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The Abel map for surface singularities II. Generic analytic structure ☆

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ABSTRACT

We study the analytic and topological invariants associated with complex normal surface singularities. Our goal is to provide topological formulae for several discrete analytic invariants whenever the analytic structure is generic (with respect to a fixed topological type), under the condition that the link is a rational homology sphere. The list of analytic invariants includes: the geometric genus, the cohomology of certain natural line bundles, the cohomology of their restrictions on effective cycles (supported on the exceptional curve of a resolution), the cohomological cycle of natural line bundles, the multivariable Hilbert and Poincaré series associated with the divisorial filtration, the analytic semi-group, the maximal ideal cycle.
The first part contains the definition of 'generic structure' based on the work of Laufer [14]. The second technical ingredient is the Abel map developed in [21].
The results can be compared with certain parallel statements from the Brill–Noether theory and from the theory of Abel

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map associated with projective smooth curves (see e.g. [1] and [6]), though the tools and machineries are very different.

1. Introduction

1.1. The main objective. Our major objects in this note are the analytic and topological invariants associated with complex normal surface singularity germs. Our goal is to provide topological formulae for several discrete analytic invariants whenever the analytic structure is *generic* (with respect to a fixed topological type). Regarding this problem very little is known in the present literature. The type of formulae of the topological characterizations of the present article are totally new, as well as the methods (based on the newly created theory of Abel map).

1.2. Discussion regarding the 'generic analytic type'. Let us comment first what kind of difficulties appear in the definition and study of 'generic' analytic type. The point is that for a fixed topological type the moduli space of all analytic structures supported by that fixed topological type, is not yet described in the literature; hence, we cannot define our generic structure as a generic point of such a space. Laufer in [15] characterized those topological types which support only one analytic type (or, more generally, countably many analytic types), but about the general cases very little is known. Usually, generic structures — when they appeared — were introduced by certain ad-hoc definitions, or only in particular situations. In a slightly different direction a remarkable progress was made by Laufer (see e.g. [14]) when he defined *local complete deformations* of (resolution of) singularities. This parameter space will be the major tool in our working definition as well (see 1.5).

However, even if one defines a certain 'genericity' notion by eliminating a discriminant from a parameter space (consisting of the pathological objects from the point of view of the discussion), the next hard major task is to exploit from the genericity some key geometric/numerical/cohomological properties. (E.g., in the present article this is done via Theorem B below.)

Regarding the problem to find the values of the analytic invariants associated with the generic analytic type, a crucial obstruction was (before the present note) the *lack of examples and experience*. E.g., Laufer in [16] proved that a generic elliptic singularity has geometric genus $p_g = 1$, but except this, almost no other example is known. Even more, using the known statements of the literature, it is almost impossible to guess what are the possible topological candidates for the invariants of the generic analytic structure. The expectation is that they should be certain sharp topological bounds, but even if some topological bound is known, usually there are no tools to prove its realization for the generic (or any) analytic structure.

The situation is exemplified rather trustworthily already by the *geometric genus*. Wagreich already in 1970 in [37] defined topologically the *arithmetical genus* p_a of a normal surface singularity and for any non–rational germ (that is, when $p_g \neq 0$) he proved that $p_a \leq p_g$ (see [37, p. 425]). Though in some (easy) cases was known that they agree, analyzing the existing proofs of the inequality (see e.g. the very short proof in [29]), one might think that this inequality for germs with complicated topological types probably is extremely week. However, the point is that in the present note we prove that (contrary to the first naive judgement) the geometric analytic structure realizes exactly this p_a. For the other invariants (see Theorem A below) even the corresponding candidates were not on the table (but we expect that they will have some relationship with lattice cohomology [26]).

In fact, even in this article we make the selection of a package of analytic invariants (organized around the cohomology of natural line bundles), for which we present the corresponding 'package of topological expressions', and we will treat, say, the Hilbert–Samuel function/multiplicity/embedded-dimension package in a forthcoming manuscript (with rather different type of combinatorial answers).

