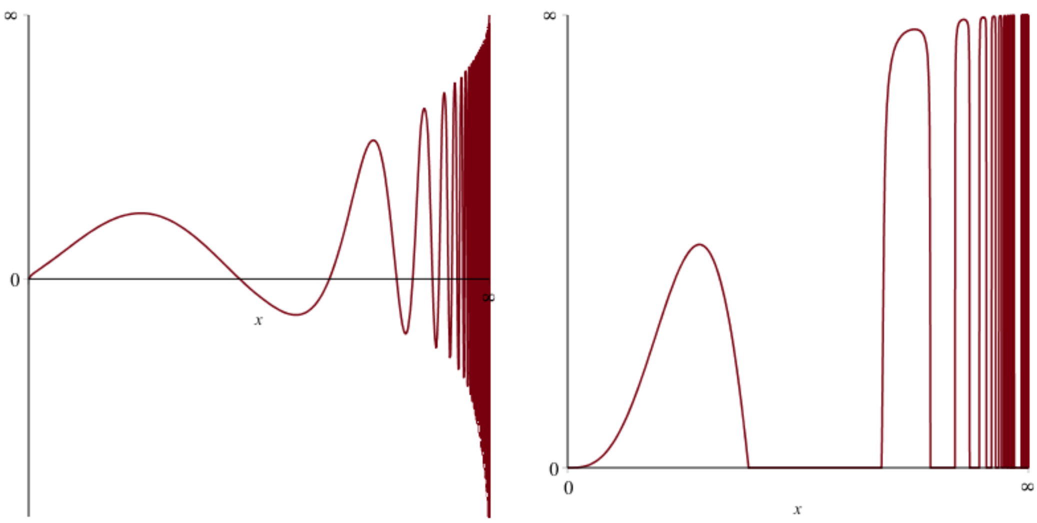

FIGURE 2. Graphs of f_∞ and G_∞ at infinity, respectively.

In the sequel, we investigate the competition at infinity concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we show that when $p \leq 1$ then the 'leading' term is the oscillatory function F, i.e., the effect of $s \mapsto \partial G(s)$ is negligible. More precisely, we prove the following result:

Theorem 2.3. (Case $p \leq 1$) *Assume that $p \leq 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^\infty) - (F_2^\infty)$ and $(G_0^\infty) - (G_1^\infty)$. If*

(i) *either $p = 1$ and $\lambda \bar{c} \leq -l_0$ (with $\lambda \geq 0$),*

(ii) *or $p < 1$ and $\lambda \geq 0$ is arbitrary,*

then the differential inclusion $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{2.3}$$

Remark 2.3. Let 2^* be the usual critical Sobolev exponent. In addition to (2.3), we also have $\lim_{i\to\infty} \|u_i^\infty\|_{H_0^1} = \infty$ whenever

$$\sup_{s\in[0,\infty)} \frac{\max\{|\xi| : \xi \in \partial F(s)\}}{1 + s^{2^*-1}} < \infty. \tag{2.4}$$

In the case when $p > 1$, it turns out that the perturbation term ∂G may compete with the oscillatory function ∂F; more precisely, we have:

Theorem 2.4. (Case $p > 1$) *Assume that $p > 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^\infty) - (F_2^\infty)$ and $(G_0^\infty) - (G_1^\infty)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^\infty > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k^\infty]$. Moreover,*

$$\|u_{i,\lambda}\|_{L^\infty} > i - 1 \quad \textit{for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty]. \tag{2.5}$$

Remark 2.4. If (2.4) holds and $p \leq 2^* - 1$ in Theorem 2.4, then we have in addition that

$$\|u_{i,\lambda}^\infty\|_{H_0^1} > i - 1 \quad \text{for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty].$$

3. Localization: a generic result

We consider the following differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{D_A^k}$$

where $k > 0$ and

(H_A^1): $A : [0, \infty) \to \mathbb{R}$ is a locally Lipschitz function with $A(0) = 0$, and there is $M_A > 0$ such that

$$\max\{|\partial A(s)|\} := \max\{|\xi| : \xi \in \partial A(s)\} \leq M_A$$

for every $s \geq 0$;

(H_A^2): there are $0 < \delta < \eta$ such that $\max\{\xi : \xi \in \partial A(s)\} \leq 0$ for every $s \in [\delta, \eta]$.

For simplicity, we extend the function A by $A(s) = 0$ for $s \leq 0$; the extended function is locally Lipschitz on the whole $\mathbb{R}$. The natural energy functional $\mathcal{T} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem (D_A^k) is defined by

$$\mathcal{T}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A(u(x))dx.$$

The energy functional $\mathcal{T}$ is well defined and locally Lipschitz on $H_0^1(\Omega)$, while its critical points in the sense of Chang (see Definition 7.3 in the Appendix) are precisely the weak solutions of the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega; \end{cases} \tag{$\mathrm{D}_A^{k,0}$}$$

note that at this stage we have no information on the sign of u.

Indeed, if $0 \in \partial\mathcal{T}(u)$, then for every $v \in H_0^1(\Omega)$ we have

$$\int_\Omega \nabla u(x)\nabla v(x)dx - k\int_\Omega u(x)v(x)dx - \int_\Omega \xi_x(x)v(x)dx = 0,$$

where $\xi_x \in \partial A(u(x))$ a.e. $x \in \Omega$, see e.g. Motreanu and Panagiotopoulos [18]. By using the divergence theorem for the first term at the left hand side (and exploring the Dirichlet boundary condition), we obtain that

$$\int_\Omega \nabla u(x)\nabla v(x)dx = -\int_\Omega \mathrm{div}(\nabla u(x))v(x)dx = -\int_\Omega \Delta u(x)v(x)dx.$$

Accordingly, we have that

$$-\int_\Omega \Delta u(x)v(x)dx + k\int_\Omega u(x)v(x) = \int_\Omega \xi_x v(x)dx$$

for every test function $v \in H_0^1(\Omega)$ which means that $-\Delta u(x) + ku(x) \in \partial A(u(x))$ in the weak sense in Ω, as claimed before.

Let us consider the number $\eta \in \mathbb{R}$ from (H_A^2) and the set

$$W^\eta = \{u \in H_0^1(\Omega) : \|u\|_{L^\infty} \leq \eta\}.$$

Our localization result reads as follows (see [12, Theorem 2.1] for its smooth form):

Theorem 3.1. *Let $k > 0$ and assume that hypotheses (H_A^1) and (H_A^2) hold. Then*

(i) *the energy functional $\mathcal{T}$ is bounded from below on W^η and its infimum is attained at some $\tilde{u} \in W^\eta$;*

(ii) *$\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$;*

(iii) *$\tilde{u}$ is a weak solution of the differential inclusion (D_A^k).*

Proof. The proof is similar to that of Kristály and Moroşanu [12]; for completeness, we provide its main steps.

(i) Due to (H_A^1), it is clear that the energy functional $\mathcal{T}$ is bounded from below on $H_0^1(\Omega)$. Moreover, due to the compactness of the embedding $H_0^1(\Omega) \subset L^q(\Omega)$, $q \in [2, 2^*)$, it turns out that $\mathcal{T}$ is sequentially weak lower semi-continuous on $H_0^1(\Omega)$. In addition, the set W^η is weakly closed, being convex and closed in $H_0^1(\Omega)$. Thus, there is $\tilde{u} \in W^\eta$ which is a minimum point of $\mathcal{T}$ on the set W^η, cf. Zeidler [24].

(ii) We introduce the set $L = \{x \in \Omega : \tilde{u}(x) \notin [0, \delta]\}$ and suppose indirectly that $m(L) > 0$. Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, \delta)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ \tilde{u}$. It is clear that γ is a Lipschitz function and $\gamma(0) = 0$. Accordingly, based on the superposition theorem of Marcus and Mizel [14], one has that $w \in H_0^1(\Omega)$. Moreover, $0 \leq w(x) \leq \delta$ for a.e. Ω. Consequently, $w \in W^\eta$.

