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RESEARCH ARTICLE

How to choose a fair delegation?

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Abstract
This paper analyzes how to choose a delegation, a committee to represent a society such as in a peace conference. We propose normative conditions and seek Pareto optimal, consistent, neutral, and non-manipulable ways to choose a delegation. We show that a class of threshold rules is characterized by these criteria. The rules do not choose a fixed number of delegates, but instead require different sizes of delegations, depending on the heterogeneity in society. Therefore the resulting delegations are very inclusive, and with t delegates the ratio of individuals whose opinions are not included is always below 0.5^t. For instance, a delegation of size two should have at least 75% support from the society and therefore only less than 25% of the opinion pool can be neglected.

Keywords Aggregation rules · Committee selection · Conflict management · Kemeny distance · Strategy-proofness

JEL Classification C70 · D71

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1 Introduction

In many situations, individuals participate in collective decision making via a committee of representatives or delegates, i.e., there is a double-layered aggregation of individual opinions. Consider, for instance, voting for political candidates in elections to represent one's opinion in a parliament. Board decisions in large corporations are also taken collectively via a committee representing different departments, albeit not every department is allocated a seat at the board. Correspondingly, peace conferences and negotiations over conflict zones require delegates to be sent by different interest groups or ethnicities involved in a civil war. The choice of which interest group or ethnicity to invite to the conference, however, is not very straightforward—especially not in extremely heterogeneous, polarized, or divided societies. In fact, the way in which peace conferences are set also signals the possible effectiveness of these talks.

In many settings, delegates representing their interest groups (or certain types) with various degrees of support in their society do not necessarily carry a weight representing their social support in the committee.[1] This is especially the case when the delegation is summoned *not* to make a decision but to deliberate on a memorandum of understanding, or exchange ideas, information, perspectives and perhaps eventually act as an advisory board. For instance, with a peace conference scenario in mind, consider a society and the issues it is facing. Delegates representing various ethnic groups are invited to the table in the hope of creating mutual understanding. Similarly, consider different interest groups who have diverse opinions on how the city council should allocate the budget across different expenditures, e.g., a public park, a new tunnel, or a citywide educational program. Of course, delegates may represent groups with different powers, and hold a varying degree of support from their supporters. However, the output of such initiatives as the first layer in a multi-layered aggregation typically consists of compiling a report or suggestion to another body that makes the decisions or mediates the process. Therefore, we avoid a consequentialist approach where the eventual outcome of the mediation process is the only factor that matters. We follow a non-consequentialist method and focus on how this delegated mediation should be formed. As proposed and characterized in Suzumura and Xu (2001), there exist situations where people care about the "features of the decision making procedure through which the consequences are brought about". It is those cases that we inquire about, where for each individual "some form of representation" of their opinion is the ultimate goal instead of the final implementation. By this, we make a clear distinction between two concepts on opinion formation, i.e., *deliberative* and *aggregative*. We believe that our results contribute to the somewhat neglected part of this research area on deliberation. [2]

Consider a vector, say (6, 2, 2, 1, 1, 0), where the values correspond to the number of people in a society who support a particular opinion out of six possibilities. What choice of opinions would be appropriate to represent this society? What should be the size of the delegation? Should we fix the size ex-ante and then choose the opinions

[1] For instance each of the 193 member states in the United Nations has a single ambassador for their permanent missions, regardless of the size of the nations they represent.

[2] Interesting aspects of consultation and deliberation also appear in the papal conclave, axiomatically analyzed by Mackenzie (2020).

according to some criteria? An ideal world in our context would obviously be forming a delegation where all five opinions with nonzero support are represented. This may be impractical, however, especially when opinions are too diverse. Considering the support for each opinion, a plausible solution would be to represent opinions that have "sufficient" support. We want to find out what limited set of opinions we could choose to represent this society in a reasonable, fair, and plausible way. We are not interested in how much relative importance an opinion in the delegation has per se, but only in whether an opinion is "relevant" enough to be invited to the table. Eventually, we might bring together some people representing those opinions on behalf of possibly very diverse interest groups and hope to achieve a fruitful exchange of information and deliberation within the delegation itself.

This paper investigates possible mechanisms through which this table can be formed. We require such mechanisms to respect some minimal normative requirements. These requirements allow us to build an axiomatic framework for the analysis of *deliberative democracy* in contrast to *aggregative democracy*. For instance, the delegation choice should respect unanimous agreements in the society and be consistent in choosing delegates when similar societies are merged. We expect it to be neutral in the way it treats the opinions, and also expect it to be non-manipulable, such that the individuals have no improvement in representation through misreporting their ideas. We propose all these norms as criteria to choose which opinions should get a seat at the table and then show that there is a unique class of rules which satisfy all these criteria, hence a characterization result. These novel rules are non-trivial and relatively simple to comprehend, making them practically usable to form the table.

We assume individuals have priority orderings (opinions) over some available issues, and those orderings form the preference profile of a society. We formalize the delegation rules as mechanisms that assign a set of orderings (opinions) to each given preference profile as representatives in a delegation. Since the delegation is not necessarily comprised of a single opinion, a delegation rule herein corresponds to a social welfare correspondence instead of a social welfare function. However, we employ the term delegation as it entails a particular interpretation. That is, a delegation, which is a set of orderings, is the collection of opinions that represents the society and should be invited to the table. In addition, these rules do not impose a fixed size of delegation, instead, the size of the delegations naturally depends on how the opinions are distributed.

We first require that if all individuals in a society agree on how to rank one issue over another, the delegation should respect that. This is also known as *Pareto optimality*. Second, we impose that when two distinct societies represented by identical delegations merge, the merged society should also be represented by the same delegation (Young 1974, 1975; Smith 1973), an idea known as *Consistency*[3]. The third condition, *Support neutrality* reflects an idea of fairness, and requires that only the support of individual opinions should matter in the delegation choice.[4] Finally, we require that no individual can manipulate the choice of delegation to their advantage.

[3] Consistency additionally implies that the delegation choice is anonymous, a condition which requires that the names of the individuals do not matter.

[4] Support neutrality additionally implies that the delegation choice is neutral, a condition which requires that relabelling the names of issues does not matter.

This condition is called *Strategy-proofness*. The first two, Pareto optimality and consistency, are very standard conditions in the literature. In what follows, we explain further support neutrality and strategy-proofness.

Support neutrality imposes neutrality towards "equivalent supports". Consider two societies of equal size, say, six individuals, facing three issues, hence six possible preferences. Now represent the opinions in both societies by the number of followers each preference has, e.g., (3, 2, 1, 0, 0, 0) and (0, 1, 2, 3, 0, 0). As it happens, the two societies have "equivalent" ballots, i.e., the support distribution is merely a shuffling of the number of followers. In this case, we require the delegation choice in each society to correspond to the support for the delegates. For instance, if the first preference in the former ballot with a support of 3 is chosen as a delegate in the first society, then the fourth preference in the latter ballot should also be chosen in the second society. Support neutrality is also a variable alternative axiom, which necessitates that increasing the number of available issues and preferences, does not influence the outcome so long as the distributions of supported opinions are equivalent.[5] We explain this further in detail in the coming section and provide an example in the appendix.

Strategy-proofness requires that the rule is not manipulable by individuals (or coalitions). Therefore a rule being strategy-proof naturally induces honest reporting of individual opinions. Consider a society and a delegation representing it. Suppose an individual misreports his opinion, and this alters the delegation such that at least one new delegate is strictly closer[6] to his opinion than any other delegate in the original delegation. This situation is considered as a successful manipulation. We require that the delegation choice should not be prone to any such manipulation. If a rule is not manipulable by any individual, then we call it individual strategy-proof, whereas if no coalition of individuals can achieve such manipulation, we call it coalitional strategy-proof. The latter is a stronger requirement than the former. Our strategy-proofness concept is fundamentally different than that of Bossert and Storcken (1992), Bossert and Sprumont (2014) and Athanasoglou (2016), since we allow multiple opinions in the outcome.[7]

We find that there exists a non-dictatorial, non-trivial, and in fact, simple class of rules which is characterized by these conditions, and which we call *threshold rules*. The threshold rules impose different sizes of delegations depending on the composition

5 For instance, increasing the number of issues to 4, and hence the number of possible preferences to 24, would still yield the first preference as the only delegate if the ballot stayed as $(3, 2, 1, 0, 0, \ldots, 0)$.

6 We use the most typical measure of closeness for orderings, i.e., the Kemeny distance (Kemeny 1959). This metric is applied in many different contexts similar (or identical) to the Kendall–Tau distance (Kendall 1938), the Damerau–Levenshtein distance (Damerau 1964; Levenshtein 1966), the Hamming distance (Hamming 1950), and swap distance among others.

7 When a profile of orderings is aggregated into a single alternative, i.e., social choice function, Gibbard (1973) and Satterthwaite (1975) show the impossibility of finding proper non-dictatorial and strategy-proof rules on unrestricted domain. See Barberà et al. (2001) and Barberà (2011) for more on strategy-proof social choice rules and Dasgupta and Maskin (2008) for robustness of the majority rule when the domain is restricted in various ways. When a profile is aggregated into a single ordering, i.e., social welfare function, the results are mixed since the definition of strategy-proofness can be quite numerous. Bossert and Storcken (1992) prove an impossibility result, Sato (2013) offers more positive news, and finally, Bossert and Sprumont (2014) uses a weaker version of strategy-proofness than in Bossert and Storcken (1992) and provides some examples of non-manipulable rules.

of the society instead of a fixed size. This is very natural as opinions in a group of people may have a different level of polarization and diversity. Therefore, threshold rules change the size of the delegation depending on the heterogeneity of the society. The rules also share a common lower bound in terms of how much representative power they require for all possible size of delegations. For instance, if a delegation is composed of t delegates, then the ratio of individuals whose opinions are not included is always below 0.5^t, hence the ratio of the individuals supporting those delegates to the whole society must be strictly higher than $1 - 0.5^t$.

The delegation rules we characterize only differ in *how high* the thresholds are set above the common lower bound. We show that for each threshold rule, there exists a *threshold function* f, which imposes how much minimal support a delegation of size t has to have to be an appropriate representation for a society. The rule orders each possible opinion/delegates according to their support in the society and chooses the lowest number of delegates t^* with a total support reaching the respective threshold, i.e., $f(t^*)$. For example, a threshold rule might require 60% of the society's support for singleton delegations, i.e., $f(1) = 0.6$. If this support is not found, then it might look for 85% of the society's support for a delegation of size 2, $f(2) = 0.85$. If this support is not found, then the process continues, with monotonically increasing thresholds for each t. We show that all threshold rules satisfy two conditions: (i) $f(1) > 0.5$ and (ii) $1 \geq f(t) \geq (f(t-1)+1)/2$ for $t \geq 2$, i.e., the minimal threshold for a singleton delegation is above 50%, and the minimal threshold for delegations of size $t \geq 2$ is at least the average of the previous level, $f(t-1)$ and 100%, but at most 100%. Of course, $f(1)$ can also start from 100% (and hence continue at that level), requiring 100% support for each possible sizes of delegations, which can only be reached by including all the reported preferences.

As Lanz (2011) argues, "Only stakeholders who add value to the process and augment the chances of reaching a sustainable settlement should be given seats at the table, [...]". The challenge, therefore, is to make the invitations to the table from a normative perspective while maintaining inclusivity and feasibility. This paper proposes a quantitative measure on how to form the table for invitations, the number of seats at the table, and finally how representative in total, the invitees must minimally be.

The paper proceeds as follows. Section 2 presents the notation and conditions. In Sect. 3, we define threshold rules and provide some examples. In Sect. 4, we provide our characterization. Section 5 concludes with some policy implication.

2 Basic notation and conditions

2.1 Model

Let $\mathcal{A}$ be a countably infinite set of alternatives, interpreted as potential issues. Given a finite nonempty subset $A \subsetneq \mathcal{A}$, *preferences* are taken to be strict priority rankings of these issues, formalized as complete, antisymmetric and transitive binary relations over the set of alternatives A. We denote the set of all preferences over A by $\mathcal{L}(A)$. Given a preference $R \in \mathcal{L}(A)$, and two distinct alternatives

a and b, the case where a is preferred to b can be denoted by $R = .a.b.$ or $(a, b) \in R$. Consider, for instance, for $A = \{a, b, c\}$ the preferences $R_1 = abc$ and $R_2 = acb$. Then $\{(a, b), (a, c), (b, c), (a, a), (b, b), (c, c)\} = R_1$ and $\{(a, b), (a, c), (c, b), (a, a), (b, b), (c, c)\} = R_2$.