1.3. The technical presentation of the results. In order to formulate the invariants and the topological characterizations in a more formal way we need some notation. Let $\widetilde{X} \to X$ be a good resolution with irreducible exceptional curves $\{E_v\}_{v\in\mathcal{V}}$, with resolution graph Γ, negative definite intersection lattice $L = H_2(\widetilde{X},\mathbb{Z})$, dual lattice $L' = H^2(\widetilde{X},\mathbb{Z}) \simeq H_2(\widetilde{X},\partial\widetilde{X},\mathbb{Z})$, and discriminant group $H = L'/L$ (for details see 2.1). We assume that the link M of (X, o) is a rational homology sphere, that is, Γ is a tree of rational E_v's. In such a case $H = H_1(M,\mathbb{Z})$ is finite. Usually Z will denote an effective cycle supported on the exceptional curve E. The dual lattice L' is also the target of the surjective first Chern class map $c_1 : \mathrm{Pic}(\widetilde{X}) \to L'$, set $c_1^{-1}(l') = \mathrm{Pic}^{l'}(\widetilde{X})$. For any Chern class one defines the 'natural line bundle' $\mathcal{O}_{\widetilde{X}}(l') \in \mathrm{Pic}^{l'}(\widetilde{X})$, and its restrictions $\mathcal{O}_Z(l')$, cf. 3.4.

In the sequel we fix a topological type, that is, a resolution graph. The topological invariants are read from Γ, or equivalently, from L. The most elementary one is the 'Riemann–Roch' expression $\chi : L' \to \mathbb{Q}$ given by $\chi(l') := -(l', l' - Z_K)/2$, where $Z_K \in L'$ is the anticanonical cycle defined combinatorially by the adjunction formulae, cf. 2.1.

The list of analytic invariants, associated with a generic analytic type (with respect to the fixed graph), which are described in the present article topologically are the following: $h^1(\mathcal{O}_Z)$, $h^1(\mathcal{O}_Z(l'))$ (with certain restriction on the Chern class l'), — this last one applied for $Z \gg 0$ provides $h^1(\mathcal{O}_{\widetilde{X}})$ and $h^1(\mathcal{O}_{\widetilde{X}}(l'))$ too —, the cohomological cycle of natural line bundles, the multivariable Hilbert and Poincaré series associated with the divisorial filtration, the analytic semigroup, the maximal ideal cycle. See [4,5,19,23,25,28,31,33] for the definitions and relationships between them. Here some definitions will be recalled in section 6.

Surprisingly, in all the topological characterization we need to use merely χ, however, it is really remarkable the level of complexity and subtlety of the combinatorial expressions/invariants carried by this 'simple' (?) quadratic function. Definitely, this can happen due to the fact that we work over the lattices L and L', and the position of the lattice points with respect to the level sets of χ play the key role. It is a real challenge now to interpret these expressions in terms of lattice cohomology [26,27] or other topological 3–manifold invariants.

Theorem A. *Fix a resolution graph and assume that the analytic type of $\widetilde{X}$ is generic. Then the following identities hold:*
(a) For any effective cycle $Z \in L_{>0}$ with $|Z|$ connected

$$h^1(\mathcal{O}_Z) = 1 - \min_{0<l\leq Z, l\in L}\{\chi(l)\}.$$

(b) If $l' = \sum_{v\in\mathcal{V}} l'_v E_v \in L'$ satisfies $l'_v < 0$ for any E_v in the support of Z then

$$h^1(Z, \mathcal{O}_Z(l')) = \chi(-l') - \min_{0\leq l\leq Z, l\in L}\{\chi(-l'+l)\}.$$

(For a characterization valid for more general Chern classes l' see section 6.)
(c) If $p_g(X,o) = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is the geometric genus of (X,o) then

$$p_g(X,o) = 1 - \min_{l\in L_{>0}}\{\chi(l)\} = -\min_{l\in L}\{\chi(l)\} + \begin{cases} 1 & \text{if } (X,o) \text{ is not rational,} \\ 0 & \text{else.}\end{cases}$$

(d) More generally, for any $l' \in L'$

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l')) = \chi(-l') - \min_{l\in L_{\geq 0}}\{\chi(-l'+l)\} + \begin{cases} 1 & \text{if } l' \in L_{\geq 0} \text{ and } (X,o) \text{ is not rational,} \\ 0 & \text{else.}\end{cases}$$