Let us introduce the sets

$$L_1 = \{x \in L : \tilde{u}(x) < 0\} \quad \text{and} \quad L_2 = \{x \in L : \tilde{u}(x) > \delta\}.$$

In particular, $L = L_1 \cup L_2$, and by definition, it follows that $w(x) = \tilde{u}(x)$ for all $x \in \Omega \setminus L$, $w(x) = 0$ for all $x \in L_1$, and $w(x) = \delta$ for all $x \in L_2$. In addition, one has

$$\begin{aligned} \mathcal{T}(w) - \mathcal{T}(\tilde{u}) &= \frac{1}{2}\left[\|w\|^2_{H^1_0} - \|\tilde{u}\|^2_{H^1_0}\right] + \frac{k}{2}\int_\Omega [w^2 - \tilde{u}^2] - \int_\Omega [A(w(x)) - A(\tilde{u}(x))] \\ &= -\frac{1}{2}\int_L |\nabla \tilde{u}|^2 + \frac{k}{2}\int_L [w^2 - \tilde{u}^2] - \int_L [A(w(x)) - A(\tilde{u}(x))]. \end{aligned}$$

On account of $k > 0$, we have

$$k\int_L [w^2 - \tilde{u}^2] = -k\int_{L_1} \tilde{u}^2 + k\int_{L_2} [\delta^2 - \tilde{u}^2] \leq 0.$$

Since $A(s) = 0$ for all $s \leq 0$, we have

$$\int_{L_1} [A(w(x)) - A(\tilde{u}(x))] = 0.$$

By means of the Lebourg's mean value theorem, for a.e. $x \in L_2$, there exists $\theta(x) \in [\delta, \tilde{u}(x)] \subseteq [\delta, \eta]$ such that

$$A(w(x)) - A(\tilde{u}(x)) = A(\delta) - A(\tilde{u}(x)) = a(\theta(x))(\delta - \tilde{u}(x)),$$

where $a(\theta(x)) \in \partial A(\theta(x))$. Due to (H^2_A), it turns out that

$$\int_{L_2} [A(w(x)) - A(\tilde{u}(x))] \geq 0.$$

Therefore, we obtain that $\mathcal{T}(w) - \mathcal{T}(\tilde{u}) \leq 0$. On the other hand, since $w \in W^\eta$, then $\mathcal{T}(w) \geq \mathcal{T}(\tilde{u}) = \inf_{W^\eta} \mathcal{T}$, thus every term in the difference $\mathcal{T}(w) - \mathcal{T}(\tilde{u})$ should be zero; in particular,

$$\int_{L_1} \tilde{u}^2 = \int_{L_2} [\tilde{u}^2 - \delta^2] = 0.$$

The latter relation implies in particular that $m(L) = 0$, which is a contradiction, completing the proof of (ii).

(iii) Since $\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$, an arbitrarily small perturbation $\tilde{u} + \epsilon v$ of $\tilde{u}$ with $0 < \epsilon \ll 1$ and $v \in C^\infty_0(\Omega)$ still implies that $\mathcal{T}(\tilde{u} + \varepsilon v) \geq \mathcal{T}(\tilde{u})$; accordingly, $\tilde{u}$ is a minimum point for $\mathcal{T}$ in the strong topology of $H^1_0(\Omega)$, thus $0 \in \partial\mathcal{T}(\tilde{u})$, cf. Remark 7.1 in the Appendix. Consequently, it follows that $\tilde{u}$ is a weak solution of the differential inclusion (D^k_A). □

In the sequel, we need a truncation function of $H^1_0(\Omega)$, see also [12]. To construct this function, let $B(x_0, r) \subset \Omega$ be the n-dimensional ball with radius $r > 0$ and center $x_0 \in \Omega$. For $s > 0$, define

$$w_s(x) = \begin{cases} 0, & \text{if} \quad x \in \Omega \setminus B(x_0, r); \\ s, & \text{if} \quad x \in B(x_0, r/2); \\ \frac{2s}{r}(r - |x - x_0|), & \text{if} \quad x \in B(x_0, r) \setminus B(x_0, r/2). \end{cases} \tag{3.1}$$

Note that that $w_s \in H^1_0(\Omega)$, $\|w_s\|_{L^\infty} = s$ and

$$\|w_s\|^2_{H^1_0} = \int_\Omega |\nabla w_s|^2 = 4r^{n-2}(1 - 2^{-n})\omega_n s^2 \equiv C(r, n)s^2 > 0; \tag{3.2}$$

hereafter ω_n stands for the volume of $B(0, 1) \subset \mathbb{R}^n$.

4. Proof of Theorems 2.1 and 2.2

Before giving the proof of Theorems 2.1 and 2.2, in the first part of this section we study the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{D_A^k}$$

where $k > 0$ and the locally Lipschitz function $A : \mathbb{R}_+ \to \mathbb{R}$ verifies

(H_0^0): $A(0) = 0$;
(H_1^0): $-\infty < \liminf_{s\to 0^+} \frac{A(s)}{s^2}$ and $\limsup_{s\to 0^+} \frac{A(s)}{s^2} = +\infty$;
(H_2^0): there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \eta_{i+1} < \delta_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and

$$\max\{\partial A(s)\} := \max\{\xi : \xi \in \partial A(s)\} \leq 0$$

for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Theorem 4.1. *Let $k > 0$ and assume hypotheses (H_0^0), (H_1^0) and (H_2^0) hold. Then there exists a sequence $\{u_i^0\}_i \subset H_0^1(\Omega)$ of distinct weak solutions of the differential inclusion problem (D_A^k) such that*

$$\lim_{i\to\infty} \|u_i^0\|_{H_0^1} = \lim_{i\to\infty} \|u_i^0\|_{L^\infty} = 0. \tag{4.1}$$

Proof. We may assume that $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0,1)$. For any fixed number $i \in \mathbb{N}$, we define the locally Lipschitz function $A_i : \mathbb{R} \to \mathbb{R}$ by

$$A_i(s) = A(\tau_{\eta_i}(s)), \tag{4.2}$$

where $A(s) = 0$ for $s \leq 0$ and $\tau_\eta : \mathbb{R} \to \mathbb{R}$ denotes the truncation function $\tau_\eta(s) = \min(\eta, s)$, $\eta > 0$. For further use, we introduce the energy functional $\mathcal{T}_i : H_0^1(\Omega) \to \mathbb{R}$ associated with problem ($\mathrm{D}_{A_i}^k$).

We notice that for $s \geq 0$, the chain rule (see Proposition 7.4 in the Appendix) gives

$$\partial A_i(s) = \begin{cases} \partial A(s) & \text{if} \quad s < \eta_i, \\ \overline{\mathrm{co}}\{0, \partial A(\eta_i)\} & \text{if} \quad s = \eta_i, \\ \{0\} & \text{if} \quad s > \eta_i. \end{cases}$$

It turns out that on the compact set $[0, \eta_i]$, the upper semicontinuous set-valued map $s \mapsto \partial A_i(s)$ attains its supremum (see Proposition 7.1 in the Appendix); therefore, there exists $M_{A_i} > 0$ such that

$$\max|\partial A_i(s)| := \max\{|\xi| : \xi \in \partial A_i(s)\} \leq M_{A_i}$$

for every $s \geq 0$, i.e., ($\mathrm{H}_{A_i}^1$) holds. The same is true for ($\mathrm{H}_{A_i}^2$) by using (H_2^0) on $[\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Accordingly, the assumptions of Theorem 3.1 are verified for every $i \in \mathbb{N}$ with $[\delta_i, \eta_i]$, thus there exists $u_i^0 \in W^{\eta_i}$ such that

$$u_i^0 \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^{\eta_i}, \tag{4.3}$$

$$u_i^0(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{4.4}$$

$$u_i^0 \text{ is a solution of } (\mathrm{D}_{A_i}^k). \tag{4.5}$$

On account of relations (4.2), (4.4) and (4.5), u_i^0 is a weak solution also for the differential inclusion problem (D_A^k).