To measure closeness, we use the well-known Kemeny distance.[8] The Kemeny distance counts the number of disagreements in two preferences. Formally, for any two preferences R_1 and R_2, the Kemeny distance is $\delta(R_1, R_2) = (|R_2 \backslash R_1| + |R_1 \backslash R_2|)/2$ where $|R_2 \backslash R_1|$ denotes the number of ordered pairs in R_2 but not in R_1, and vice versa. For instance, for R_1 and R_2 above, the only disagreement stems from how to rank the alternatives b and c. For the rest, both preferences are aligned. This disagreement is exposed by the Kemeny distance by summing $|R_2 \backslash R_1| = |\{(c, b)\}|$ and $|R_1 \backslash R_2| = |\{(b, c)\}|$ and dividing by 2 to account for the symmetry in these pairs. All in all, this is interpreted as half of the symmetric set difference, i.e., $\delta(R_1, R_2) = 1$ for $R_1 = abc$ and $R_2 = acb$.

Let $\mathcal{N}$ be a countably infinite set of agents, interpreted as potential individuals. Given a finite nonempty subset $N \subsetneq \mathcal{N}$ with cardinality n, $\mathcal{L}(A)^n$ denotes the set of all preference profiles P, i.e., preferences of n agents where $P(i)$ refers to the preference of agent $i \in N$ and $P(S)$ refers to the preference profile, say a subprofile, of a subset of agents $S \subseteq N$. Given a profile $P \in \mathcal{L}(A)^n$, and $R \in \mathcal{L}(A)$, we denote the number of agents who reported R in this profile as $p(R) = |\{i \in N \mid P(i) = R\}|$.

Given any finite $A \subsetneq \mathcal{A}$, let $R_1, R_2, \ldots, R_{|A|!}$ be an *enumeration* of preferences in $\mathcal{L}(A)$, e.g., the lexicographic enumeration for $A = \{a, b, c\}$ is "$R_1 = abc, R_2 = acb, R_3 = bac, R_4 = bca, R_5 = cab, R_6 = cba$". Let $\mathbb{Z}_+$ denote the set of non-negative integers. Given any such enumeration, a profile $P \in \mathcal{L}(A)^n$ can also be interpreted as a vector composed of the number of followers each preference has, e.g., $p = (p_1, p_2, , p_3, \ldots, p_{|A|!})$ on $\mathbb{Z}_+^{|A|!}$ with the interpretation that $p_t = |\{i \in N \mid P(i) = R_t\}|$ is the *support* for preference $R_t \in \mathcal{L}(A)$ and p is the support for the preference profile P. As an example, for 3 alternatives and the lexicographic enumeration given above, the support for the following preference profile,

$$P = \{\underbrace{abc, abc, abc}_{R_1}, \underbrace{bac, bac}_{R_3}, \underbrace{cab}_{R_5}\} \in \mathcal{L}(A)^6$$

can be denoted by $p = (3, 0, 2, 0, 1, 0)$. For simplicity, we also denote the *normalized support* for the same profile similarly, e.g., $p = (0.5, 0, 0.\bar{3}, 0, 0.1\bar{6}, 0)$.

Consider two disjoint finite sets of agents N, N', and preference profiles $P \in \mathcal{L}(A)^n$, and $P' \in \mathcal{L}(A)^{n'}$. Then, $\bar{P} = (P, P') \in \mathcal{L}(A)^{n+n'}$ denotes the merging of two profiles, i.e, $\bar{P}(i) = P(i)$ if $i \in N$ and $\bar{P}(i) = P'(i)$ if $i \in N'$. If P and P' are such that there exists a bijection $\sigma : N \leftrightarrow N'$ such that $P(i) = P'(\sigma(i))$ for all $i \in N$, then we call $\bar{P} = (P, P')$ as a two-fold replica of P and denote it by $2P$. The definition naturally extends to all c-fold replicas cP of P, for any $c \geq 2$ for $c \in \mathbb{Z}_+$.

[8] Kemeny (1959) introduced this distance. For a recent characterization of this distance, revealing a flaw in Kemeny (1959), see Can and Storcken (2018).

We investigate delegation rules, collection of social welfare correspondences that can be defined for any finite population and any finite set of issues and assign a nonempty subset of opinions to each preference profile. Formally, a delegation rule is denoted by φ:

$$\varphi\colon \bigcup_{N \subsetneq \mathcal{N}, A \subsetneq \mathcal{A}} \mathcal{L}(A)^N \to 2^{\mathcal{L}(A)} \backslash \{\emptyset\} \tag{2.1}$$

Given a preference profile $P \in \mathcal{L}(A)^n$, the set of opinions $\varphi(P) \subseteq \mathcal{L}(A)$ is interpreted as the *set of delegates* or the *delegation* for this society.

2.2 Conditions

Next, we introduce some conditions on how to choose a delegation. The first condition requires that if everyone prefers an alternative over another, then no delegate should say otherwise.

Definition 1 (*Pareto optimality*) A rule φ is *Pareto Optimal* whenever for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, for all $P \in \mathcal{L}(A)^n$ and for all $a, b \in A$, if for all $i \in N$, $(a, b) \in P(i)$, then for all $R \in \varphi(P)$, $(a, b) \in R$.

The second condition we impose concerns merging of two societies each endowed with the same delegation. In such situations, the delegation assigned to the merged society should remain the same. This concept is well known in many contexts under varying names with slight changes, including reinforcement, homogeneity,[9] etc.

Definition 2 (*Consistency*) A rule φ is *consistent* whenever for all $A \subsetneq \mathcal{A}$, for all two disjoint finite sets $N, N' \subsetneq \mathcal{N}$ (with cardinality n and n' respectively) and for all profiles, $P \in \mathcal{L}(A)^n$ and $P' \in \mathcal{L}(A)^{n'}$, if $\varphi(P) = \varphi(P')$ then $\varphi((P, P')) = \varphi(P) = \varphi(P')$.

The third condition we impose concerns variable alternative scenarios, wherein the fixed set of individuals face more issues to report their preferences on. Consider, for instance, two sets of alternatives $A \subsetneq \bar{A}$ such that $|A| = 3$ and $|\bar{A}| = 4$. Consider two profiles on these sets with the following frequency supports:

$$P \in \mathcal{L}(A)^n \text{ with } p = (3, 2, 1, 0, 0, 0) \text{ and } \bar{P} \in \mathcal{L}(\bar{A})^n \text{ with } \bar{p} = (0, 1, 2, 3, \underbrace{0, \ldots, 0}_{20 \text{ entries}})$$

Note that the nonzero entries in each vector are identical (except for the shuffling). The condition requires that shuffling the support for preferences should shuffle the delegates in the exact same way.[10] Formally, take any two sets of alternatives such

[9] Homogeneity is a milder version of this concept, which requires that result would be insensitive to replicating the population (Fishburn 1977).

[10] This condition is, in fact, an amalgamation of two well-known conditions, neutrality and anonymity, and stronger than both.

that $A \subseteq \bar{A} \subsetneq \mathcal{A}$, a profile $P \in \mathcal{L}(A)^n$ and an injection[11] $\pi : \{1, 2, \ldots, |A|!\} \to \{1, 2, \ldots, |\bar{A}|!\}$ such that it injects each ranking on A to some unique ranking on $\bar{A}$. We say $\bar{P} \in \mathcal{L}(\bar{A})^n$ is an "*expansion of* P *by* π" if for all $t \in \{1, 2, \ldots, |A|!\}$ we have $p_t = \bar{p}_{\pi(t)}$. We consider such profiles $P, \bar{P}$ to have *equivalent supports* and refer to π as a corresponding injection.[12]

Definition 3 (*Support neutrality*) A rule is *support neutral* whenever for any two sets of alternatives $A \subseteq \bar{A} \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, for all $P \in \mathcal{L}(A)^n$ and $\bar{P} \in \mathcal{L}(\bar{A})^n$ with equivalent supports and for all corresponding injections π, we have:

$$R_i \in \varphi(P) \text{ if and only if } \bar{R}_{\pi(i)} \in \varphi(\bar{P}).$$

We provide an example in Appendix A.2 to illustrate this condition.[13] Next, we show that support neutrality, together with Pareto optimality, implies that the delegation can only be chosen from preferences that are reported. Thus we do not have to worry about finding a delegate whose role would be to represent some "compromised" preference. Let $RP(P) = \{R \in \mathcal{L}(A) \mid p(R) > 0\}$, denote the set of *reported preferences*, preferences which are reported by at least one agent in profile P.

Proposition 2.1 *If a rule φ satisfies Pareto optimality and support neutrality, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$, $\varphi(P) \subseteq RP(P)$.*

Proof The proof is in Appendix B.1. □

The following remark says that we can always find an expansion for a profile in which delegates and non-delegates are clustered, that is, each agent whose preference is not in the delegation will prefer any non-delegate to any delegate. In other words, every agent who supports a preference which is not part of the delegation would like to enlarge the delegation set.

Let us extend the definition of injections to sets of preferences. Given $A \subseteq \bar{A} \subsetneq \mathcal{A}$ and any injection $\pi : \{1, 2, \ldots |A|!\} \to \{1, 2, \ldots |\bar{A}|!\}$, and any $X \subseteq \mathcal{L}(A)$,

$$\pi(X) = \{\bar{R}_{\pi(i)} \in \mathcal{L}(\bar{A}) \mid R_i \in X\}.$$

Remark 2.1 Note that, since $\mathcal{A}$ is infinite, for any $A \subsetneq \mathcal{A}$, for any preference profile on A and for any two disjoint sets $X, Y \subsetneq \mathcal{L}(A)$, we can always find an expansion P by some π of the initial preference profile such that the injections of the two sets X and Y (denoted respectively by $\pi(X)$ and $\pi(Y)$), form clusters that are "far away" from each other. Formally:

[11] For $A = \bar{A}$, π is a permutation.

[12] There may be more than one corresponding injection for two equivalent supports.

[13] Note that the definition of support neutrality even extends to profiles on two disjoint sets of alternatives. For instance, let $A = \{x, y, z\}$ and $B = \{a, b, c\}$, and consider two profiles $P \in \mathcal{L}(A)^n$ and $\bar{P} \in \mathcal{L}(B)^n$ with identical ballots. Consider expansions of P and $\bar{P}$, say P' and $\bar{P}'$ respectively, to $A \cup B$ by some injection. Support neutrality applies between P and P' (and between $\bar{P}$ and $\bar{P}'$). By construction, P' and $\bar{P}'$ have equivalent supports. Therefore support neutrality applies between P' and $\bar{P}'$. This, in turn, imposes support neutrality between P and $\bar{P}$.

$$\max_{R, R' \in \pi(X)} \delta(R, R') < \min_{R \in \pi(X), R' \in \pi(Y)} \delta(R, R')$$

(The example in Appendix A.3 illustrates this remark. We would like to note, however, that the remark is quite general and the proofs in the sequel do not use the axiomatic properties of the Kemeny distance specifically except for betweenness.)

The fourth condition, strategy-proofness, implies that no agent should "benefit" from misreporting his preference, i.e., truth telling is a weakly dominant strategy. We say an agent i weakly prefers a delegate R_1 to another delegate R_2, whenever $P(i)$ is weakly closer to R_1 than it is to R_2 in terms of the Kemeny distance, i.e., $\delta(P(i), R_1) \leq \delta(P(i), R_2)$. Similarly, we say an agent i weakly prefers a delegation D_1 to another delegation D_2, whenever $P(i)$ is weakly closer to the most preferred delegate in D_1 than it is to the most preferred delegate in D_2, i.e., $\min\{\delta(P(i), R_1) \mid R_1 \in D_1\} \leq \min\{\delta(P(i), R_2) \mid R_2 \in D_2\}$. Strategy-proofness means that every agent weakly prefers the delegation they get under true preferences to any delegation they achieve by misreporting. In other words, there is no possibility of misreporting and getting a new delegate in the delegation which is closer to the agent's preference. Here, we take the closest delegate as the only relevant one for the agents, meaning agents do not care about the distance to other delegates.[14] We first discuss the usual individual strategy-proofness and afterward the coalitional version of it. In the sequel we shall only use the former. However, we show later in Proposition 2.2 that the latter is implied by the former under support neutrality.