(e) Let $H(\mathbf{t}) = \sum_{l'\in L'} \mathfrak{h}(l')\mathbf{t}^{l'}$ be the multivariable equivariant Hilbert series associated with the divisorial filtration. Write l' as $r_h + l_0$ for some $l_0 \in L$ and $r_h \in L'$ the unique representative of $h = [l']$ in the semi-open cube of L'. Then $\mathfrak{h}(r_h) = 0$ for $l_0 = 0$. Furthermore, for $l_0 > 0$ and $h \neq 0$

$$\mathfrak{h}(l') = \min_{l\in L_{\geq 0}}\{\chi(l'+l)\} - \min_{l\in L_{\geq 0}}\{\chi(r_h+l)\}.$$

For $h = 0$ and $l' = l_0 > 0$

$$\mathfrak{h}(l_0) = \min_{l\in L_{\geq 0}}\{\chi(l_0+l)\} - \min_{l\in L_{\geq 0}}\{\chi(l)\} + \begin{cases} 1 & \text{if } (X,o) \text{ is not rational,} \\ 0 & \text{else.}\end{cases}$$

(f) Write the multivariable equivariant Poincaré series $P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_{v\in\mathcal{V}}(1-t_v^{-1})$ as $\sum_{l'\in\mathcal{S}'} \mathfrak{p}(l')\mathbf{t}^{l'}$. It is supported in the Lipman (antinef) cone, in particular in $L'_{\geq 0}$. Then $\mathfrak{p}(0) = 1$ and for $l' > 0$ one has

$$\mathfrak{p}(l') = \sum_{I \subset \mathcal{V}} (-1)^{|I|+1} \min_{l \in L_{\geq 0}} \chi(l' + l + E_I).$$

(g) Consider the analytic semigroup $\mathcal{S}'_{an} := \{l' \in L' : \mathcal{O}_{\widetilde{X}}(-l')$ *has no fixed components*$\}$*. Then*

$$\mathcal{S}'_{an} = \{l' : \chi(l') < \chi(l' + l) \text{ for any } l \in L_{>0}\} \cup \{0\}.$$

(h) Assume that Γ *is a non–rational graph and set* $\mathcal{M} = \{Z \in L_{>0} : \chi(Z) = \min_{l \in L} \chi(l)\}$.

Then the unique minimal element of $\mathcal{M}$ *is the cohomological cycle, while the unique maximal element of* $\mathcal{M}$ *is the maximal ideal cycle of* $\widetilde{X}$.

1.4. The Abel map. The main tool of the present note is the Abel map constructed and studied in [21]. Though in [21] we also listed several applications, the present note shows its power, its applicability in a really difficult problem, with a priori unexpected answers which become totally natural and motivated from the perspective of this new approach.

Let us recall shortly this object (for details see [21] or §2 and 3.4 here). Let (X, o), $\widetilde{X} \to X$, L and L' as above. Then for any effective cycle Z supported on E and for any (possible) Chern class $l' \in L'$ we consider the space $\mathrm{ECa}^{l'}(Z)$ of effective Cartier divisors D supported on Z, whose associated line bundles $\mathcal{O}_Z(D)$ have first Chern class l'. Furthermore, we also consider the Abel map $c^{l'}(Z) : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$, $D \mapsto \mathcal{O}_Z(D)$.

Using the Abel map, in [21, Th. 5.3.1] we have shown that for any analytic singularity and resolution with fixed resolution graph, and for any $\mathcal{L} \in \mathrm{Pic}^{l'}(Z)$, one has $h^1(Z, \mathcal{L}) \geq \chi(-l') - \min_{0 \leq l \leq Z,\, l \in L} \chi(-l' + l)$, and equality holds for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$. In particular, for any analytic type, $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ can be expressed combinatorially. Now, the expectation and our guiding principle is the following: for a generic analytic structure the natural line bundle $\mathcal{O}_Z(l')$ should have the same h^1 as the generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ (associated with any analytic structure). This is the key technical statement of the note (for notations see 5.1).