We are going to prove that there are infinitely many distinct elements in the sequence $\{u_i^0\}_i$. To conclude it, we first prove that

$$\mathcal{T}_i(u_i^0) < 0 \quad \text{for all } i \in \mathbb{N}; \text{ and} \tag{4.6}$$

$$\lim_{i\to\infty} \mathcal{T}_i(u_i^0) = 0. \tag{4.7}$$

The left part of (H_1^0) implies the existence of some $l_0 > 0$ and $\zeta \in (0, \eta_1)$ such that

$$A(s) \geq -l_0 s^2 \text{ for all } s \in (0, \zeta). \tag{4.8}$$

One can choose $L_0 > 0$ such that

$$\frac{1}{2}C(r,n) + \left(\frac{k}{2} + l_0\right) m(\Omega) < L_0 (r/2)^n \omega_n, \tag{4.9}$$

where $r > 0$ and $C(r,n) > 0$ come from (3.2). Based on the right part of (H_1^0), one can find a sequence $\{\tilde{s}_i\}_i \subset (0, \zeta)$ such that $\tilde{s}_i \leq \delta_i$ and

$$A(\tilde{s}_i) > L_0 \tilde{s}_i^2 \quad \text{for all } i \in \mathbb{N}. \tag{4.10}$$

Let $i \in \mathbb{N}$ be a fixed number and let $w_{\tilde{s}_i} \in H_0^1(\Omega)$ be the function from (3.1) corresponding to the value $\tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W^{\eta_i}$, and due to (4.8), (4.10) and (3.2) one has

$$\begin{aligned}
\mathcal{T}_i(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega w_{\tilde{s}_i}^2 - \int_\Omega A_i(w_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,n)\tilde{s}_i^2 + \frac{k}{2}\int_\Omega w_{\tilde{s}_i}^2 - \int_{B(x_0,r/2)} A(\tilde{s}_i)dx - \int_{B(x_0,r)\setminus B(x_0,r/2)} A(w_{\tilde{s}_i}(x))dx \\
&\leq \left[\frac{1}{2}C(r,n) + \frac{k}{2}m(\Omega) - L_0(r/2)^n\omega_n + l_0 m(\Omega)\right]\tilde{s}_i^2.
\end{aligned}$$

Accordingly, with (4.3) and (4.9), we conclude that

$$\mathcal{T}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{T}_i \leq \mathcal{T}_i(w_{\tilde{s}_i}) < 0 \tag{4.11}$$

which completes the proof of (4.6).

Now, we prove (4.7). For every $i \in \mathbb{N}$, by using the Lebourg's mean value theorem, relations (4.2) and (4.4) and (H_0^0) , we have

$$\mathcal{T}_i(u_i^0) \geq -\int_\Omega A_i(u_i^0(x))dx = -\int_\Omega A_1(u_i^0(x))dx \geq -M_{A_1} m(\Omega)\delta_i.$$

Since $\lim_{i\to\infty}\delta_i = 0$, the latter estimate and (4.11) provides relation (4.7).

Based on (4.2) and (4.4), we have that $\mathcal{T}_i(u_i^0) = \mathcal{T}_1(u_i^0)$ for all $i \in \mathbb{N}$. This relation with (4.6) and (4.7) means that the sequence $\{u_i^0\}_i$ contains infinitely many distinct elements.

We now prove (4.1). One can prove the former limit by (4.4), i.e. $\|u_i^0\|_{L^\infty} \leq \delta_i$ for all $i \in \mathbb{N}$, combined with $\lim_{i\to\infty}\delta_i = 0$. For the latter limit, we use $k > 0$, (4.11), (4.2) and (4.4) to get for all $i \in \mathbb{N}$ that

$$\frac{1}{2}\|u_i^0\|_{H_0^1}^2 \leq \frac{1}{2}\|u_i^0\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega (u_i^0)^2 < \int_\Omega A_i(u_i^0(x)) = \int_\Omega A_1(u_i^0(x)) \leq M_{A_1} m(\Omega)\delta_i,$$

which completes the proof. □

Proof of Theorem 2.1. We split the proof into two parts.
(i) *Case* $p=1$. Let $\lambda \geq 0$ with $\lambda\bar{c} < -l_0$ and fix $\tilde{\lambda}_0 \in \mathbb{R}$ such that $\lambda\bar{c} < \tilde{\lambda}_0 < -l_0$. With these choices we define

$$k := \tilde{\lambda}_0 - \lambda\bar{c} > 0 \ \text{ and } A(s) := F(s) + \frac{\tilde{\lambda}_0}{2}s^2 + \lambda\left(G(s) - \frac{\bar{c}}{2}s^2\right) \text{ for every } s \in [0,\infty). \tag{4.12}$$

It is clear that $A(0)=0$, i.e., (H_0^0) is verified. Since $p=1$, by (G_1^0) one has

$$\underline{c} = \liminf_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s} \leq \limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = \bar{c}.$$

In particular, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max\{\partial G(s)\} - \bar{c}s < \epsilon s, \ \forall s \in [0,\gamma],$$

and

$$\min\{\partial G(s)\} - \underline{c}s > -\epsilon s, \ \forall s \in [0,\gamma].$$

For $s \in [0,\gamma]$, Lebourg's mean value theorem and $G(0)=0$ implies that there exists $\xi_s \in \partial G(\theta_s s)$ for some $\theta_s \in [0,1]$ such that $G(s) - G(0) = \xi_s s$. Accordingly, for every $s \in [0,\gamma]$ we have that

$$(\underline{c} - \epsilon)s^2 \leq G(s) \leq (\bar{c} + \epsilon)s^2. \tag{4.13}$$

By (4.13) and (F_1^0) we have that

$$\liminf_{s\to 0^+} \frac{A(s)}{s^2} \geq \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_0 - \lambda\bar{c}}{2} + \lambda \liminf_{s\to 0^+} \frac{G(s)}{s^2} \geq \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_0 - \lambda\bar{c}}{2} + \lambda\underline{c} > -\infty$$

and

$$\limsup_{s\to 0^+} \frac{A(s)}{s^2} \geq \limsup_{s\to 0^+} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_0 - \lambda\bar{c}}{2} + \lambda \liminf_{s\to 0^+} \frac{G(s)}{s^2} = +\infty,$$

i.e., (H_1^0) is verified.

Since

$$\partial A(s) \subseteq \partial F(s) + \tilde{\lambda}_0 s + \lambda(\partial G(s) - \bar{c}s), \tag{4.14}$$

and $\lambda \geq 0$, we have that

$$\max\{\partial A(s)\} \leq \max\{\partial F(s) + \tilde{\lambda}_0 s\} + \lambda \max\{\partial G(s) - \bar{c}s\}. \tag{4.15}$$

Since

$$\limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = \bar{c},$$

cf. (G_1^0), and

$$\liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} = l_0 < 0,$$

cf. (F_2^0), it turns out by (4.15) that

$$\liminf_{s\to 0^+} \frac{\max\{\partial A(s)\}}{s} \leq \liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} + \tilde{\lambda}_0 - \lambda\bar{c} + \lambda \limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} \leq l_0 + \tilde{\lambda}_0 < 0.$$

Therefore, one has a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that $\frac{\max\{\partial A(s_i)\}}{s_i} < 0$ i.e., $\max\{\partial A(s_i)\} < 0$ for all $i \in \mathbb{N}$. By using the upper semicontinuity of $s \mapsto \partial A(s)$, we may choose two numbers $\delta_i, \eta_i \in (0,1)$ with $\delta_i < s_i < \eta_i$ such that $\partial A(s) \subset \partial A(s_i) + [-\epsilon_i, \epsilon_i]$ for every $s \in [\delta_i, \eta_i]$, where $\epsilon_i := -\max\{\partial A(s_i)\}/2 > 0$. In particular, $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$. Thus, one may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Accordingly, (H_2^0) is verified as well. Let us apply Theorem 4.1 with the choice (4.12), i.e., there exists a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + (\tilde{\lambda}_0 - \lambda\bar{c})u_i(x) \in \partial F(u_i(x)) + \tilde{\lambda}_0 u_i(x) + \lambda(\partial G(u_i(x)) - \bar{c}u_i(x)) & x \in \Omega, \\ u_i(x) \geq 0 & x \in \Omega, \\ u_i(x) = 0 & x \in \partial\Omega, \end{cases}$$

where we used the inclusion (4.14). In particular, u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$, which completes the proof of (i).