Definition 4 (*Strategy-proofness*) A rule φ is *strategy-proof* whenever for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, for all $P \in \mathcal{L}(A)^n$ and for all $i \in N$, there exists no $P' = (P'(i), P(N \backslash \{i\})) \in \mathcal{L}(A)^n$ such that

$$\min_{R \in \varphi(P)} \delta(P(i), R) > \min_{R \in \varphi(P')} \delta(P(i), R).$$

Definition 5 (*Coalitional strategy-proofness*) A rule is *coalitional strategy-proof* whenever for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, for all $P \in \mathcal{L}(A)^n$ and for all coalitions $S \subseteq N$, there exists no $P' = (P'(S), P(N \backslash S)) \in \mathcal{L}(A)^n$ such that:

$$\min_{R \in \varphi(P)} \delta(P(i), R) > \min_{R \in \varphi(P')} \delta(P(i), R)$$
$$\text{for all } i \in S.$$

[14] Here we do not assume any negative externality in representativeness, i.e., agents only care about the delegate(s) that are closest to them in terms of representation. We are thankful to an anonymous referee pointing out that this formulation actually corresponds to the standard metric between two sets, where the singleton set $P(i)$ and the set of delegates D is compared. Note, however, that other methods, e.g., averaging the distances to set D, or taking the median preference in D would give perfectly valid but different scenarios of representation.

Remark 2.2 Note that our individual strategy-proofness concept is fundamentally different than that of Bossert and Storcken (1992) and of Athanasoglou (2016). We allow multiple preferences in the outcome (in the case of single-valued delegation rules they are equivalent). A recent paper by Bossert and Sprumont (2014) differs from the former two interpretations since the manipulation is based on a concept known as betweenness (see also Grandmont (1978), Kemeny (1959), and Sato (2013)). In their interpretation, an agent can benefit only when the outcome is manipulated to somewhere between herself and the preference corresponding to truth telling. In our interpretation agents can benefit when the outcome is manipulated to anywhere, resulting a closer preference. This makes the strategy-proofness we propose, ceteris paribus, stronger and harder to satisfy. We provide an example in Appendix A.1 which is strategy-proof in the sense of Bossert and Sprumont (2014), but not in the way we interpret it.

Next, we show that under support neutrality strategy-proofness implies coalitional strategy-proofness. We use this implication throughout the proofs.

Proposition 2.2 *If a rule φ is strategy-proof and support neutral, then it is also coalitional strategy-proof.*

Proof Let φ be a strategy-proof and support neutral rule. For any $A \subsetneq \mathcal{A}$, any $N \subsetneq \mathcal{N}$ and any $P \in \mathcal{L}(A)^n$, and for any $S \subseteq \{i \in N \mid P(i) \notin \varphi(P)\}$, let us denote any deviation from P by agents in S as $P' = (P'(S), P(N\backslash S))$. Let $W = \varphi(P)$ and $O = \mathcal{L}(A)\backslash\varphi(P)$ denote a partition of $\mathcal{L}(A)$.

By Remark 2.1, there exists an expansion of P by π, say $\bar{P}$, where $\bar{W} = \pi(W)$ and $\bar{O} = \pi(O)$ such that

$$\max_{\bar{R}, R' \in \bar{O}} \delta(\bar{R}, R') < \min_{\bar{R} \in \bar{O}, R' \in \bar{W}} \delta(\bar{R}, R'). \tag{2.2}$$

Consider any enumeration of $i \in S$, i.e. $S = \{1, 2, \ldots, s\}$. Let us construct expanded profiles, $\bar{P}_0, \bar{P}_1, \ldots, \bar{P}_s$, with $\bar{P}_0 = \bar{P}$, $\bar{P}_s = \bar{P}'$ (the expansion of P' by π, i.e., $\bar{P}' = (\bar{P}'(S), \bar{P}(N\backslash S))$, and for all $i \in \{1, 2, \ldots, s\}$, $\bar{P}_i = (\bar{P}'(\{1, 2, \ldots, i\}), \bar{P}(N\backslash\{1, 2, \ldots, i\}))$. This is a formalization of the idea that any deviation by a coalition can be constructed as a result of consecutive unilateral deviations by a sequence of agents.

By Proposition 2.1, $\varphi(\bar{P}_i) \subseteq RP(\bar{P}_i)$ for all $i \in \{1, 2, \ldots, s\}$. Note that from $\bar{P}_0$ to $\bar{P}_1$, there cannot be a preference $\bar{R} \in \bar{O}$ that becomes a new delegate for $\bar{P}_1$. This is because by Inequality 2.2, we have that for all $R' \in \bar{W}$, $\delta(\bar{R}, \bar{P}(1)) < \delta(R', \bar{P}(1))$ and this would contradict individual strategy-proofness. A similar argument holds from $\bar{P}_i$ to $\bar{P}_{i+1}$ for any $i \in \{1, 2, \ldots, s-1\}$. As the choice of enumeration of agents in S is arbitrary, eventually this implies that there exists no $\bar{R} \in \bar{O}$ such that $\bar{R} \in \varphi(\bar{P}_s)$. As $\bar{P}_s = \bar{P}'$, and $\bar{P}'$ is an expansion of P', then there exists no $R \in O$, such that $R \in \varphi(P')$, since there exists no $\bar{R} \in O$ with $\bar{R} \in \varphi(\bar{P}')$. Then $\varphi(P') \subseteq W = \varphi(P)$. As $\varphi(P')$ is a subset of $\varphi(P)$, this implies that no agent in S has become strictly better off, i.e.,

$$\text{There exists no } i \in S \text{ such that } \min_{R \in \varphi(P)} \delta(P(i), R) > \min_{R \in \varphi(P')} \delta(P(i), R).$$

Since all agents whose preferences are already included in the delegation, i.e, $P(i) \in W$ has distance of zero to the delegation, they will not have any incentive to deviate or to join a coalition. This means that there cannot be any coalition S which can successfully manipulate. □

3 Using thresholds for delegation rules

3.1 Threshold rules

In this section, we introduce a large class of delegation rules which we call *threshold rules*. Every threshold rule is associated with a particular *threshold function* which we introduce below. Thereafter we show that the rules are well-defined and provide some examples within this special class of delegation rules. Let $\mathbb{Z}_{++}$ denote the set of positive integers.

Definition 6 (*Threshold Function*) A *threshold function* is a function $f : \mathbb{Z}_{++} \to (\frac{1}{2}, 1]$ such that for all t:

$$f(t+1) \geq \frac{f(t)+1}{2}.$$

These functions simply assign a threshold for each possible delegation of size t. Let us introduce some additional notation to define the threshold rules. Given any $P \in \mathcal{L}(A)^n$, consider an enumeration which orders preferences according to their support from the agents from the strongest to weakest, i.e., $p_i \geq p_{i+1}$. For example, let $p = (0.5, 0.\bar{3}, 0.1\bar{6}, 0, 0, \ldots, 0)$ be the normalized support for P. Let us also denote the corresponding preferences as $R_1, R_2, \ldots, R_{|A|!}$ i.e., R_1 is the preference with the strongest support and so forth.[15] Then we can define the *cumulative support* ρ as the cumulative vector of p, i.e., for all i, $\rho_i = p_1 + \ldots + p_i$. For instance, the cumulative support for the aforementioned P is: $\rho = (0.5, 0.8\bar{3}, 1, 1, \ldots, 1)$.

We first introduce the threshold rules as an algorithm, then proceed with the formal definition.

[15] Note that some preferences in profiles might have equal support with a tie. In that case, the enumeration of those preferences can be chosen arbitrarily.

Take any profile P and the cumulative support for it as ρ. Consider any threshold function f. Let $R_1, R_2, \ldots R_{|A|!}$ denote an ordering of preferences according to their support, with ties broken arbitrarily.

Step 1: Check whether $\rho_1 \geq f(1)$. If yes, $\varphi^f(P) = \{R_1\}$ and the algorithm stops. Otherwise, go to the next step.

Step 2: Check whether $\rho_2 \geq f(2)$. If yes, $\varphi^f(P) = \{R_1, R_2\}$ and the algorithm stops. Otherwise, go to the next step.

$$\vdots$$

Step t: Check whether $\rho_t \geq f(t)$. If yes, $\varphi^f(P) = \{R_1, R_2, R_3, \ldots, R_t\}$ and the algorithm stops. Otherwise, go to the next step.

$$\vdots$$

Note that the algorithm stops after finite steps since we are dealing with a finite subset A of $\mathcal{A}$. Next, we propose the formal definition. Again, given any profile P, we use the enumeration $R_1, R_2, \ldots R_{|A|!}$ which orders according to the size of the support.

Definition 7 (*Threshold Rule*) Given a threshold function f, a *threshold rule corresponding to* f is defined for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ as

$$\varphi^f(P) = \{R_1, R_2, \ldots, R_{t^*}\}$$

where $t^* = \arg\min_t \{t \in \mathbb{Z}_{++} \mid \rho_t \geq f(t)\}$.

Thus the threshold rule selects the lowest number of delegates at which the corresponding threshold for total support is reached via the threshold function f. Note that an equivalent formulation for f in Definition 6 is as follows:

$$f(t+1) - f(t) \geq \frac{1}{2}(1 - f(t)). \tag{3.1}$$

Under this formulation, it is easy to see that the delegation is not complete until the last additional delegate joining the table actually accounts for representing at least half of the previously excluded opinions.[16] In addition, all the threshold functions obey

[16] We thank an anonymous referee for this alternative formulation and pointing out an interesting interpretation.

the following lower bounds:

$$f(t) > \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^t = \sum_{i=1}^{t} \left(\frac{1}{2}\right)^i . \tag{3.2}$$

On top of this common lower bound feature, each of the threshold rules we characterize differs in the amount of representation they require from a delegation. For instance, if a single delegate is sent by the rule to the table, then that delegate, and the opinion she represents must have strictly more than 50% support in the society. However, let us consider a more demanding threshold rule, such as one that requires a 60% for a single delegate representation. In that case, the minimal required support for a delegation of size 2 becomes at least 80% (averaging 60% and 100%). The thresholds for larger delegations quickly increase by averaging each threshold with 100% to find the next threshold, making it harder for small delegations to reach. Therefore, these rules are fairly inclusive in most of the cases. Essentially, the size of the delegations under the threshold rules depend on the diversity and the modality of the preferences in the society.

However, there are two immediate concerns about these delegations rules. The first is whether we can always find a delegation that exceeds the threshold. The second is what happens when the algorithm stops at t^*, where two preferences have equal support, i.e., $p_{t^*} = p_{t^*+1}$ and $R_{t^*} \in \varphi^f(P)$ but $R_{t^*+1} \notin \varphi^f(P)$. We address both concerns in Proposition 3.1 which shows that the rules are well-defined.

Proposition 3.1 *For all threshold functions f, the threshold rule φ^f is well-defined.*

Proof The proof is in Appendix B.2. □

3.2 Illustrations and comparison of exclusiveness

Next, we demonstrate with some examples how threshold rules assign delegates to different preference profiles. Thereafter, we propose a measure of social exclusion based on the threshold functions.

3.2.1 Some illustrations for threshold rules

Example 3.1 We will show four different delegation rules by their threshold rules in the case of three alternatives. Let us consider the three different preference profiles, P^1, P^2, and P^3 denoted below by the normalized support (on the left side) and the cumulative support (on the right side) for preferences. Note that for the sake of simplicity we use profiles with the same enumeration wherein the support for R_i is decreasing in i.

$$\begin{array}{ll}
p^1 = (0.31, 0.29, 0.29, 0.11, 0, 0) & \rho^1 = (0.31, 0.6, 0.89, 1, 1, 1) \\
p^2 = (0.78, 0.12, 0.1, 0, 0, 0) & \rho^2 = (0.78, 0.9, 1, 1, 1, 1) \\
p^3 = (0.55, 0.12, 0.11, 0.11, 0.11, 0) & \rho^3 = (0.55, 0.67, 0.78, 0.89, 1, 1)
\end{array}$$

The illustrations which are provided below capture the essence of threshold rules. Even though neither cumulative supports nor the relevant thresholds for each cardinality are continuous values, connecting discrete values via lines makes the visualization easier. In the following graphs, the first number of delegates that a cumulative support is above the corresponding threshold indicates the number of delegates in the delegation.

Throughout these examples, let us denote the threshold function as $f = (., \ldots, 1, \ldots)$, where the i^{th} entry corresponds to $f(i)$. Since this function is increasing and has the bound of 1, once the value of 1 is reached, all further values is equal to 1.

- The first rule, φ^1 is defined by the threshold function $f^1 = (0.51, 0.76, 0.89, 0.95, 1, \ldots)$. This rule checks whether the total support for some delegation reaches the relevant threshold for the size of the delegation, and if it does, picks that delegation with the smallest number of delegates.
- The second rule we deal with, φ^2 is characterized by the threshold vector, $f^2 = (0.51, 1, \ldots)$. This rule checks whether there exists any preference that is supported by more than at least 51% of the agents and if it is the case makes it the singleton delegate. If it is not the case, the rule picks all reported preferences instead.
- The third rule we deal with, φ^3 is characterized by the threshold vector, $f^3 = (0.\bar{6}, 1, \ldots)$. This rule checks whether there exists any preference that is supported by more than at least two-thirds of the agents (a.k.a. qualified majority), and if it is the case, then makes it the singleton delegate. If it is not the case, the rule picks all reported preferences instead.
- The last rule we deal with, φ^4 is the reported preference rule $RP(P)$, which chooses all preferences reported. The relevant threshold vector is $f^4 = (1, \ldots)$, i.e. the total support for any delegation should be at least 100%.