Theorem B. *Assume that* $\widetilde{X}$ *is generic. Under some (necessary) negativity restriction on the Chern class* l' *(see Theorem 5.1.1 and Remark 6.1.1(b)) the following facts hold.*

(I) The following facts are equivalent:

(a) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, *where* $\mathcal{O}_Z(l')$ *is the natural line bundle with Chern class* l'*;*

(b) $\mathcal{L}_{gen} \in \mathrm{im}(c^{\tilde{l}})$, *where* $\mathcal{L}_{gen}$ *is a generic line bundle in* $\mathrm{Pic}^{\tilde{l}}(Z)$ *(that is,* $c^{\tilde{l}}$ *is dominant);*

(c) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, *and for any* $D \in (c^{\tilde{l}})^{-1}(\mathcal{O}_Z(l'))$ *the tangent map* $T_D c^{\tilde{l}} : T_D \mathrm{ECa}^{\tilde{l}}(Z) \to T_{\mathcal{O}_Z(l')} \mathrm{Pic}^{\tilde{l}}(Z)$ *is surjective.*

(II) $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ *for* $i = 0, 1$ *and for a generic line bundle* $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$.

The proof is long and technical, it fills in all section 5 (the 'hard' part is (a) $\Rightarrow$ (c)). It uses the explicit description of tangent map of $c^{l'}$ in terms of Laufer duality (integration of forms along divisors, cf. 2.2). In this section certain familiarity with [21] might help the reading.

By this result, if $\widetilde{X}$ has generic analytic structure, then the cohomology of natural line bundles can be expressed by the very same topological formula as $\mathcal{L}_{gen}$ with the same Chern class. Then all the formulae of Theorem A above follow directly.

In the next paragraph we say a few words about 'generic analytic type'.

1.5. The working definition of the generic analytic type. Usually when we have a parameter space for a family of geometric objects, the 'generic object' might depend essentially on the fact that what kind of geometrical problem we wish to solve, or, what kind of anomalies we wish to avoid. Accordingly, we determine a discriminant space of the non–wished objects, and generic means its complement. In the present article all the discrete analytic invariants we treat are basically guided by the cohomology groups of the natural line bundles (for their definition see [24], [30] or 3.4 here, they associate in a canonical way a line bundle to any given Chern class). Hence, the discriminant spaces (sitting in the base space of complete deformation spaces of Laufer [14]) are defined as the 'jump loci' of the cohomology groups of the natural line bundles. In section 3 we recall the needed results of Laufer regarding complete deformations of some $\widetilde{X}$, and we build on this our working definition of general analytic type.

Note that the natural line bundles are well–defined only if the link is a rational homology sphere. Furthermore, this assumption appeared in the theory of Abel maps as well. Hence, in the article we also impose this topological restriction.

2. Preliminaries and notations

2.1. Notations regarding a good resolution. [23,24,28,11,21] Let (X, o) be the germ of a complex analytic normal surface singularity, and let us fix a good resolution $\phi : \widetilde{X} \to X$ of (X, o). Let E be the exceptional curve $\phi^{-1}(0)$ and $\cup_{v\in\mathcal{V}} E_v$ be its irreducible decomposition. Define $E_I := \sum_{v\in I} E_v$ for any subset $I \subset \mathcal{V}$.

We will assume that each E_v is rational, and the dual graph is a tree. This happens exactly when the link M of (X, o) is a rational homology sphere.

The $\mathbb{Z}$–module $L := H_2(\widetilde{X}, \mathbb{Z})$ is a lattice endowed with the natural negative definite intersection form $(,)$. It is freely generated by the classes of $\{E_v\}_{v\in\mathcal{V}}$. The dual lattice is $L' = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \{l' \in L \otimes \mathbb{Q} : (l', L) \in \mathbb{Z}\}$. It is generated by the (anti)dual classes $\{E^*_v\}_{v\in\mathcal{V}}$ defined by $(E^*_v, E_w) = -\delta_{vw}$ (where δ_{vw} stands for the Kronecker symbol). It is also identified with $H^2(\widetilde{X}, \mathbb{Z})$. The anticanonical cycle $Z_K \in L'$ is defined via the adjunction identities $(Z_K, E_v) = E_v^2 + 2$ for all v.

All the E_v–coordinates of any E^*_u are strict positive. We define the (rational) Lipman cone as $\mathcal{S}' := \{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$. As a monoid it is generated over $\mathbb{Z}_{\geq 0}$ by $\{E^*_v\}_v$.

The lattice L embeds into L' with $L'/L \simeq H_1(M,\mathbb{Z})$. The quotient L'/L is abridged by H. Each class $h \in H = L'/L$ has a unique representative $r_h \in L'$ in the semi-open cube $\{\sum_v r_v E_v \in L' : r_v \in \mathbb{Q} \cap [0,1)\}$, such that its class $[r_h]$ is h.