(ii) *Case* $p > 1$. Let $\lambda \geq 0$ be arbitrary fixed and choose a number $\lambda_0 \in (0, -l_0)$. Let

$$k := \lambda_0 > 0 \ \text{ and } A(s) := F(s) + \lambda G(s) + \lambda_0 \frac{s^2}{2} \text{ for every } s \in [0, \infty). \tag{4.16}$$

Since $F(0) = G(0) = 0$, hypothesis (H_0^0) clearly holds. By (G_1^0) one has

$$\underline{c} = \liminf_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s^p} \leq \limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s^p} = \bar{c}.$$

In particular, since $p > 1$, then

$$\lim_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s} = \lim_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = 0 \tag{4.17}$$

and for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max\{\partial G(s)\} - \bar{c}s^p < \epsilon s^p, \ \forall s \in [0, \gamma]$$

and

$$\min\{\partial G(s)\} - \underline{c}s^p > -\epsilon s^p, \ \forall s \in [0, \gamma].$$

For a fixed $s \in [0, \gamma]$, by Lebourg's mean value theorem and $G(0) = 0$ we conclude again that $G(s) - G(0) = \xi_s s$. Accordingly, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that $(\underline{c} - \epsilon)s^{p+1} \leq G(s) \leq (\bar{c} + \epsilon)s^{p+1}$ for every $s \in [0, \gamma]$. Thus, since $p > 1$,

$$\lim_{s\to 0^+} \frac{G(s)}{s^2} = \lim_{s\to 0^+} \frac{G(s)}{s^{p+1}} s^{p-1} = 0.$$

Therefore, by using (4.16) and (F_1^0), we conclude that

$$\liminf_{s\to 0^+} \frac{A(s)}{s^2} = \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \lambda \lim_{s\to 0^+} \frac{G(s)}{s^2} + \frac{\lambda_0}{2} > -\infty,$$

and

$$\limsup_{s\to 0^+} \frac{A(s)}{s^2} = \infty,$$

i.e., (H_0^1) holds. Since

$$\partial A(s) \subseteq \partial F(s) + \lambda \partial G(s) + \lambda_0 s,$$

and $\lambda \geq 0$, we have that

$$\max\{\partial A(s)\} \leq \max\{\partial F(s)\} + \max\{\lambda \partial G(s) + \lambda_0 s\}.$$

Since

$$\limsup_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s^p} = \bar{c},$$

cf. (G_1^0), and

$$\liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} = l_0,$$

cf. (F_2^0), by relation (4.17) it turns out that

$$\liminf_{s\to 0^+}\frac{\max\{\partial A(s)\}}{s}=\liminf_{s\to 0^+}\frac{\max\{\partial F(s)\}}{s}+\lambda\lim_{s\to 0^+}\frac{\max\{\partial G(s)\}}{s}+\lambda_0=l_0+\lambda_0<0,$$

and the upper semicontinuity of ∂A implies the existence of two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i \subset (0,1)$ such that $0<\eta_{i+1}<\delta_i<s_i<\eta_i$, $\lim_{i\to\infty}\eta_i=0$, and $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Therefore, hypothesis (H_2^0) holds. Now, we can apply Theorem 4.1, i.e., there is a sequence $\{u_i\}_i\subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x)+\lambda_0 u_i(x)\in\partial A(u_i(x))\subseteq \partial F(u_i(x))+\lambda\partial G(u_i(x))+\lambda_0 u_i(x) & x\in\Omega,\\ u_i(x)\geq 0 & x\in\Omega,\\ u_i(x)=0 & x\in\partial\Omega,\end{cases}$$

which means that u_i solves problem $(\mathcal{D}_\lambda)$, $i\in\mathbb{N}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. The proof is done in two steps:
(i) Let $\lambda_0\in(0,-l_0)$, $\lambda\geq 0$ and define

$$k:=\lambda_0>0 \ \text{ and } A^\lambda(s):=F(s)+\lambda G(s)+\lambda_0\frac{s^2}{2} \text{ for every } s\in[0,\infty). \tag{4.18}$$

One can observe that $\partial A^\lambda(s)\subseteq\partial F(s)+\lambda_0 s+\lambda\partial G(s)$ for every $s\geq 0$. On account of (F_2^0), there is a sequence $\{s_i\}_i\subset(0,1)$ converging to 0 such that

$$\max\{\partial A^{\lambda=0}(s_i)\}\leq\max\{\partial F(s_i)\}+\lambda_0 s_i<0.$$

Thus, due to the upper semicontinuity of $(s,\lambda)\mapsto\partial A^\lambda(s)$, we can choose three sequences $\{\delta_i\}_i,\{\eta_i\}_i,\{\lambda_i\}_i\subset(0,1)$ such that $0<\eta_{i+1}<\delta_i<s_i<\eta_i, \lim_{i\to\infty}\eta_i=0$, and

$$\max\{\partial A^\lambda(s)\}\leq 0 \text{ for all } \lambda\in[0,\lambda_i], s\in[\delta_i,\eta_i],\ i\in\mathbb{N}.$$

Without any loss of generality, we may choose

$$\delta_i\leq\min\{i^{-1},2^{-1}i^{-2}[1+m(\Omega)(\max_{s\in[0,1]}|\partial F(s)|+\max_{s\in[0,1]}|\partial G(s)|)]^{-1}\}. \tag{4.19}$$

For every $i\in\mathbb{N}$ and $\lambda\in[0,\lambda_i]$, let $A_i^\lambda:[0,\infty)\to\mathbb{R}$ be defined as

$$A_i^\lambda(s)=A^\lambda(\tau_{\eta_i}(s)), \tag{4.20}$$

and the energy functional $\mathcal{T}_{i,\lambda}:H_0^1(\Omega)\to\mathbb{R}$ associated with the differential inclusion problem($D^k_{A_i^\lambda}$) is given by

$$\mathcal{T}_{i,\lambda}(u)=\frac{1}{2}\|u\|^2_{H_0^1}+\frac{k}{2}\int_\Omega u^2dx-\int_\Omega A_i^\lambda(u(x))dx.$$

One can easily check that for every $i\in\mathbb{N}$ and $\lambda\in[0,\lambda_i]$, the function A_i^λ verifies the hypotheses of Theorem 3.1. Accordingly, for every $i\in\mathbb{N}$ and $\lambda\in[0,\lambda_i]$:

$$\mathcal{T}_{i,\lambda} \text{ attains its infinum on } W^{\eta_i} \text{ at some } u^0_{i,\lambda}\in W^{\eta_i} \tag{4.21}$$

$$u^0_{i,\lambda}(x)\in[0,\delta_i] \text{ for a.e. } x\in\Omega; \tag{4.22}$$

$$u^0_{i,\lambda} \text{ is a weak solution of } (D^k_{A_i^\lambda}). \tag{4.23}$$

By the choice of the function A^λ and $k>0$, $u^0_{i,\lambda}$ is also a solution to the differential inclusion problem ($D^k_{A^\lambda}$), so $(\mathcal{D}_\lambda)$.