The illustrations of the rules and the delegations for each example profile are provided below. The bold numbers for ρ_i, indicates that R_i is a chosen delegate for the profile p under the rule φ^f.

As can be seen from Fig. 1, $\varphi^1(P^1) = \{R_1, R_2, R_3\}$, while $\varphi^1(P^2) = \varphi^1(P^3) = \{R_1\}$.

As can be seen from Fig. 2, $\varphi^2(P^1) = \{R_1, R_2, R_3, R_4\}$, while $\varphi^2(P^2) = \varphi^2(P^3) = \{R_1\}$.

As can be seen from Fig. 3, $\varphi^3(P^1) = \{R_1, R_2, R_3, R_4\}$, $\varphi^3(P^2) = \{R_1\}$, and $\varphi^3(P^3) = \{R_1, R_2, R_3, R_4, R_5\}$.

As can be seen from Fig. 4, $\varphi^4(P^1) = \{R_1, R_2, R_3, R_4\}$, $\varphi^4(P^2) = \{R_1, R_2, R_3\}$, and $\varphi^4(P^3) = \{R_1, R_2, R_3, R_4, R_5\}$.

3.2.2 A measure of exclusion

Given a threshold function f, as a measure of social exclusion for a threshold function can be considered as follows:

$$e(f) = \sum_t (1 - f(t))$$

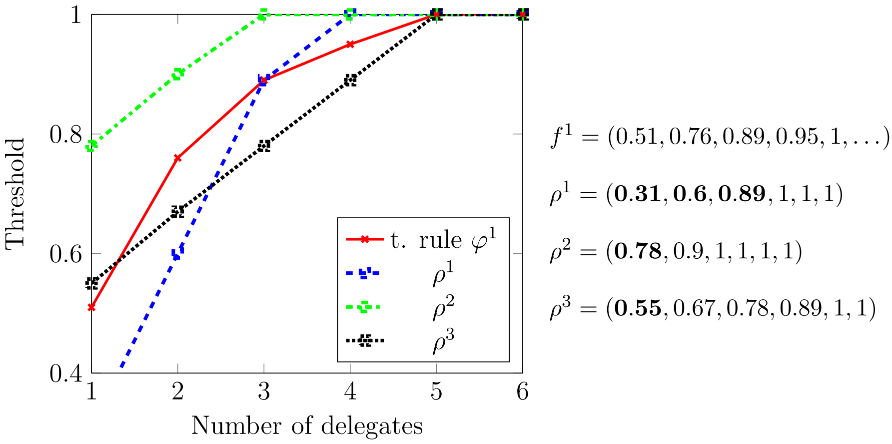


Fig. 1 Delegations chosen by threshold rule φ^1

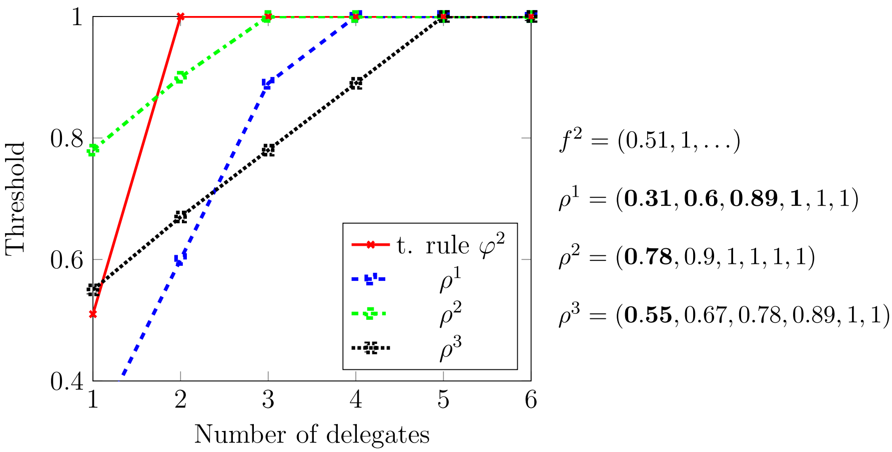


Fig. 2 Delegations chosen by threshold rule φ^2

The exclusion measure $e(f)$ gives a value of almost 1 for the most exclusive rule discussed as $f^1 = (0.51, 0.76, 0.89, 0.95, 0.98, 1, \ldots)$ and it is zero for the most inclusive rule $f^4 = (1, 1, \ldots)$. in Sect. 3. The resemblance to Lorenz curves under this setting is not superficial. Firstly, similar to Lorenz curves, these two thresholds lead to the two extreme values for the exclusion measure $e(f^1) = 1$ and $e(f^4) = 0$. Secondly, analogous to Lorenz domination, these two curves f^1 and f^4 in Fig. 5, serve as the lower and upper bounds for all the other threshold functions, e.g., f^2 and f^3 in Figs. 2 and 3 in Sect. 3.

If a threshold curve is above another, then it is more inclusive. Nevertheless, there are still interesting situations in terms of exclusion. These cases arise from intersecting threshold curves, similar to those in the inequality literature and Lorenz curves where Lorenz dominance does not apply (see, for instance, Davies and Hoy (1995) and Aaberge (2000)). Consider, for example, the following threshold functions which are

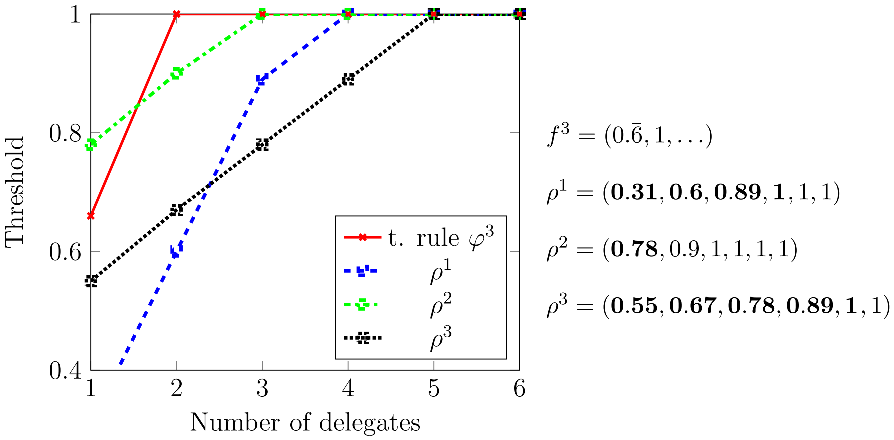


Fig. 3 Delegations chosen by threshold rule φ^3

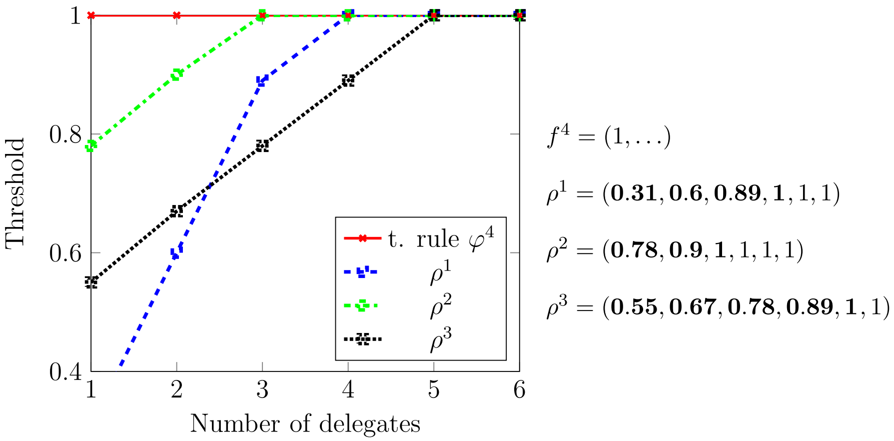
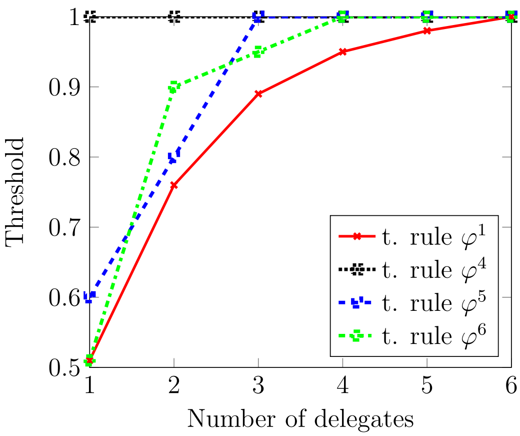


Fig. 4 Delegations chosen by threshold rule φ^4

also depicted in Fig. 5:

$$f^5 = (0.6, 0.8, 1, \dots)$$
$$f^6 = (0.51, 0.9, 0.95, 1, \dots)$$

The derived exclusion values for these functions are: $e(f^5) = 0.4 + 0.2 = 0.6$, and $e(f^6) = 0.49 + 0.1 + 0.05 = 0.064$. The exclusion measure e gives a higher value of exclusivity for f^6, hence f^5 is more inclusive. However, this does not mean that f^6 always has a lower number of delegates. As explained, the number of delegates (and inclusivity) depends also on the heterogeneity of the profile. Therefore depending on the composition of the preference profile of the society, the size of the delegation can play out either way. For example, consider two preference profiles $\rho^1 = (0.5, 0.3, 0.2)$ and $\rho^2 = (0.51, 0.49)$. Despite the higher exclusion value of f^6, for profile ρ^1, f^6



$$f^1 = (0.51, 0.76, 0.89, 0.95, 0.98, 1, \ldots)$$

$$f^4 = (1, \ldots)$$

$$f^5 = (0.6, 0.8, 1, \ldots)$$

$$f^6 = (0.51, 0.9, 0.95, 1, \ldots)$$

Fig. 5 Intersecting threshold curves

assigns 3 delegates while f^5 assigns only 2. Conversely, for profile ρ^2, f^6 has 1 delegate while f^5 has 2.

We are thankful to an anonymous referee for pointing out to this very interesting analogy to intersecting Lorenz curves. We believe it to be worthwhile to analyze exclusion measures in terms of expected number of delegates. An interesting open question to that end is whether with a higher exclusivity measure, one would have a lower expected number of delegates for different profile distributions.

4 Characterization of the threshold delegation rules

In this section, we show that the conditions of Pareto optimality, consistency, support neutrality, and strategy-proofness characterize the class of delegation rules which we explained in the previous section. We show that these conditions lead to some implications concerning the behavior of the delegation rules. The first four lemmas shape the structure of the rules concerning the support of delegates. Another four lemmas prove the existence of a series of critical thresholds for choosing delegates and set forth the structure of these thresholds. We conclude the section with our main theorem which states that the only rules satisfying the conditions we demand are the threshold delegation rules.

4.1 Delegates and their support in the society

In what follows, Lemma 4.1 shows that if a preference is chosen as a delegate, then any other preference with stronger support in the society should also be chosen. Lemma 4.2 argues that rules should only care about the percentage of the support, i.e., only the normalized support of preference profiles matter. Lemma 4.3 proves that i) equal redistribution of the total support for the delegates among themselves does not change the delegation, and ii) equal redistribution of the total support for the rest among themselves also does not modify the delegation.

Lemma 4.1 *If a rule φ satisfies consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$ if $R \in \varphi(P)$ and $p(R') \geq p(R)$, we have $R' \in \varphi(P)$.*

Proof The proof is in Appendix B.3. □

This lemma and Proposition 2.1 implies that any rule satisfying these conditions will have a delegation composed of preferences with relatively higher support compared to preferences that are not in the delegation. The following lemma proves the delegations to be the same for two different societies with identical normalized supports. Let $p/n = (\frac{p_1}{n}, \frac{p_2}{n}, \ldots, \frac{p_{|A|!}}{n})$ denote the normalized support by the number of agents.

Lemma 4.2 *If a rule φ satisfies consistency and support neutrality, then for all $A \subsetneq \mathcal{A}$, for all $N, N' \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ and $P' \in \mathcal{L}(A)^{n'}$ such that $p/n = p'/n'$, we have $\varphi(P) = \varphi(P')$.*

Proof The proof is in Appendix B.4. □

The following lemma proves that neither averaging between supports of chosen delegates nor averaging between supports of non-delegates will change the delegation.