There is a natural (partial) ordering of L' and L: we write $l'_1 \geq l'_2$ if $l'_1 - l'_2 = \sum_v r_v E_v$ with all $r_v \geq 0$. We set $L_{\geq 0} = \{l \in L : l \geq 0\}$ and $L_{>0} = L_{\geq 0} \setminus \{0\}$.

The support of a cycle $l = \sum n_v E_v$ is defined as $|l| = \cup_{n_v \neq 0} E_v$.

2.2. The Abel map. [21] Let $\mathrm{Pic}(\widetilde{X}) = H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$ be the group of isomorphism classes of holomorphic line bundles on $\widetilde{X}$. The first Chern map $c_1 : \mathrm{Pic}(\widetilde{X}) \to L'$ is surjective; write $\mathrm{Pic}^{l'}(\widetilde{X}) = c_1^{-1}(l')$. Since $H^1(M,\mathbb{Q}) = 0$, by the exponential exact sequence on $\widetilde{X}$ one has $\mathrm{Pic}^0(\widetilde{X}) \simeq H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \simeq \mathbb{C}^{p_g}$, where p_g is the geometric genus.

Similarly, if Z is an effective non–zero integral cycle supported by E, then $\mathrm{Pic}(Z) = H^1(Z, \mathcal{O}^*_Z)$ denotes the group of isomorphism classes of invertible sheaves on Z. Again, it appears in the exact sequence $0 \to \mathrm{Pic}^0(Z) \to \mathrm{Pic}(Z) \xrightarrow{c_1} L'(|Z|) \to 0$, where $\mathrm{Pic}^0(Z)$ is identified with $H^1(Z, \mathcal{O}_Z)$ by the exponential exact sequence. Here $L(|Z|)$ denotes the sublattice of L generated by the base element $E_v \subset |Z|$, and $L'(|Z|)$ is its dual lattice.

For any $Z \in L_{>0}$ let $\mathrm{ECa}(Z)$ be the space of (analytic) effective Cartier divisors on Z. Their supports are zero–dimensional in E. Taking the class of a Cartier divisor provides the *Abel map* $c : \mathrm{ECa}(Z) \to \mathrm{Pic}(Z)$. Let $\mathrm{ECa}^{\tilde{l}}(Z)$ be the set of effective Cartier divisors with Chern class $\tilde{l} \in L'(|Z|)$, i.e. $\mathrm{ECa}^{\tilde{l}}(Z) := c^{-1}(\mathrm{Pic}^{\tilde{l}}(Z))$. The restriction of c is denoted by $c^{\tilde{l}}(Z) : \mathrm{ECa}^{\tilde{l}}(Z) \to \mathrm{Pic}^{\tilde{l}}(Z)$.

We also use the notation $\mathrm{ECa}^{l'}(Z) := \mathrm{ECa}^{R(l')}(Z)$ and $\mathrm{Pic}^{l'}(Z) := \mathrm{Pic}^{R(l')}(Z)$ for any $l' \in L'$, where $R : L' \to L'(|Z|)$ is the cohomological restriction, dual to the inclusion $L(|Z|) \hookrightarrow L$. (This means that $R(E^*_v) =$ the (anti)dual of E_v in the lattice $L'(|Z|)$ if $E_v \subset |Z|$ and $R(E^*_v) = 0$ otherwise.)

A line bundle $\mathcal{L} \in \mathrm{Pic}^{\tilde{l}}(Z)$ is in the image $\mathrm{im}(c^{\tilde{l}})$ if and only if it has a section without fixed components, that is, if $H^0(Z,\mathcal{L})_{reg} \neq \emptyset$, where $H^0(Z,\mathcal{L})_{reg} := H^0(Z,\mathcal{L}) \setminus \cup_v H^0(Z - E_v, \mathcal{L}(-E_v))$. Here the inclusion of $H^0(Z - E_v, \mathcal{L}(-E_v))$ into $H^0(Z,\mathcal{L})$ is given by the long cohomological exact sequence associated with $0 \to \mathcal{L}(-E_v)|_{Z-E_v} \to \mathcal{L} \to \mathcal{L}|_{E_v} \to 0$, and it represents the subspace of sections, whose fixed components contain E_v.