(ii) It is clear that for $\lambda = 0$, the set-valued map $\partial A_i^\lambda = \partial A_i^0$ verifies the hypotheses of Theorem 4.1. In particular, $\mathcal{T}_i := \mathcal{T}_{i,0}$ is the energy functional associated with problem $(\mathrm{D}^k_{A_i^0})$. Consequently, the elements $u_i^0 := u_{i,0}^0$ verify not only (4.21)-(4.23) but also

$$\mathcal{T}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{T}_i \leq \mathcal{T}_i(w_{\tilde{s}_i}) < 0 \text{ for all } i \in \mathbb{N}. \tag{4.24}$$

Similarly to Kristály and Moroşanu [12], let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty} \theta_i = 0$. Due to (4.24) we may assume that

$$\theta_i < \mathcal{T}_i(u_i^0) \leq \mathcal{T}_i(w_{\tilde{s}_i}) < \theta_{i+1}. \tag{4.25}$$

Let us choose

$$\lambda_i^{'} = \frac{\theta_{i+1} - \mathcal{T}_i(w_{\tilde{s}_i})}{m(\Omega)\max_{s\in[0,1]}|G(s)| + 1} \text{ and } \lambda_i^{''} = \frac{\mathcal{T}_i(u_i^0) - \theta_i}{m(\Omega)\max_{s\in[0,1]}|G(s)| + 1}\ ,\ i \in \mathbb{N}, \tag{4.26}$$

and for a fixed $k \in \mathbb{N}$, set

$$\lambda_k^0 = \min(1, \lambda_1, ..., \lambda_k, {\lambda_1}', ..., {\lambda_k}', {\lambda_1}'', ..., {\lambda_k}'') > 0. \tag{4.27}$$

Having in our mind these choices, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$ one has

$$\begin{aligned}
\mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) &\leq \mathcal{T}_{i,\lambda}(w_{\tilde{s}_i}) = \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H_0^1} - \int_\Omega F(w_{\tilde{s}_i}(x))dx - \lambda\int_\Omega G(w_{\tilde{s}_i}(x))dx \\
&= \mathcal{T}_i(w_{\tilde{s}_i}) - \lambda\int_\Omega G(w_{\tilde{s}_i}(x))dx \\
&< \theta_{i+1},
\end{aligned} \tag{4.28}$$

and due to $u_{i,\lambda}^0 \in W^{\eta_i}$ and to the fact that u_i^0 is the minimum point of $\mathcal{T}_i$ on the set W^{η_i}, by (4.25) we also have

$$\mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{T}_i(u_{i,\lambda}^0) - \lambda\int_\Omega G(u_{i,\lambda}^0(x))dx \geq \mathcal{T}_i(u_i^0) - \lambda\int_\Omega G(u_{i,\lambda}^0(x))dx > \theta_i. \tag{4.29}$$

Therefore, by (4.28) and (4.29), for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$, one has

$$\theta_i < \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1},$$

thus

$$\mathcal{T}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{T}_{k,\lambda}(u_{k,\lambda}^0) < 0.$$

We notice that $u_i^0 \in W^{\eta_1}$ for every $i \in \{1, ..., k\}$, so $\mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{T}_{1,\lambda}(u_{i,\lambda}^0)$ because of (4.20). Therefore, we conclude that for every $\lambda \in [0, \lambda_k^0]$,

$$\mathcal{T}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{T}_{1,\lambda}(u_{k,\lambda}^0) < 0 = \mathcal{T}_{1,\lambda}(0).$$

Based on these inequalities, it turns out that the elements $u_{1,\lambda}^0, ..., u_{k,\lambda}^0$ are distinct and non-trivial whenever $\lambda \in [0, \lambda_k^0]$.

Now, we are going to prove the estimate (2.2). We have for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$:

$$\mathcal{T}_{1,\lambda}(u_{i,\lambda}^0) = \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1} < 0.$$

By Lebourg's mean value theorem and (4.19), we have for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$ that

$$\begin{aligned} \frac{1}{2}\|u_{i,\lambda}^0\|_{H_0^1}^2 &< \int_\Omega F(u_{i,\lambda}^0(x))dx + \lambda \int_\Omega G(u_{i,\lambda}^0(x))dx \\ &\leq m(\Omega)\delta_i[\max_{s\in[0,1]} |\partial F(s)| + \max_{s\in[0,1]} |\partial G(s)|] \\ &\leq \frac{1}{2i^2}. \end{aligned}$$

This completes the proof of Theorem 2.2. □

5. Proof of Theorems 2.3 and 2.4

We consider again the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \qquad (\mathrm{D}_A^k),$$

where $k > 0$ and the locally Lipschitz function $A : \mathbb{R}_+ \to \mathbb{R}$ verifies

(H_0^∞): $A(0) = 0$;
(H_1^∞): $-\infty < \liminf_{s\to\infty} \frac{A(s)}{s^2}$ and $\limsup_{s\to\infty} \frac{A(s)}{s^2} = +\infty$;
(H_2^∞): there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \delta_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and

$$\max\{\partial A(s)\} := \max\{\xi : \xi \in \partial A(s)\} \leq 0$$

for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

The counterpart of Theorem 4.1 reads as follows.

Theorem 5.1. *Let $k > 0$ and assume the hypotheses (H_0^∞), (H_1^∞) and (H_2^∞) hold. Then the differential inclusion problem (D_A^k) admits a sequence $\{u_i^\infty\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{5.1}$$

Proof. The proof is similar to the one performed in Theorem 4.1; we shall show the differences only. We associate the energy functional $\mathcal{T}_i : H_0^1(\Omega) \to \mathbb{R}$ with problem ($\mathrm{D}_{A_i}^k$), where $A_i : \mathbb{R} \to \mathbb{R}$ is given by

$$A_i(s) = A(\tau_{\eta_i}(s)), \tag{5.2}$$

with $A(s) = 0$ for $s \leq 0$. One can show that there exists $M_{A_i} > 0$ such that

$$\max |\partial A_i(s)| := \max\{|\xi| : \xi \in \partial A_i(s)\} \leq M_{A_i}$$

for all $s \geq 0$, i.e, hypothesis ($\mathrm{H}_{A_i}^1$) holds. Moreover, ($\mathrm{H}_{A_i}^2$) follows by (H_2^∞). Thus Theorem 4.1 can be applied for all $i \in \mathbb{N}$, i.e., we have an element $u_i^\infty \in W^{\eta_i}$ such that

$$u_i^\infty \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^{\eta_i}, \tag{5.3}$$

$$u_i^\infty(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{5.4}$$

$$u_i^\infty \text{ is a weak solution of } (\mathrm{D}_{A_i}^k). \tag{5.5}$$

By (5.2), u_i^∞ turns to be a weak solution also for differential inclusion problem (D_A^k).

We shall prove that there are infinitely many distinct elements in the sequence $\{u_i^\infty\}_i$ by showing that

$$\lim_{i\to\infty} \mathcal{T}_i(u_i^\infty) = -\infty. \tag{5.6}$$

By the left part of (H_1^∞) we can find $l_\infty^A > 0$ and $\zeta > 0$ such that

$$A(s) \geq -l_\infty^A \text{ for all } s > \zeta. \tag{5.7}$$

Let us choose $L_\infty^A > 0$ large enough such that

$$\frac{1}{2}C(r,n) + \left(\frac{k}{2} + l_\infty^A\right) m(\Omega) < L_\infty^A (r/2)^n \omega_n. \tag{5.8}$$

On account of the right part of (H_1^∞), one can fix a sequence $\{\tilde{s}_i\}_i \subset (0,\infty)$ such that $\lim_{i\to\infty} \tilde{s}_i = \infty$ and

$$A(\tilde{s}_i) > L_\infty^A {\tilde{s}_i}^2 \text{ for every } i \in \mathbb{N}. \tag{5.9}$$

We know from (H_2^∞) that $\lim_{i\to\infty} \delta_i = \infty$, therefore one has a subsequence $\{\delta_{m_i}\}_i$ of $\{\delta_i\}_i$ such that $\tilde{s}_i \leq \delta_{m_i}$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$, and recall $w_{s_i} \in H_0^1(\Omega)$ from (3.1) with $s_i := \tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W^{\eta_{m_i}}$ and according to (3.2), (5.7) and (5.9) we have

$$\begin{aligned}
\mathcal{T}_{m_i}(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega w_{\tilde{s}_i}^2 - \int_\Omega A_{m_i}(w_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,n)\tilde{s}_i^2 + \frac{k}{2}\int_\Omega w_{\tilde{s}_i}^2 - \int_{B(x_0,r/2)} A(\tilde{s}_i)dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{w_{\tilde{s}_i}>\zeta\}} A(w_{\tilde{s}_i}(x))dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{w_{\tilde{s}_i}\leq\zeta\}} A(w_{\tilde{s}_i}(x))dx \\
&\leq \left[\frac{1}{2}C(r,n) + \frac{k}{2}m(\Omega) - L_\infty^A (r/2)^n\omega_n + l_\infty^A m(\Omega)\right]\tilde{s}_i^2 + \tilde{M}_A m(\Omega)\zeta,
\end{aligned}$$

where $\tilde{M}_A = \max\{|A(s)| : s \in [0,\zeta]\}$ does not depend on $i \in \mathbb{N}$. This estimate combined by (5.8) and $\lim_{i\to\infty} \tilde{s}_i = \infty$ yields that

$$\lim_{i\to\infty} \mathcal{T}_{m_i}(w_{\tilde{s}_i}) = -\infty. \tag{5.10}$$

By equation (5.3), one has

$$\mathcal{T}_{m_i}(u_{m_i}^\infty) = \min_{W^{\eta_{m_i}}} \mathcal{T}_{m_i} \leq \mathcal{T}_{m_i}(w_{\tilde{s}_i}). \tag{5.11}$$

It follows by (5.10) that $\lim_{i\to\infty} \mathcal{T}_{m_i}(u_{m_i}^\infty) = -\infty$.