Lemma 4.3 *If a rule φ satisfies consistency and support neutrality, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$, denoting $|\varphi(P)| = t$, and picking an enumeration on $\mathcal{L}(A)$ such that $p_i \geq p_j$ for all $i < j$, the following holds:*

(i) *For any $P' \in \mathcal{L}(A)^n$ such that $\frac{p'_j}{n} = \sum_{i=1}^{t} \frac{p_i}{nt}$ for all $j \in \{1, 2, \ldots, t\}$ and $\frac{p'_j}{n} = \frac{p_j}{n}$ for all $j \in \{t+1, t+2, \ldots, |A|!\}$ we have $\varphi(P) = \varphi(P')$.*
(ii) *For any $P'' \in \mathcal{L}(A)^n$ such that $\frac{p''_j}{n} = \frac{p_j}{n}$ for all $j \in \{1, 2, \ldots, t\}$ and $\frac{p''_j}{n} = \sum\limits_{i=t+1}^{|A|!} \frac{p_i}{n(|A|!-t)}$ for all $j \in \{t+1, t+2, \ldots, |A|!\}$ we have $\varphi(P) = \varphi(P'')$.*

Proof The proof is in Appendix B.5. □

Example 4.1 As an example for those two cases, let us take some A with $|A| = 3$. Let us take $P \in \mathcal{L}(A)^n$ with support $p = (\mathbf{8}, \mathbf{7}, \mathbf{6}, 3, 0, 0)$ where bold numbers indicate the support for the chosen delegates. As an example for two subcases of the Lemma 4.3, let us take $P', P'' \in \mathcal{L}(A)^n$ with supports $p' = (7, 7, 7, 3, 0, 0)$, $p'' = (8, 7, 6, 1, 1, 1)$ respectively. Then, Lemma 4.3 implies $\varphi(P) = \varphi(P') = \varphi(P'') = \{R_1, R_2, R_3\}$.

Remark 4.1 Using permutations and merging as in the proof of Lemma 4.3, it is straightforward to see that the lemma also applies to any subset of delegates or non-delegates. That is, averaging between supports of some subset of chosen delegates or some subset of non-delegates will not change the delegation.

In the next lemma, we show that if a preference is chosen as a delegate, it must have more support than the total support for all the preferences which are not in the delegation. This is mainly due to the strategy-proofness condition.

Lemma 4.4 *If a rule φ satisfies support neutrality and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ if $R \in \varphi(P)$, then we have*

$$p(R) > \sum_{R' \notin \varphi(P)} p(R').$$

Proof The proof is in Appendix B.6. □

4.2 When to choose a delegate, and when not to?

As seen in the previous four lemmas, any well-defined delegation rule satisfying the conditions of Pareto optimality, consistency, support neutrality, and strategy-proofness takes the most supported preference in the delegation. However, for this preference to be the only delegate, it has to be powerful enough to eliminate all the other opinions. We need a new tool to capture this. Take any φ which satisfies all the conditions. Categorize all $P \in \mathcal{L}(A)^n$ for any $N \subsetneq \mathcal{N}$ and $A \subsetneq \mathcal{A}$ according to the size of the delegations as follows: $\mathcal{P}_t = \{P \in \mathcal{L}(A)^n \mid N \subsetneq \mathcal{N}, A \subsetneq \mathcal{A} \text{ and } |\varphi(P)| = t\}$. Lemmas 4.1 and 4.2 imply that we only have to focus on the normalized support of the profiles from stronger to the weaker. Therefore, we can define a *corresponding vector* for this φ for any $A \subsetneq \mathcal{A}$ as

$$k^{\varphi}(A) = [k_1, k_2, \ldots, k_{|A|!}] \text{ ,where each } k_t = \min_{P \in \mathcal{P}_t} \left(\sum_{i=1}^{t} p_i \right) / n.$$

To ease the notation, we will omit φ from k^{φ} whenever it is clear. Furthermore, by support neutrality, we know that for these rules the vector k is the same for every A with equal cardinality. To understand these vectors, consider all profiles which end up with a single delegate under φ. Then k_1 gives the relative support of the delegate with minimal value, among all the profiles with a single delegation. Similarly, k_t gives the total relative support of the delegation with the minimal value, among all the profiles with a delegation of size t. In what follows, we discuss some features of these vectors.

Lemma 4.5 shows how k_t values relate to one another. Lemma 4.6 shows if a preference has more relative support than k_1 it has to be chosen uniquely. Lemma 4.7 shows how the choice of delegates depends on k in general. Finally, Lemma 4.8 shows how the vectors for sets of alternatives of different sizes relate to each other.

Lemma 4.5 *If a rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, and for all $N \subsetneq \mathcal{N}$, the corresponding vector satisfies that $k_t^{\varphi}(A) \geq \frac{k_{t-1}^{\varphi}(A)+1}{2}$ for all $t \in \{2, 3, \ldots, |A|!\}$.*

Proof The proof is in Appendix B.7. □

In the following two lemmas, Lemmas 4.6 and 4.7, we assume an enumeration which orders preferences according to their support from the agents from the strongest to weakest, i.e., $p_i \geq p_j$ for all $i < j$. Lemma 4.6 shows that for profiles in which a strongest single preference has a relative support p_1/n more than k_1, the delegation should only consist of this preference, R_1. Lemma 4.7 extends this to larger delegation sizes, i.e., the delegation should comprise of the first t strongest preferences whose relative total support surpasses their corresponding threshold k_t while no smaller sub-delegation satisfies this.

Lemma 4.6 *If a rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ such that $p_1 \geq nk_1^{\varphi}(A)$, we have that $\varphi(P) = \{R_1\}$.*

Proof The proof is in Appendix B.8. □

Lemma 4.7 *If a rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$ such that*

(i) *for some $t > 1$, $\sum_{i=1}^{t} p_i \geq nk_t^{\varphi}(A)$ and,*

(ii) *for all $l < t$, $\sum_{i=1}^{l} p_i < nk_l^{\varphi}(A)$*

we have: $\varphi(P) = \{R_1, R_2, \ldots, R_t\}$.

Proof The proof is in Appendix B.9. □

The next lemma shows that the corresponding vectors of the rules are independent of the number of alternatives.

Lemma 4.8 *If a rule φ satisfies support neutrality, then for all $A \subsetneq \bar{A} \subsetneq \mathcal{A}$, the corresponding vector satisfies that $k^{\varphi}(A)_t = k^{\varphi}(\bar{A})_t$ for all $t \in \{1, 2, \ldots, |A|!\}$.*

Proof The proof is in Appendix B.10. □

Note that Lemma 4.8 has further implications. In fact, for any two sets of alternatives, A, B the $k^{\varphi}(A)_t$ and $k^{\varphi}(B)_t$ values will always be the same. That can be achieved by extending each of the sets to $A \cup B$ by separately by implementing the lemma above.

Next, our main theorem finalizes the result by showing there is only one class of delegation rules, i.e., the threshold rules, associated with a threshold function that satisfies all the conditions we have imposed.

Theorem 4.1 *A rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness if and only if for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$ we have that $\varphi(P) = \varphi^f(P)$ for some threshold function f.*

Proof We delegate the "if" part to Appendix B.11 and prove the "only if" part here. Proposition 2.1 and Lemma 4.1 together imply that only preferences with higher support will be assigned as delegates as opposed to those with lower support. Lemma 4.2 implies that the degree of this relative power will be independent of the number of agents. Lemma 4.3 states that averaging of supports for delegates will not change the delegation. This implies that only the normalized supports for the delegations matter.

Given any rule φ satisfying the conditions, we can then construct a corresponding vector $k(A)$ where, again, the i^{th} entry is the minimum support needed for a delegation of size i across all preference profiles on A. By consistency, k is constant across all possible subsets of agents $N \subsetneq \mathcal{N}$, and by Lemma 4.8, $k(A)_i = k(B)_i$, i.e., k is also constant across all possible sets of alternatives $A \subsetneq \mathcal{A}$. Hence one can construct a function f on positive integers such that $f(i) = k_i$ for all possible delegations of size i. By Lemma 4.4, we have that $f(1) > 1/2$. By Lemma 4.5, for all $t \geq 1$ we have that

$$f(t+1) \geq \frac{f(t)+1}{2}.$$

Thus, by construction, $f : \mathbb{Z}_{++} \to (\frac{1}{2}, 1]$ is the unique threshold function induced by φ as in Definition 6, and hence $\varphi = \varphi^f$ is a threshold delegation rule. $\square$

5 Conclusion

This paper brings about a novel class of rules for choosing a delegation, characterized by intuitive fairness, efficiency, and non-manipulability properties. The nature of these threshold delegation rules is such that they provide a good compromise in at least three aspects, inclusivity, minimalism, and non-manipulability. Inclusivity is often deemed as crucial since it results in the legitimacy of the political settlement (Dudouet and Lundström 2016). Minimalism, in the sense that not everyone can be invited to the table, is an important parameter in the simplicity of design in conflict resolution. Finally, non-manipulability of a delegation rule is essential so that people's true opinions are always reflected in the conflict resolution, preventing further re-escalation of post-truce conflicts.

There are directions that we foresee for future research concerning the delegation choice. For instance, our selection of minimal Kemeny distance as a measure of representation of agents leads to a particular definition of strategy-proofness, which implies non-externality in representation. That is, we define a manipulation to be beneficial for an individual when the individual can successfully alter the "minimal distance" to the delegation. This interpretation implies that agents do not benefit or get harmed by other farther opinions in the delegation. Of course, one might define strategy-proofness as the inability of individuals to manipulate the "average distance to the delegation" or "distance to the farthest delegate" instead. Both these interpretations include externality in representation, i.e., introducing additional delegates might reduce the representation of agents.

Another direction is the choice of metric used to define strategy-proofness. Although Kemeny distance is quite standard in the literature, one might employ other metrics to see the robustness of these findings. As long as the clustering result in Remark 2.1 can be reached, we expect that our results can be generalized. We suspect that any metric that satisfies a minimal set of common features/axioms[17] will lead to similar results.

Concerning the limitations of the results, as discussed in the introduction, our investigation is more on the *deliberative* aspect of opinion formation than on the *aggregative* aspect. Therefore, our results are about representation of opinions for "deliberation", rather than power distribution of opinions for "aggregation". Hence, these rules do not provide an explanation to both sides of the coin in collective decision making. Nevertheless, our model can still be used to check whether the result of a parliamentary election can be supported by a fair, threshold rule based delegation rule. In this case, the agents are the citizens casting votes, and political parties represent the preferences like delegates. Since we only care whether the opinion of a party is considered, what matters is if a party makes it to the parliament or not. As long as a party is in, it does not matter how many representatives or how much power it has for deliberation. A party with non-negligible support can only end up with no seats in a legislature if there is a minimum electoral threshold as a percentage of votes to be reached. Without an electoral threshold, essentially, all parties make it to the parliament, and there is maximal inclusivity. If there is an electoral threshold, then depending on the votes, a certain number of parties make it to the parliament, and some minorities are not represented. In this case, depending on the number of parties, the most exclusive threshold rule specifies the maximum percentage of votes that can be neglected.

We conclude the paper by suggesting the policymakers that when a peace delegation, a committee or a board of size t is summoned from different opinions, for a fair and inclusive representation, the ratio of uninvited opinions should be less than 0.5^t of the population, e.g., if only 3 delegates are invited, then uninvited groups should account for less than 12.5% of the population. Similarly, in a parliament with six parties and a 5% electoral threshold, the total percentage of the votes for the parties below the electoral threshold should have in total less than $0.5^6 = 1.56\%$. This provides a minimal bound for inclusivity, furthermore it can help us to quantify "negligible minorities".

[17] We suspect one of these particular axioms to be the betweenness axiom introduced in Kemeny (1959)

A Examples

A.1 Example for showing the Kemeny rule is not strategy-proof

Example A.1 Given a profile $P \in \mathcal{L}(A)^n$, a preference R is a *Kemeny ranking* for P, if for all $R' \in \mathcal{L}(A)$, we have that $\sum_{i \in N} \delta(R, P(i)) \leq \sum_{i \in N} \delta(R', P(i))$. A rule which assigns all Kemeny rankings to each profile is called the Kemeny rule. More formally, the Kemeny rule, denoted by φ_K, assigns to a profile $P \in \mathcal{L}(A)^n$: $\varphi_K(P) = \{R \in \mathcal{L}(A) \mid R \text{ is a Kemeny ranking for } P\}$.

Our counterexample is just with 4 alternatives, and 11 agents. P is as follows:

$$\begin{array}{ccccccccccc} d & d & d & a & c & c & a & b & b & b & b \\ a & a & a & d & a & a & b & c & c & c & c \\ b & b & c & c & b & b & d & d & d & d & a \\ c & c & b & b & d & d & c & a & a & a & d \end{array}$$

It can be seen that $\varphi_K(P) = \{abcd\}$. $\delta(abcd, bcad) = 2$.