We notice that the sequence $\{\mathcal{T}_i(u_i^\infty)\}_i$ is non-increasing. Indeed, let $i < k$; due to (5.2) one has that

$$\mathcal{T}_i(u_i^\infty) = \min_{W^{\eta_i}} \mathcal{T}_i = \min_{W^{\eta_i}} \mathcal{T}_k \geq \min_{W^{\eta_k}} \mathcal{T}_k = \mathcal{T}_k(u_k^\infty), \tag{5.12}$$

which completes the proof of (5.6).

The proof of (5.1) goes in a similar way as in [12]. □

Proof of Theorem 2.3. We split the proof into two parts.
(i) *Case* $p = 1$. Let $\lambda \geq 0$ with $\lambda\bar{c} < -l_\infty$ and fix $\tilde{\lambda}_\infty \in \mathbb{R}$ such that $\lambda\bar{c} < \tilde{\lambda}_\infty < -l_\infty$. With these choices, we define

$$k := \tilde{\lambda}_\infty - \lambda\bar{c} > 0 \text{ and } A(s) := F(s) + \frac{\tilde{\lambda}_\infty}{2}s^2 + \lambda\left(G(s) - \frac{\bar{c}}{2}s^2\right) \text{ for every } s \in [0,\infty). \tag{5.13}$$

It is clear that $A(0)=0$, i.e., (H_0^∞) is verified. A similar argument for the p-order perturbation ∂G as before shows that

$$\liminf_{s\to\infty}\frac{A(s)}{s^2}\geq\liminf_{s\to\infty}\frac{F(s)}{s^2}+\frac{\tilde\lambda_\infty-\lambda\overline{c}}{2}+\lambda\liminf_{s\to\infty}\frac{G(s)}{s^2}\geq\liminf_{s\to\infty}\frac{F(s)}{s^2}+\frac{\tilde\lambda_\infty-\lambda\overline{c}}{2}+\lambda\underline{c}>-\infty,$$

and

$$\limsup_{s\to\infty}\frac{A(s)}{s^2}\geq\limsup_{s\to\infty}\frac{F(s)}{s^2}+\frac{\tilde\lambda_\infty-\lambda\overline{c}}{2}+\lambda\liminf_{s\to\infty}\frac{G(s)}{s^2}=+\infty,$$

i.e., (H_1^∞) is verified.

Since

$$\partial A(s)\subseteq\partial F(s)+\tilde\lambda_\infty s+\lambda(\partial G(s)-\overline{c}s),\quad s\geq 0, \tag{5.14}$$

it turns out that

$$\liminf_{s\to\infty}\frac{\max\{\partial A(s)\}}{s}\leq\liminf_{s\to\infty}\frac{\max\{\partial F(s)\}}{s}+\tilde\lambda_\infty-\lambda\overline{c}+\lambda\limsup_{s\to\infty}\frac{\max\{\partial G(s)\}}{s}=l_\infty+\tilde\lambda_\infty<0.$$

By using the upper semicontinuity of $s\mapsto\partial A(s)$, one may fix two sequences $\{\delta_i\}_i,\{\eta_i\}_i\subset(0,\infty)$ such that $0<\delta_i<s_i<\eta_i<\delta_{i+1}$, $\lim_{i\to\infty}\delta_i=\infty$, and $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Thus, (H_2^∞) is verified as well. By applying the inclusion (5.14) and Theorem 4.1 with the choice (5.13), there exists a sequence $\{u_i\}_i\subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases}-\triangle u_i(x)+(\tilde\lambda_\infty-\lambda\overline{c})u_i(x)\in\partial F(u_i(x))+\tilde\lambda_\infty u_i(x)+\lambda(\partial G(u_i(x))-\overline{c}u_i(x)) & x\in\Omega,\\ u_i(x)\geq 0 & x\in\Omega,\\ u_i(x)=0 & x\in\partial\Omega,\end{cases}$$

i.e., u_i solves problem ($\mathcal{D}_\lambda$), $i\in\mathbb{N}$.

(ii) *Case* $p<1$. Let $\lambda\geq 0$ be arbitrary fixed and choose a number $\lambda_\infty\in(0,-l_\infty)$. Let

$$k:=\lambda_\infty>0\ \text{ and } A(s):=F(s)+\lambda G(s)+\lambda_\infty\frac{s^2}{2}\ \text{for every } s\in[0,\infty). \tag{5.15}$$

Since $F(0)=G(0)=0$, hypothesis (H_0^∞) clearly holds. Moreover, by (G_1^∞), for sufficiently small $\epsilon>0$ there exists $s_0>0$, such that $(\underline{c}-\epsilon)s^{p+1}\leq G(s)\leq(\overline{c}+\epsilon)s^{p+1}$ for every $s>s_0$. Thus, since $p<1$,

$$\lim_{s\to\infty}\frac{G(s)}{s^2}=\lim_{s\to\infty}\frac{G(s)}{s^{p+1}}s^{p-1}=0.$$

Accordingly, by using (5.15) we obtain that hypothesis (H_1^∞) holds. A similar argument as above implies that

$$\liminf_{s\to\infty}\frac{\max\{\partial A(s)\}}{s}\leq l_0+\lambda_\infty<0,$$

and the upper semicontinuity of ∂A implies the existence of two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i\subset(0,1)$ such that $0<\delta_i<s_i<\eta_i<\delta_{i+1}$, $\lim_{i\to\infty}\delta_i=\infty$, and $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Therefore, hypothesis (H_2^∞) holds. Now, we can apply Theorem 4.1, i.e., there is a sequence $\{u_i\}_i\subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases}-\triangle u_i(x)+\lambda_\infty u_i(x)\in\partial A(u_i(x))\subseteq\partial F(u_i(x))+\lambda\partial G(u_i(x))+\lambda_\infty u_i(x) & x\in\Omega,\\ u_i(x)\geq 0 & x\in\Omega,\\ u_i(x)=0 & x\in\partial\Omega,\end{cases}$$

which means that u_i solves problem ($\mathcal{D}_\lambda$), $i\in\mathbb{N}$, which completes the proof. □

Proof of Theorem 2.4. The proof is done in two steps:

(i) Let $\lambda_\infty \in (0, -l_\infty), \lambda \geq 0$ and define

$$k := \lambda_\infty > 0 \quad \text{and } A^\lambda(s) := F(s) + \lambda G(s) + \lambda_\infty \frac{s^2}{2} \text{ for every } s \in [0, \infty). \tag{5.16}$$

One has clearly that $\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_\infty s + \lambda \partial G(s)$ for every $s \in \mathbb{R}$. On account of (F_2^∞), there is a sequence $\{s_i\}_i \subset (0, \infty)$ converging to ∞ such that

$$\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_\infty s_i < 0.$$

By the upper semicontinuity of $(s, \lambda) \mapsto \partial A^\lambda(s)$, we can choose the sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0, \infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}, \lim_{i\to\infty} \delta_i = \infty$, and

$$\max\{\partial A^\lambda(s)\} \leq 0$$

for all $\lambda \in [0, \lambda_i], s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$.

For every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, let $A_i^\lambda : [0, \infty) \to \mathbb{R}$ be defined by

$$A_i^\lambda(s) = A^\lambda(\tau_{\eta_i}(s)), \tag{5.17}$$

and accordingly, the energy functional $\mathcal{T}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem$(\mathrm{D}^k_{A_i^\lambda})$ is

$$\mathcal{T}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A_i^\lambda(u(x))dx.$$

Then for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, the function A_i^λ clearly verifies the hypotheses of Theorem 3.1. Accordingly, for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$ there exists

$$\mathcal{T}_{i,\lambda} \text{ attains its infimum at some } \tilde{u}^\infty_{i,\lambda} \in W^{\eta_i} \tag{5.18}$$

$$\tilde{u}^\infty_{i,\lambda} \in [0, \delta_i] \text{ for a.e. } x \in \Omega; \tag{5.19}$$

$$\tilde{u}^\infty_{i,\lambda}(x) \text{ is a weak solution of } (\mathrm{D}^k_{A_i^\lambda}). \tag{5.20}$$

Due to (5.17), $\tilde{u}^\infty_{i,\lambda}$ is not only a solution to $(\mathrm{D}^k_{A_i^\lambda})$ but also to the differential inclusion problem $(\mathrm{D}^k_{A^\lambda})$, so $(\mathcal{D}_\lambda)$.

(ii) For $\lambda = 0$, the function $\partial A_i^\lambda = \partial A_i^0$ verifies the hypotheses of Theorem 4.1. Moreover, $\mathcal{T}_i := \mathcal{T}_{i,0}$ is the energy functional associated with problem $(\mathrm{D}^k_{A_i^0})$. Consequently, the elements $u_i^\infty := u^\infty_{i,0}$ verify not only (5.18)-(5.20) but also

$$\mathcal{T}_{m_i}(u^\infty_{m_i}) = \min_{W^{\eta m_i}}(\mathcal{T}_{m_i}) \leq \mathcal{T}_{m_i}(w_{\tilde{s}_i}) \text{ for all } i \in \mathbb{N}, \tag{5.21}$$

where the subsequence $\{u^\infty_{m_i}\}_i$ of $\{u_i^\infty\}_i$ and $w_{\tilde{s}_i} \in W^{\eta_i}$ appear in the proof of Theorem 5.1.

Similarly to Kristály and Moroşanu [12], let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty} \theta_i = -\infty$. On account of (5.21) we may assume that

$$\theta_{i+1} < \mathcal{T}_{m_i}(u^\infty_{m_i}) \leq \mathcal{T}_{m_i}(w_{\tilde{s}_i}) < \theta_i. \tag{5.22}$$

Let

$$\lambda_i^{'} = \frac{\theta_i - \mathcal{T}_{m_i}(w_{\tilde{s}_i})}{m(\Omega)\max_{s\in[0,1]}|G(s)| + 1} \text{ and } \lambda_i^{''} = \frac{\mathcal{T}_{m_i}(u^\infty_{m_i}) - \theta_{i+1}}{m(\Omega)\max_{s\in[0,1]}|G(s)| + 1}, \ i \in \mathbb{N}, \tag{5.23}$$

and for a fixed $k \in \mathbb{N}$, we set

$$\lambda_k^\infty = \min(1, \lambda_1, ..., \lambda_k, \lambda_1{}^{'}, ..., \lambda_k{}^{'}, \lambda_1{}^{''}, ..., \lambda_k{}^{''}) > 0. \tag{5.24}$$

Then, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^\infty]$, due to (5.22) we have that

$$\begin{aligned} \mathcal{T}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) &\le \mathcal{T}_{m_i,\lambda}(w_{\tilde{s}_i}) = \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H^1_0} - \int_\Omega F(w_{\tilde{s}_i}(x))dx - \lambda \int_\Omega G(w_{\tilde{s}_i}(x))dx \\ &= \mathcal{T}_{m_i}(w_{\tilde{s}_i}) - \lambda \int_\Omega G(w_{\tilde{s}_i}(x))dx \\ &< \theta_i. \end{aligned} \tag{5.25}$$

Similarly, since $\tilde{u}^\infty_{m_i,\lambda} \in W^{\eta_{m_i}}$ and $u^\infty_{m_i}$ is the minimum point of $\mathcal{T}_i$ on the set $W^{\eta_{m_i}}$, on account of (5.22) we have

$$\mathcal{T}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) = \mathcal{T}_{m_i}(\tilde{u}^\infty_{m_i,\lambda}) - \lambda \int_\Omega G(\tilde{u}^\infty_{m_i,\lambda})dx \ge \mathcal{T}_{m_i}(u^\infty_{m_i}) - \lambda \int_\Omega G(\tilde{u}^\infty_{m_i,\lambda})dx > \theta_{i+1}. \tag{5.26}$$

Therefore, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^\infty]$,

$$\theta_{i+1} < \mathcal{T}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) < \theta_i < 0, \tag{5.27}$$

thus

$$\mathcal{T}_{m_k,\lambda}(\tilde{u}^\infty_{m_k,\lambda}) < ... < \mathcal{T}_{m_1,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0. \tag{5.28}$$

Because of (5.17), we notice that $\tilde{u}^\infty_{m_i,\lambda} \in W^{\eta_{m_k}}$ for every $i \in \{1, ..., k\}$, thus $\mathcal{T}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) = \mathcal{T}_{m_k,\lambda}(\tilde{u}^\infty_{i,\lambda})$. Therefore, for every $\lambda \in [0, \lambda_k^\infty]$,

$$\mathcal{T}_{m_k,\lambda}(\tilde{u}^\infty_{m_k,\lambda}) < ... < \mathcal{T}_{m_k,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0 = \mathcal{T}_{m_k,\lambda}(0),$$

i.e, the elements $\tilde{u}^\infty_{m_1,\lambda}, ..., \tilde{u}^\infty_{m_k,\lambda}$ are distinct and non-trivial whenever $\lambda \in [0, \lambda_k^\infty]$. The estimate (2.5) follows in a similar manner as in [12]. □

6. Concluding remarks

1. Suitable modification of our arguments provide multiplicity results for the differential inclusion problem

$$\begin{cases} -\Delta u(x) + u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in } \mathbb{R}^n; \\ u \ge 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{$\tilde{\mathcal{D}}_\lambda$}$$

   where ∂F and ∂G behave in a similar manner as before. The main difficulty in the investigation of $(\tilde{\mathcal{D}}_\lambda)$ is the lack of compact embedding of the Sobolev space $H^1(\mathbb{R}^n)$ into the Lebesgue spaces $L^q(\mathbb{R}^n)$, $n \ge 2$, $q \in [2, 2^*)$. However, by using Strauss-type estimates and Lions-type embedding results for radially symmetric functions of $H^1(\mathbb{R}^n)$ (see e.g. Willem [23]), the principle of symmetric criticality for non-smooth functionals (see Kobayashi and Ôtani [13] and Squassina [22]) provides the expected results. A related result in the smooth setting can be found in Kristály [11].
2. Assume that ∂F oscillates at a point $l \in [0, +\infty]$ and ∂G has a p-order growth at l. We are wondering if our results, valid for $l = 0$ and $l = +\infty$, can be extended to any $l \in (0, \infty)$, even in the smooth framework.

7. Appendix: Locally Lipschitz functions

In this part we collect those notions and properties of locally Lipschitz functions which are used in the proofs; for details, see Clarke [7] and Chang [6]. Let $(X, \|\cdot\|)$ be a real Banach space and $U \subset X$ be an open set; we denote by $\langle \cdot, \cdot \rangle$ the duality mapping between $X^\star$ and X.

Definition 7.1. (see [7]) *A function $f : X \to \mathbb{R}$ is* locally Lipschitz *if, for every $x \in X$, there exist a neighborhood U of x and a constant $L > 0$ such that*

$$|f(x_1) - f(x_2)| \leq L\|x_1 - x_2\| \quad \textit{for all} \quad x_1, x_2 \in U.$$

Definition 7.2. (see [7]) *Let f be a locally Lipschitz function near the point x and let v be any arbitrary vector in X. The* generalized directional derivative in the sense of Clarke *of f at the point $x \in X$ in the direction $v \in X$ is*

$$f^\circ(x; v) = \limsup_{z \to x, \ \tau \to 0^+} \frac{f(z + \tau v) - f(z)}{\tau}.$$

The generalized gradient *of f at $x \in X$ is the set*

$$\partial f(x) = \{x^\star \in X^\star : \ \langle x^\star, v\rangle \leq f^\circ(x; v) \text{ for all } v \in X\}.$$

For all $x \in X$, the functional $f^\circ(x, \cdot)$ is finite and positively homogeneous. Moreover, we have the following properties.