The last agent can manipulate to reach the following P':

$$\begin{array}{ccccccccccc} d & d & d & a & c & c & a & b & b & b & b \\ a & a & a & d & a & a & b & c & c & c & c \\ b & b & c & c & b & b & d & d & d & d & d \\ c & c & b & b & d & d & c & a & a & a & a \end{array}$$

It can be seen that $\varphi_K(P') = \{bcda\}$. $\delta(bcda, bcad) = 1$. So, the last agent is in a better position with reporting a false profile. So, even though both the Kemeny rule and our strategy-proofness condition is defined upon minimal Kemeny distance, we show that the Kemeny rule is not strategy-proof.

A.2 Example for a rule which is support neutral

Example A.2 Let us take two sets, A and $\bar{A}$ with $A \subsetneq \bar{A}$, $|A| = 3$ and $|\bar{A}| = 4$. Let us have 7 agents. For this example, we use the lexicographic enumeration where alternatives are ordered with their place in the alphabet, so $R_1 = abc$, $R_2 = acb$ and so on, and $\bar{R}_1 = abcd$, $\bar{R}_2 = abdc$ and so on.

Let us define $P \in \mathcal{L}(A)^7$ as

$$\begin{array}{ccccccc} a & a & a & a & a & b & b \\ b & b & b & c & c & a & c \\ c & c & c & b & b & c & a \end{array} \qquad \text{Here, } p = (3, 2, 1, 1, 0, 0).$$

Take some φ where $\varphi(P) = \{abc\} = \{R_1\}$. Next, we define $\bar{P} \in \mathcal{L}(\bar{A})^7$ as

$$\begin{array}{ccccccc} a & a & a & a & a & a & a \\ b & b & b & b & b & c & c \\ c & c & c & d & d & b & d \\ d & d & d & c & c & d & b \end{array} \qquad \text{Here, } \bar{p} = (3, 2, 1, 1, \underbrace{0\ 0 \ldots 0}_{\text{20 zeros.}}).$$

There is an injective function from A to $\bar{A}$ that satisfies our condition. For $\pi(t) = t$, we have $p_t = \bar{p}_{\pi(t)} = \bar{p}_t$ for all $t \in \{1, 2, \ldots, 6\}$. So, for φ to satisfy support neutrality, we must have $\varphi(\bar{P}) = \{\bar{R}_{\pi(1)}\} = \{\bar{R}_1\} = \{abcd\}$.

Or, since support neutrality is binding two ways, we can claim that if $\varphi(\bar{P}) = \{abcd\} = \{\bar{R}_{\pi(1)}\} = \{\bar{R}_1\}$, then we must have that $\varphi(P) = \{abc\} = \{R_1\}$.

A.3 Example for a profile in which support neutrality is used to increase coalitional options

Example A.3 Consider $A = \{a, b, c\}$, $N = 16$, and a profile P whose support is $p = (5, 4, 3, 2, 1, 1)$. If $\varphi(P) = \{R_1, R_2, R_3, R_4\}$, then agents whose preferences are not in the delegation have distance of 1 to the nearest delegate in all possible permutations. So, there is not a possible clustering, in which every agent strictly prefers some other non-delegate to the closest delegate for this particular alternative set. However, when we move to $\bar{A} = \{a, b, c, d\}$, it is easy to see that there is such a cluster with correct injection.

B Proofs

B.1 Proof of Proposition 2.1

Proposition 2.1 *If a rule φ satisfies Pareto optimality and support neutrality, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$, $\varphi(P) \subseteq RP(P)$.*

Proof Take some finite $A \subsetneq \mathcal{A}$ and $N \subsetneq \mathcal{N}$. Take some preference profile $P \in \mathcal{L}(A)^n$ with $RP(P) \subseteq \mathcal{L}(A)$. First, note that with equality, we are done. Suppose, for a contradiction, that there exists a preference $R \in \varphi(P)$ with $p(R) = 0$ that is, a preference with zero support. Consider any $x \in \mathcal{A} \backslash A$ and let us construct the expansion of P to $\bar{A} = A \cup x$ by $\bar{P} \in \mathcal{L}(\bar{A})^n$ as follows: for all $i \in N$, $\bar{P}(i) = P(i)||x$ where $P(i)||x = P(i) \cup (x, x) \cup \{(a, x) \mid a \in A\}$, i.e., concatenation of x with $P(i)$. Clearly, for any $a \in A$ and $i \in N$, $(a, x) \in \bar{P}(i)$. Take any $a \in A$, and $R^* \in \mathcal{L}(\bar{A})$ such that $(x, a) \in R^*$. By Pareto optimality, $R^* \notin \varphi(\bar{P})$. Note that $\bar{p}(R^*) = 0 = \bar{p}(R||x)$. Then by support neutrality $R||x \notin \varphi(\bar{P})$. As $\bar{P}$ is an expansion of P, again by support neutrality, we conclude that $R \notin \varphi(P)$. □

B.2 Proof of Proposition 3.1

Proposition 3.1 *For all threshold functions f, the threshold rule φ^f is well-defined.*

Proof Let us pick some threshold function f, and consider the corresponding threshold rule $\varphi^f(P)$. To show that the rule is well-defined, we need to show the following.

1. There is always a t that satisfies $\rho_t \geq f(t)$.

For any $p \in \mathcal{L}(A)^n$, $t = |A|!$, satisfies this. Since any reported preference should be within $\mathcal{L}(A)$, including all preferences in the delegation ensures to have cumulative support of 1. Since by definition any $f(t) \leq 1$, we have: $\rho_{|A|!} \geq f(|A|!)$. This shows that the universal set is always guaranteed to pass the relevant threshold.

2. There is always a unique way to pick the first t preferences.

Case 1: For all distinct i, j, we have $p_i \neq p_j$.

If this is the case, powers of the preferences are well ordered, there is a unique ordering enumeration, so there is always a unique way to pick the first t preferences.

Case 2: For some distinct i, j, we have $p_i = p_j$.

We will show that for any $p_i = p_j$ it is either $R_i, R_j \in \varphi^f(P)$ or $R_i, R_j \notin \varphi^f(P)$.

First, consider the case where the number of delegates is one. In that case, we must have $\rho_1 = \frac{p_1}{n} \geq f(1) > \frac{1}{2}$. This directly shows that any enumeration has the same preference as its first, whenever the first preference is passing the relevant threshold.

Second, consider the case when $t^* > 1$. By definition, for all $t > 1$ we have that

$$\begin{aligned}
&f(t) \geq \frac{1 + f(t-1)}{2}. \quad \text{Multiply both sides by 2 to get}\\
&2f(t) \geq 1 + f(t-1). \quad \text{Subtract } f(t) + f(t-1) \text{ from both sides to get}\\
&f(t) - f(t-1) \geq 1 - f(t).
\end{aligned} \tag{B.1}$$

From definition of the rule, we know the following is true for some t^*:

$$\rho_{t^*-1} < f(t^* - 1), \tag{B.2a}$$
$$\rho_{t^*} \geq f(t^*). \tag{B.2b}$$

Multiplying both sides of B.2b by -1 and adding 1 to both sides we get

$$1 - \rho_{t^*} \leq 1 - f(t^*). \tag{B.3}$$

Subtracting B.2a from B.2b leads to

$$\frac{p(t^*)}{n} > f(t^*) - f(t^* - 1). \tag{B.4}$$

If we combine B.3, B.4, and B.1 we get

$$\frac{p(t^*)}{n} > f(t^*) - f(t^* - 1) \geq 1 - f(t^*) \geq 1 - \rho_{t^*} = \sum_{i=t^*+1}^{|A|!} \frac{p_i}{n}. \tag{B.5}$$

The rightmost term is the total support for the preferences which are not part of the delegation, where the leftmost term is the support for the weakest delegate. This implies that not only the weakest delegate has strictly more support than the next preference, but he also has strictly more support than the total support for non-delegates. This says that there is again a unique way to select the topmost t preferences as delegates, even though the relevant enumeration is not unique this time. In other words, when there is any tie between support for some preferences, by design all of those preferences belong to the delegation or none of them. $\square$

B.3 Proof of Lemma 4.1

Lemma 4.1 *If a rule φ satisfies consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$ if $R \in \varphi(P)$ and $p(R') \geq p(R)$, we have $R' \in \varphi(P)$.*

Proof Suppose, for a contradiction, that there exists two preferences R^{h} (preference with (h)igher support), and R^{l} (preference with (l)ower support) with $R^{\mathrm{l}} \in \varphi(P)$ and $R^{\mathrm{h}} \notin \varphi(P)$. Let $p(R^{\mathrm{h}}) = h$ and $p(R^{\mathrm{l}}) = l$. Without loss of generality, we can assume that $h + l$ is even since by consistency we can replicate the profile once by using the two-fold replica with no changes in the delegation. This will ensure that $h - l$ can also be assumed to be even.

First, assume that $h - l = 0$. This contradicts support neutrality since $h = l$. Next, assume that $h - l = 2$. In that case, an agent whose original preference is R^{h} may misreport R^{l}. Denoting the modified profile by P', that will cause $p'(R^{\mathrm{h}}) = p'(R^{\mathrm{l}})$. From support neutrality, either both of R^{h} and R^{l} will be in the delegation, or none will be included. If both are included, this means that the agent deviated to his benefit, contradicting strategy-proofness. If none is included, some agent with R^{l} as his original preference may report R^{h} to get back to the original preference profile, resulting in $R^{\mathrm{l}} \in \varphi(P)$ again, this also contradicts strategy-proofness. So for $h-l = 2$, we showed that with the original preference profile, if R^{l} is included in the delegation, so must R^{h} be.

Assume that our hypothesis holds for $h-l = k$ for some even k, that is, if $R^{\mathrm{l}} \in \varphi(P)$, then $R^{\mathrm{h}} \in \varphi(P)$. Now, let $h - l = k + 2$. Then an agent whose original preference is R^{h} can report R^{l} to trigger the situation with $h - l = k$. Since this violates strategy-proofness, we must have $R^{\mathrm{h}} \in \varphi(P)$ even when $h - l = k + 2$. By induction, this completes the proof. $\square$

B.4 Proof of Lemma 4.2

Lemma 4.2 *If a rule φ satisfies consistency and support neutrality, then for all $A \subsetneq \mathcal{A}$, for all $N, N' \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ and $P' \in \mathcal{L}(A)^{n'}$ such that $p/n = p'/n'$, we have $\varphi(P) = \varphi(P')$.*

Proof From consistency, we know that $\varphi(P) = \varphi(2P) = \varphi(3P) = \ldots = \varphi(nP)$. So, $\varphi(P) = \varphi(n'P)$ and $\varphi(P') = \varphi(nP')$. Since $n'p = np'$, support neutrality implies that $\varphi(n'P) = \varphi(nP')$, completing the proof. $\square$

B.5 Proof of Lemma 4.3

Lemma 4.3 *If a rule φ satisfies consistency and support neutrality, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$, denoting $|\varphi(P)| = t$, and picking an enumeration on $\mathcal{L}(A)$ such that $p_i \geq p_j$ for all $i < j$, the following holds:*

i) *For any $P' \in \mathcal{L}(A)^n$ such that $\frac{p'_j}{n} = \sum_{i=1}^{t} \frac{p_i}{nt}$ for all $j \in \{1, 2, \ldots, t\}$ and $\frac{p'_j}{n} = \frac{p_j}{n}$ for all $j \in \{t+1, t+2, \ldots, |A|!\}$ we have $\varphi(P) = \varphi(P')$.*
ii) *For any $P'' \in \mathcal{L}(A)^n$ such that $\frac{p''_j}{n} = \frac{p_j}{n}$ for all $j \in \{1, 2, \ldots, t\}$ and $\frac{p''_j}{n} = \sum_{i=t+1}^{|A|!} \frac{p_i}{n(|A|!-t)}$ for all $j \in \{t+1, t+2, \ldots, |A|!\}$ we have $\varphi(P) = \varphi(P'')$.*

Proof i) Let P and $p = (p_1, p_2, \ldots, p_{|A|!})$ be as in the Lemma with $|\varphi(P)| = t$ and P' as defined in the Lemma. Consider the following profiles with the same enumeration on $\mathcal{L}(A)$ where $p_1, p_2, \ldots, p_t$ rotates and bold numbers indicate the support for the chosen delegates:

$$\begin{aligned}
P^1 &\in \mathcal{L}(A)^n \text{ such that } p^1 = (\mathbf{p_t}, \mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_{t-1}}, p_{t+1}, p_{t+2}, \ldots, p_{|A|!})\\
P^2 &\in \mathcal{L}(A)^n \text{ such that } p^2 = (\mathbf{p_{t-1}}, \mathbf{p_t}, \mathbf{p_1}, \ldots, \mathbf{p_{t-2}}, p_{t+1}, p_{t+2}, \ldots, p_{|A|!})\\
&\vdots\\
P^{t-1} &\in \mathcal{L}(A)^n \text{ such that } p^{t-1} = (\mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}, \ldots, \mathbf{p_1}, p_{t+1}, p_{t+2}, \ldots, p_{|A|!})
\end{aligned}$$