Proposition 7.1. (see [7]) *Let X be a real Banach space, $U \subset X$ an open subset and $f, g : U \to \mathbb{R}$ be locally Lipschitz functions. The following properties hold:*

- (a) *For every $x \in U$, $\partial f(x)$ is a nonempty, convex and weakly*-compact subset of $X^\star$ which is bounded by the Lipschitz constant $L > 0$ of f near x;*
- (b) *$f^\circ(x; v) = \max\{\langle \xi, v\rangle : \ \xi \in \partial f(x)\}$ for all $v \in X$;*
- (c) *$(f + g)^\circ(x; v) \leq f^\circ(x; v) + g^\circ(x; v)$ for all $x \in U$, $v \in X$;*
- (d) *$\partial(f + g)(u) \subset \partial f(u) + \partial g(u)$ for all $u \in U$;*
- (e) *$(-f)^\circ(x; v) = f^\circ(x; -v)$ for all $x \in U$;*
- (f) *The function $(x, v) \mapsto f^\circ(x; v)$ is upper semicontinuous;*
- (g) *The set-valued map $\partial f : U \to 2^{X^\star}$ is weakly*-closed, that is, if $\{x_i\} \subset U$ and $\{w_i\} \subset X^*$ are sequences such that $x_i \to x$ strongly in X and $w_i \in \partial f(x_i)$ with $w_i \rightharpoonup z$ weakly* in X^*, then $z \in \partial f(x)$. In particular, if X is finite dimensional, then ∂f is upper semicontinuous, i.e., for every $\epsilon > 0$ there exists $\gamma > 0$ such that $\partial f(x') \subseteq \partial f(x) + B_{X^*}(0, \epsilon)$, $\forall x' \in B_X(x, \gamma)$;*

Proposition 7.2. (see [6]) *The number $\lambda_f(u) = \inf\limits_{w \in \partial f(u)} ||w||_{X^\star}$ is well defined and*

$$\liminf_{u \to u_0} \lambda_f(u) \geq \lambda_f(u_0).$$

Definition 7.3. (see [6]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. We say that $u \in X$ is a* critical point *(in the sense of Chang) of f, if $\lambda_f(u) = 0$, i.e., $0 \in \partial f(u)$.*

Remark 7.1. (see [7]) (a) $u \in X$ is a critical point of f if $f^\circ(u; v) \geq 0$ for all $v \in X$.

(b) If $x \in U$ is a local minimum or maximum of the locally Lipschitz function $f : X \to \mathbb{R}$ on an open set of a Banach space, then x is a critical point of f.

Proposition 7.3. (see [7]) (Lebourg's mean value theorem) *Let X be a Banach space, $x, y \in X$ and $f : X \to \mathbb{R}$ be Lipschitz on an open set containing the line segment $[x, y]$. Then there is a point $a \in (x, y)$ such that*

$$f(y) - f(x) \in \langle \partial f(a), y - x \rangle.$$

Proposition 7.4. (see [7]) (Chain Rule) *Let X be Banach space, let us consider the composite function $f = g \circ h$ where $h : X \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. Let denote h_i, $i \in \{1, ..., n\}$ be the component functions of h. We assume h_i is locally Lipschitz near x and g is too near $h(x)$. Then f is locally Lipschitz near x as well. Let us denote by α_i the elements of ∂g, and let $\alpha = (\alpha_1, ..., \alpha_n)$; then*

$$\partial f(x) \subset \overline{\text{co}}\{\sum \alpha_i \xi_i : \xi_i \in \partial h_i(x), \alpha \in \partial g(h(x))\},$$

where $\overline{\text{co}}$ denotes the weak-closed convex hull.

References

[1] A. Ambrosetti, H. Brezis, G. Cerami: Combined effects of concave and convex nonlinearities in some elliptic problems. *J. Funct. Anal.* 122(1994), no. 2, 519–543.

[2] A. Ambrosetti, P. Rabinowitz: Dual variational methods in critical point theory and applications. *J. Funct. Anal.* 14(1973), 349–381.

[3] V. Ambrosio, L. D'Onofrio, G. Molica Bisci: On nonlocal fractional Laplacian problems with oscillating potentials. *Rocky Mountain J. Math.* 48(2018), no. 5, 1399–1436.

[4] G. Autuori, P. Pucci: Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$. *J. Differential Equations* 255(2013), no. 8, 2340–2362.

[5] M. F. Bocea, P. D. Panagiotopoulos, V. D. Rădulescu: A perturbation result for a double eigenvalue hemivariational inequality with constraints and applications. *J. Global Optim.* 14(1999), 137–156.

[6] K.-C. Chang: Variational methods for nondifferentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.* 80(1981), 102–129.

[7] F.H. Clarke: Optimization and Nonsmooth Analysis. *Wiley*, 1983.

[8] D.G. De Figueiredo, J.-P. Gossez, P. Ubilla: Local superlinearity and sublinearity for indefinite semilinear elliptic problems. *J. Funct. Anal.* 199(2003), no. 2, 452–467.

[9] F. Gazzola, V. Rădulescu: A nonsmooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^N$. *Differential Integral Equations*, 13(2000), 47–60.

[10] A. Kristály: Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^n$, *J. Differential Equations*, 220(2006), no. 2, 511–530.

[11] A. Kristály: Detection of arbitrarily many solutions for perturbed elliptic problems involving oscillatory terms. *J. Differential Equations* 245(2008), no. 12, 3849–3868.

[12] A. Kristály, G. Moroşanu: New competition phenomena in Dirichlet problems. *J. Math. Pures Appl.* 94(2010) no. 6, 555–570.

[13] J. Kobayashi, M. Ôtani: The principle of symmetric criticality for non-differentiable mappings. *J. Funct. Anal.* 214(2004), no. 2, 428–449.

[14] M. Marcus, V. Mizel: Every superposition operator mapping one Sobolev space into another is continuous. *J. Funct. Anal.* 33(1979), 217–229.

[15] M. Mălin, V. Rădulescu: Infinitely many solutions for a nonlinear difference equation with oscillatory nonlinearity. *Ric. Mat.* 65(2016), no. 1, 193–208.

[16] G. Molica Bisci, P. F. Pizzimenti: Sequences of weak solutions for non-local elliptic problems with Dirichlet boundary condition. *Proc. Edinb. Math. Soc.* 57(2014), no. 3, 779–809.

[17] G. Molica Bisci, V. Rădulescu, R. Servadei: Competition phenomena for elliptic equations involving a general operator in divergence form. *Anal. Appl. (Singap.)* 15(2017), no. 1, 51–82.

[18] D. Motreanu, P.D. Panagiotopoulos: Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities. *Springer-Science Business Media*, B.V., 1999.

[19] P. Omari, F. Zanolin: Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential. *Comm. Partial Differential Equations*, 21(1996), 721–733.

[20] P.D. Panagiotopoulos: Inequality problems in mechanics and applications. Convex and nonconvex energy functionals. *Birkhäuser-Verlag*, Basel, 1985.

[21] J. Saint Raymond: On the multiplicity of the solutions of the equation $-\Delta u = \lambda f(u)$. *J. Differential Equations*, 180(2002), 65–88.

[22] M. Squassina: On Palais' principle for non-smooth functionals. *Nonlinear Anal.* 74(2011), no. 11, 3786–3804.

[23] M. Willem: Minimax Theorems. *Birkhäuser Boston*, 1996.

[24] E. Zeidler: Nonlinear functional analysis and its applications. vol. III, *Springer*, 1985.

Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary & Department of Economics, Babeş-Bolyai University, 400591 Cluj-Napoca, Romania

E-mail address: kristaly.alexandru@nik.uni-obuda.hu; alex.kristaly@econ.ubbcluj.ro

Department of Mathematics and Informatics Babeş-Bolyai University 400084 Cluj-Napoca, Romania

E-mail address: ildiko.mezei@math.ubbcluj.ro

Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary

E-mail address: karoly.szilak@gmail.com