From support neutrality, we know that $\varphi(P) = \varphi(P^i)$ for any $i \in \{1, \ldots, t-1\}$. By design, merging all these profiles $(P, P^1, P^2, \ldots, P^{t-1})$ gives tP', and from consistency, we get that $\varphi(tP') = \varphi(P)$. From Lemma 4.2, $\varphi(P) = \varphi(tP') = \varphi(P')$ is guaranteed.

ii) Let P and $p = (p_1, p_2, \ldots, p_{|A|!})$ be as in the Lemma with $|\varphi(P)| = t$ and P'' as defined in the Lemma. Consider the following profiles with the same enumeration on $\mathcal{L}(A)$ where $p_{t+1}, p_{t+2}, \ldots, p_{|A|!}$ rotates and bold numbers indicate the support for the chosen delegates:

$$\begin{aligned}
P^1 &\in \mathcal{L}(A)^n \text{ such that } p^1 = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_t}, p_{|A|!}, p_{t+1}, p_{t+2}, \ldots, p_{|A|!-1})\\
P^2 &\in \mathcal{L}(A)^n \text{ such that } p^2 = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_t}, p_{|A|!-2}, p_{|A|!}, p_{t+1}, \ldots, p_{|A|!-2})\\
&\vdots\\
P^{t-1} &\in \mathcal{L}(A)^n \text{ such that } p^{t-1} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_t}, p_{t+2}, p_{t+3}, p_{t+4}, \ldots, p_{t+1})
\end{aligned}$$

From support neutrality, we know that $\varphi(P) = \varphi(P^i)$ for any $i \in \{1, \ldots, t-1\}$. By design, merging all these profiles $(P, P^1, P^2, \ldots, P^{t-1})$ gives tP'', and from consistency we get that $\varphi(tP'') = \varphi(P)$. From Lemma 4.2, $\varphi(P) = \varphi(tP'') = \varphi(P'')$ is guaranteed.

□

B.6 Proof of Lemma 4.4

Lemma 4.4 *If a rule φ satisfies support neutrality and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ if $R \in \varphi(P)$, then we have*

$$p(R) > \sum_{R' \notin \varphi(P)} p(R').$$

Proof Let us denote by $W = \varphi(P)$ the preferences of "winning" agents, and by $L = RP(P) \backslash \varphi(P)$ the preferences of "losing" agents, where $RP(P) = \{R \in \mathcal{L}(A) \mid p(R) > 0\}$ is again the set of reported preferences. Suppose, for a contradiction, that there exists a profile $P \in \mathcal{L}(A)^n$ and a preference $R_t \in \varphi(P)$ such that:

$$p(R_t) \leq \sum_{\bar{R} \in L} p(\bar{R}).$$

That is, a preference R_t in the delegation has weakly less support than the total support for all preferences of losing agents combined. By Remark 2.1, there exists an expansion of P, in which the corresponding injections of W and L are clustered far away from each other. By the same logic, one can find an expansion by π, say $\bar{P}$, if needed to an even larger alternative set, in which in addition to having $\pi(W)$ and $\pi(L)$ as far away clusters, $R_{\pi(t)}$ is relatively closer to $\pi(L)$ than to the other preferences in $\pi(W)$. Formally:

$$\max_{R, R' \in \pi(L)} \delta(R, R') < \min_{R \in \pi(L), R' \in \pi(W)} \delta(R, R') \tag{B.6}$$

and for any $R, R' \in \pi(L)$, $\tilde{R} \in \pi(W)$,

$$\delta(R, R') < \delta(R, R_{\pi(t)}) < \delta(R, \tilde{R}) \tag{B.7}$$

Now consider a transformation of this expansion, denoted by $\bar{P}'$, where all losing agents concentrate on a preference of a fellow losing agent, say R^s. By construction:

$$\sum_{R \in L} p(R) = \bar{p}'(R^s)$$

By supposition, $\bar{p}(R_{\pi(t)}) \leq \bar{p}(R^s)$. Note that by coalitional strategy-proofness, we have $R^s \notin \varphi(\bar{P}')$. Then we have two cases:

Case 1: If $R_{\pi(t)} \in \varphi(\bar{P}')$, then by Lemma 4.1, R^s should also be in the delegation $\varphi(\bar{P}')$, which is a contradiction.

Case 2: If $R_{\pi(t)} \notin \varphi(\bar{P}')$, furthermore by Inequality B.7, $R_{\pi(t)}$ is a favorable preference for all agents with $\bar{P}(i) \in \pi(L)$. Then the agents in L can misreport (and disperse back to their preferences in $\bar{P}$). As $R_{\pi(t)} \in \varphi(\bar{P})$, this contradicts strategy-proofness. □

B.7 Proof of Lemma 4.5

Lemma 4.5 *If a rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, and for all $N \subsetneq \mathcal{N}$, the corresponding vector satisfies that $k_t^{\varphi}(A) \geq \frac{k_{t-1}^{\varphi}(A)+1}{2}$ for all $t \in \{2, 3, \ldots, |A|!\}$.*

Proof Let us start with some $A \subsetneq \mathcal{A}$ and some $t < |A|! - 1$. Using consistency, we can pick some $N \subsetneq \mathcal{N}$ with $|N| = n$ divisible by all numbers up to $t + 1$ without loss of generality. Let us take a specific $P \in \mathcal{L}(A)^n$, which is defined as $p_i = a$ for all $i \in \{1, 2, \ldots, t + 1\}$ and $p_i = 0$ for $i \in \{t + 2, t + 3, \ldots, |A|!\}$. From Proposition 2.1 and support neutrality, we know that the delegation includes only the first $t + 1$ preferences. Let us denote this profile as (with bold numbers indicating the support for the chosen delegates)

$$p = (\underbrace{\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}}_{t+1 \text{ times}}, \underbrace{0, 0, \ldots, 0}_{|A|!-t-1 \text{ times}}).$$

Now, let us deal with a modification $P' \in \mathcal{L}(A)^n$ for this profile which is defined as $p'_i = a' = \frac{nk_t}{t}$ for all $i \in \{1, 2, \ldots, t\}$, $p'_{t+1} = b = n(1 - k_t)$ and $p'_i = 0$ for $i \in \{t + 2, t + 3, \ldots, |A|!\}$. Since the first t preferences have k_t support in total, by definition of k_t and by Lemma 4.3, we know that the delegation includes only the first t preferences. Let us denote this profile as

$$p' = (\underbrace{\mathbf{a}', \mathbf{a}', \ldots, \mathbf{a}'}_{t \text{ times}}, b, \underbrace{0, 0, \ldots, 0}_{|A|!-t-1 \text{ times}}).$$

Another relevant modification of this profile, $P'' \in \mathcal{L}(A)^n$ will be defined as $p''_i = a'' = \frac{n-2b}{t-1}$ for all $i \in \{1, 2, \ldots, t\}$, $p''_i = b = n(1 - k_t)$ for $i \in \{t, t + 1\}$ and $p''_i = 0$ for $i \in \{t + 2, t + 3, \ldots, |A|!\}$, e.g. $p'' = (a'', a'', \ldots, a'', b, b, 0, 0, \ldots, 0)$. By Proposition 2.1 we have that $\varphi(P'') \subseteq RP(P)$, and by support neutrality either i) $\varphi(P'') = \{R_1, R_2, \ldots, R_{t+1}\}$ or ii) $\varphi(P'') = \{R_1, R_2, \ldots, R_{t-1}\}$. Suppose, for a contradiction, that the former is the case. Let us take the average of support for the first t preferences to get P'. By Remark 4.1, this should not change the delegation. However, $\varphi(P') = \{R_1, R_2, \ldots, R_t\} \neq \{R_1, R_2, \ldots, R_{t+1}\} = \varphi(P'')$, which is a contradiction. So, it must be that ii) is the case, $\varphi(P'') = \{R_1, R_2, \ldots, R_{t-1}\}$. Let us denote this profile as

$$p'' = (\underbrace{\mathbf{a}'', \mathbf{a}'', \ldots, \mathbf{a}''}_{t-1 \text{ times}}, b, b, \underbrace{0, 0, \ldots, 0}_{|A|!-t-1 \text{ times}}).$$

By definition, k_{t-1} is the minimal support for all delegations with size $t - 1$. Since only the first $t - 1$ preferences are in the delegation, total support for the first $t - 1$ preferences could be at least k_{t-1}. Then, $(t-1)a'' = (t-1)\frac{n-2b}{t-1} = n(1-2(1-k_t)) \geq$

nk_{t-1}. After rearranging we get that

$$k_t \geq \frac{k_{t-1}+1}{2}.$$

$\square$

B.8 Proof of Lemma 4.6

Lemma 4.6 *If a rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$, and for all $P \in \mathcal{L}(A)^n$ such that $p_1 \geq nk_1^{\varphi}(A)$, we have that $\varphi(P) = \{R_1\}$.*

Proof Let $P^* \in \mathcal{L}(A)^{n^*}$ be one of the profiles where $|\varphi(P^*)| = 1$ and $p_1^* = n^* k_1^{\varphi}(A)$, i.e., one of the profiles wherein only a single delegate is assigned whose relative support defines k_1 in the corresponding vector. Consider now any $P \in \mathcal{L}(A)^n$ with $p_1 \geq nk_1^{\varphi}(A)$. By Lemma 4.1, $R_1 \in \varphi(P)$ and by support neutrality we can assume that $\varphi(P^*) = \{R_1\}$, i.e., the strongest ranking is the same both in P and P^*. Next, we show that R_1 is the only delegate assigned to P, i.e., $\{R_1\} = \varphi(P)$.

By consistency, we can replicate profiles P and P^* (n^* and n times respectively) with no changes in the delegation. With abuse of notation, let us denote these replicated profiles by P, $P^* \in \mathcal{L}(A)^{n \times n^*}$. So, we have $p_1^* = nn^* k_1^{\varphi}(A)$ and $p_1 \geq nn^* k_1^{\varphi}(A)$.

Suppose, for a contradiction, that $\varphi(P) \supsetneq \{R_1\}$, so there is another delegate, say R_k in the delegation. Let us partition $\mathcal{L}(A)$ into two sets, $X = \mathcal{L}(A) \backslash \{R_1\}$ and $Y = \{R_1\}$. By Remark 2.1, there exists an expansion of P by π, say $\bar{P}$, in which the injection of X, i.e., $\pi(X)$ is clustered far away from the injection of R_1, i.e., $R_{\pi(1)}$. Formally:

$$\max_{R, R' \in \pi(X)} \delta(R, R') < \min_{R \in \pi(X)} \delta(R, R_{\pi(1)}) \tag{B.8}$$

Note that $R_{\pi(1)} = \pi(Y)$ and $R_{\pi(k)} \in \pi(X)$. Let $\bar{P}^*$ denote the expansion of P^* by the same injection, π. By support neutrality, i) $R_{\pi(k)} \notin \varphi(\bar{P}^*)$, implying k^{th} strongest preference of $\bar{P}^*$ is not in the delegation of $\bar{P}^*$, and ii) $R_{\pi(k)} \in \varphi(\bar{P})$, implying that the k^{th} strongest preference of $\bar{P}$ is in the delegation of $\bar{P}$. Note that as $\bar{p}_1^* \leq \bar{p}_1$, from $\bar{P}^*$ to $\bar{P}$ this means that there is a coalition of agents moving from $\pi(X)$ to $R_{\pi(1)}$, resulting in $R_\pi(k) \in \varphi(\bar{P})$. As $R_{\pi(k)} \in \pi(X)$, by Inequality B.8, this contradicts coalitional strategy-proofness. Hence $R_{\pi(k)} \notin \varphi(\bar{P})$. Support neutrality then implies that $R_k \notin \varphi(P)$. $\square$

B.9 Proof of Lemma 4.7

Lemma 4.7 *If a rule φ satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness, then for all $A \subsetneq \mathcal{A}$, for all $N \subsetneq \mathcal{N}$ and for all $P \in \mathcal{L}(A)^n$ such that*

(i) *for some $t > 1$, $\sum_{i=1}^{t} p_i \geq nk_t^{\varphi}(A)$ and,*

(ii) *for all* $l < t$, $\sum_{i=1}^{l} p_i < nk_l^{\varphi}(A)$

we have: $\varphi(P) = \{R_1, R_2, \ldots, R_t\}$.

Proof Take any $P \in \mathcal{L}(A)^n$ as defined in the lemma. As $\sum_{i=1}^{l} p_i < nk_l$ for all $l < t$, by definiton of the corresponding vector, we have

$|\varphi(P)| \neq l$. This means that $|\varphi(P)| \geq t$. By Lemma 4.1 we get that $\varphi(P) \supseteq \{R_1, R_2, \ldots, R_t\}$. Next, we show that $\varphi(P) = \{R_1, R_2, \ldots, R_t\}$.

By consistency, we can assume that $\sum_{i=1}^{t} p_i$ is divisible by t without loss of generality. By Remark 4.1, we can take the average support for the first t preferences without changing the delegation. Let us denote this modified profile by P', which is defined as $p_i' = a' = \sum_{l=1}^{t} \frac{p_l}{t}$ if $i \leq t$ and $p_i' = p_i$ if $i > t$.

Let $P^* \in \mathcal{L}(A)^{n^*}$ be one of the profiles where $|\varphi(P^*)| = t$, $\sum_{i=1}^{t} p_i^* = n^* k_t$, i.e., one of the profiles wherein only the strongest t delegates are assigned whose relative total support defines k_t in the corresponding vector.

By consistency, we can assume that $\sum_{i=1}^{t} p_i^*$ is divisible by t without loss of generality. By Remark 4.1, we can take the average support for the first t preferences without changing the delegation. Let us denote this modified profile also by P^*, where $p_i^* = a = \frac{n^* k_t}{t}$ for all $i \leq t$.

Using consistency, we can replicate profiles P' and P^* (n^* and n times respectively) with no changes in the delegation. With abuse of notation, let us denote these replicated profiles by $P', P^* \in \mathcal{L}(A)^{n \times n^*}$. By construction, the total support for the strongest t preferences in P' is larger than those in P^*, i.e., $n^* a' t \geq nat$.

Suppose, for a contradiction, that $\varphi(P') \supsetneq \{R_1, R_2, \ldots, R_t\}$ so there is another delegate, say R_k with $k > t$ in the delegation. Let us partition $\mathcal{L}(A)$ into two sets, $X = \mathcal{L}(A) \backslash \{R_1, R_2, \ldots, R_t\}$ and $Y = \{R_1, R_2, \ldots, R_t\}$. By Remark 2.1, there exists an expansion of P' by π, say $\bar{P}'$, in which the injection of Y is clustered far away from the injection of X. Formally:

$$\max_{R, R' \in \pi(X)} \delta(R, R') < \min_{R \in \pi(X), R' \in \pi(Y)} \delta(R, R') \tag{B.9}$$

Note that $\{R_{\pi(1)}, R_{\pi(2)}, \ldots, R_{\pi(t)}\} = \pi(Y)$ and $R_{\pi(k)} \in \pi(X)$. Let $\bar{P}^*$ denote the expansion of P^* by the same injection, π. By support neutrality, $R_{\pi(k)} \notin \varphi(\bar{P}^*)$ while $R_{\pi(k)} \in \varphi(\bar{P}')$. Note that from $\bar{P}^*$ to $\bar{P}'$ there is a coalition of agents moving from $\pi(X)$ to $\pi(Y)$, resulting in $R_\pi(k) \in \varphi(\bar{P}')$. As $R_{\pi(k)} \in \pi(X)$, by Inequality B.9, this contradicts coalitional strategy-proofness. □

B.10 Proof of Lemma 4.8

Lemma 4.8 *If a rule* φ *satisfies support neutrality, then for all* $A \subsetneq \bar{A} \subsetneq \mathcal{A}$, *the corresponding vector satisfies that* $k^{\varphi}(A)_t = k^{\varphi}(\bar{A})_t$ *for all* $t \in \{1, 2, \ldots, |A|!\}$.

Proof We denote corresponding threshold vectors as

$$k^{\varphi}(A) = [k_1, k_2, \ldots, k_{|A|!}],$$
$$k^{\varphi}(\bar{A}) = [k'_1, k'_2, \ldots, k'_{|\bar{A}|!}].$$

Suppose, for a contradiction, that there exist $A, \bar{A} \subsetneq \mathcal{A}$ such that $A \subsetneq \bar{A}$ with for some $t \in \{1, 2, \ldots, |A|!\}$, $k_t \neq k'_t$ while for all $i \in \{1, 2, \ldots, t-1\}$, $k_i = k'_i$. Without loss of generality, assume that $k_t < k'_t$.

Then we construct a profile $P \in \mathcal{L}(A)^n$ and an expansion of P, by some π, denoted by $\bar{P}$ such that $\rho_t > k_t$, $\rho_t < k'_t$ and $\rho_i < k_i$ for all $i \in \{1, 2, \ldots, t-1\}$.

By Lemmas 4.6 and 4.7 we have that $\varphi(P) = \{R_1, R_2, \ldots, R_t\}$, while $\varphi(P') \supsetneq \{R_{\pi(1)}, R_{\pi(2),\ldots,R_{\pi(t)}}\}$. Since $\varphi(P')$ includes other elements than the image of $\varphi(P)$ under π, this contradicts support neutrality. □

B.11 Proof of Theorem 4.1 (if part)

We will prove that for all threshold functions f, the rule φ^f satisfies Pareto optimality, consistency, support neutrality, and strategy-proofness. Take any threshold function f.

Pareto optimality: Take any $A \subsetneq \mathcal{A}$, $N \subsetneq \mathcal{N}$, and $P \in \mathcal{L}(A)^n$. Suppose for a contradiction, there exists (a, b) such that $(a, b) \in P(i)$ for all $i \in N$ and there exists $R \in \varphi^f(P)$ such that $(b, a) \in R$. Consider an enumeration of rankings with respect to their support, i.e., for any two ranking R_i, R_j, $i \geq j$ if and only if $p_i \geq p_j$. Suppose $R = R_k$ for some $k > 1$. Note that $p_k = 0$. Consider the cumulative support $\rho_k = p_1 + \ldots + p_{k-1} + p_k$. Note that $\rho_k = \rho_{k-1}$. By construction $\varphi^f(P) = \{R_1, R_2, \ldots, R_{t^*}\}$ where $t^* = \arg\min_t\{t \in \mathbb{Z}_{++} \mid \rho_t \geq f(t)\}$. However, as $f(k-1) \leq f(k)$ and $\rho_{k-1} \geq f(k-1)$, it follows that $t^* < k$, which contradicts $R_k \in \varphi^f(P)$.

Consistency: Take any $A \subsetneq \mathcal{A}$, any two disjoint finite sets $N, N' \subsetneq \mathcal{N}$ (with cardinality n and n' respectively), and any $P \in \mathcal{L}(A)^n$ and $P' \in \mathcal{L}(A)^{n'}$ such that $\varphi^f(P) = \varphi^f(P')$. Let k denote the number of delegates in the two distinct societies, that is $|\varphi^f(P)| = |\varphi^f(P')| = k$. For the two profiles P and P', it follows from the definition of threshold rule that $\rho_k \geq f(k)$ and $\rho'_k \geq f(k)$, and for all $l \in \{1, \ldots, k-1\}$ we have that $\rho_l < f(l)$ and $\rho'_l < f(l)$. Merging the two profiles into $P'' = (P, P')$ will result in a new cumulative support $\rho''_i = (n\rho_i + n'\rho'_i)/(n+n')$ for all $i = 1, 2, \ldots, |A|!$. This ensures $\rho''_k \geq f(k)$ and $\rho''_l < f(l)$ for all $l \in \{1, \ldots, k-1\}$, implying the same delegation for the merged societies.

Support neutrality: Take any $A, \bar{A} \subsetneq \mathcal{A}$ such that $A \subseteq \bar{A}$ and any $N \subsetneq \mathcal{N}$. Take any two profiles, $P \in \mathcal{L}(A)^n$ and $\bar{P} \in \mathcal{L}(\bar{A})^n$ with equivalent supports and take any corresponding injection π. Consider enumerations of rankings with respect to their support in P and in $\bar{P}$. By construction for R_i and $\bar{R}_{\pi(i)}$ the supports in the respective profiles are equal i.e., $p_i = \bar{p}_{\pi(i)}$. This implies that the cumulative supports ρ_i and $\bar{\rho}_i$ are identical for all i. Therefore $R_i \in \varphi^f(P)$ if and only if $\bar{R}_{\pi(i)} \in \varphi^f(\bar{P})$ since shuffling the support for the preferences also shuffles the enumerations in the same way.

Strategy-proofness: Take any $A \subsetneq \mathcal{A}$, and any $N \subsetneq \mathcal{N}$. Suppose for contradiction there exists a profile $P \in \mathcal{L}(A)^n$ which is manipulable by some agent $i \in N$ with a profile $P' = (P'(i), P(N\backslash\{i\})) \in \mathcal{L}(A)^n$, i.e., $\min_{R \in \varphi^f(P)} \delta(P(i), R) > \min_{R \in \varphi^f(P')} \delta(P(i), R)$. Let $\varphi^f(P) = \{R_1, R_2, \ldots, R_{t^*}\}$ be the set of delegates for P in the order of support. Consider the steps of the threshold rule φ^f for the profile P, and let ρ denote the corresponding cumulative support. Then, we have: $\rho_j < f(j)$ for all $j \in \{1, 2, t^* - 1\}$ and $\rho_{t^*} \geq f(t^*)$, since the rule ends at t^*.

Case 1: Suppose that the manipulation is done via reporting a preference in the delegation, i.e., $P'(i) \in \{R_1, R_2, \ldots, R_{t^*}\}$. Without loss of generality let us assume $P'(i) = R_s$ such that $s^* \leq t^*$. Then, for the new profile P' let ρ' denote the corresponding cumulative support. Then we have either of the two:

1. $\rho'_j < f(j)$ for all $j \in \{1, 2, t^* - 1\}$ and $\rho'_{t^*} \geq f(t^*)$, or
2. $\rho_j < f(j)$ for all $j \in \{1, 2, s^* - 1\}$ and $\rho_{s^*} \geq f(s^*)$

In both cases, $\varphi^f(P') \subseteq \varphi^f(P)$ which doesn't decrease $\min_{R \in \varphi^f(P')} \delta(P(i), R)$.

Case 2: Suppose that the manipulation is done via reporting a preference outside the delegation, i.e., $P'(i) \notin \{R_1, R_2, \ldots, R_{t^*}\}$. Note that the threshold rules are well-defined and therefore by Inequality B.5 in the proof of Proposition 3.1, the relative support for the last (weakest) delegate t^* in profile P is larger than the sum of relative supports for all preferences outside the delegation. Formally:

$$\frac{p(t^*)}{n} > \sum_{i=t^*+1}^{|A|!} \frac{p_i}{n}.$$

Note that the preference of i was not in the original delegation, $P(i) \notin \{R_1, R_2, \ldots, R_{t^*}\}$, otherwise $\min_{R \in \varphi^f(P)} \delta(P(i), R)$ would be zero. Therefore for the new profile P', after misreporting a preference outside the delegation, we still have:

$$\frac{p'(t^*)}{n} = \frac{p(t^*)}{n} > \sum_{i=t^*+1}^{|A|!} \frac{p_i}{n} = \sum_{i=t^*+1}^{|A|!} \frac{p'_i}{n}.$$

Therefore, $\varphi^f(P') = \varphi^f(P)$, which doesn't decrease $\min_{R \in \varphi^f(P')} \delta(P(i), R)$.

C Independence of the conditions

The conditions used in the characterization were: Pareto optimality, consistency, strategy-proofness, and support neutrality. Below, to put forward the logical independence of those, let us take a look at the following four social welfare correspondences.

- **All but Pareto optimality**: $\varphi(P) = \mathcal{L}(A)$ for any $P \in \mathcal{L}(A)^n$.

- **All but consistency**:

$$\varphi(P)=\begin{cases} R, & \text{if } n \text{ is odd and } \exists R \text{ with } p(R)>\frac{N}{2}\\ R, & \text{if } n \text{ is even and } \exists R \text{ with } p(R)>\frac{2N}{3}\\ RP(P), & \text{otherwise.}\end{cases}$$

- **All but strategy-proofness**: $\varphi(P)=\{R \mid p(R)\geq p(R') \text{ for all } R'\in\mathcal{L}(A)\}$.
- **All but support neutrality**:

$$\varphi(P)=\begin{cases} \mathcal{L}(A), & \text{if } |\mathcal{L}(A)\backslash RP(P)|=1\\ RP(P), & \text{otherwise.}\end{cases}$$

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