By this definition (see (3.1.5) of [21] and the discussion before it) $\mathrm{ECa}^{\tilde{l}}(Z) \neq \emptyset$ if and only if $-\tilde{l} \in \mathcal{S}'(|Z|) \setminus \{0\}$. It is advantageous to have a similar statement for $\tilde{l} = 0$ too, hence we redefine $\mathrm{ECa}^0(Z)$ as $\{\emptyset\}$, a set/space with one element (the empty divisor), and $c^0 : \mathrm{ECa}^0(Z) \to \mathrm{Pic}^0(Z)$ by $c^0(\emptyset) = \mathcal{O}_Z$. Then

$$H^0(Z,\mathcal{L})_{reg} \neq \emptyset \;\Leftrightarrow\; \mathcal{L} = \mathcal{O}_Z \;\Leftrightarrow\; \mathcal{L} \in \mathrm{im}(c^0) \qquad (c_1(\mathcal{L}) = 0). \tag{2.2.1}$$

Then the 'extended equivalence' reads as: $\mathrm{ECa}^{\tilde{l}}(Z) \neq \emptyset$ if and only if $-\tilde{l} \in \mathcal{S}'(|Z|)$. In such a case $\mathrm{ECa}^{\tilde{l}}(Z)$ is a smooth complex algebraic variety of dimension $(\tilde{l}, Z)$, cf. [21, Th. 3.1.10]. Furthermore, the Abel map is an algebraic regular map. It can be described using Laufer's duality as follows, cf. [13], [16, p. 1281] or [21]. First, by Serre duality,

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* \simeq H^1_c(\widetilde{X}, \Omega^2_{\widetilde{X}}) \simeq H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}). \tag{2.2.2}$$

An element of $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ can be represented by the class of a form $\widetilde{\omega} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$. Furthermore, an element $[\alpha]$ of $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ can be represented by a *Čech cocyle* $\alpha_{ij} \in \mathcal{O}(U_i \cap U_j)$, where $\{U_i\}_i$ is an open cover of E, $U_i \cap U_j \cap U_k = \emptyset$, and each connected component of the intersections $U_i \cap U_j$ is either a coordinate bidisc $B = \{|u| < 2\epsilon,\ |v| < 2\epsilon\}$ with coordinates (u, v), such that $E \cap B \subset \{uv = 0\}$, or a punctured coordinate bidisc $B = \{\epsilon/2 < |v| < 2\epsilon,\ |u| < 2\epsilon\}$ with coordinates (u, v), such that $E \cap B = \{u = 0\}$. Then, Laufer's realization of the duality $H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}}) \otimes H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to \mathbb{C}$ is

$$\langle [\alpha], [\widetilde{\omega}] \rangle = \sum_B \int\limits_{|u|=\epsilon,\ |v|=\epsilon} \alpha_{ij} \widetilde{\omega}. \tag{2.2.3}$$

In particular, if $\widetilde{\omega}$ has no pole along E in B, then the B–contribution in the above sum is zero.

This duality, via the isomorphism $\exp : H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to c_1^{-1}(0) \subset H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}}) = \mathrm{Pic}(\widetilde{X})$, can be transported as follows, cf. [21]. (Here we present the case of a peculiar divisor due to the fact that this version will be used later.) Consider the following situation. We fix a smooth point p on E ($p \in E_v$), a local bidisc $B \ni p$ with local coordinates (u, v) such that $B \cap E = \{u = 0\}$, $B = \{|u|, |v| < \epsilon\}$. We assume that a certain form $\widetilde{\omega} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$ has local equation $\widetilde{\omega} = \sum_{i \in \mathbb{Z}, j \geq 0} a_{i,j} u^i v^j du \wedge dv$ in B. In the same time, we fix a divisor $\widetilde{D}$ on $\widetilde{X}$, whose local equation in B is v^ℓ, $\ell \geq 1$. Let $\widetilde{D}_{\underline{t}}$ be another divisor, which is the same as $\widetilde{D}$ in the complement of B and in B its local equation is $(v + t + \sum_{k \geq 1, l \geq 0} t_{k,l} u^k v^l)^\ell$, where all t, $t_{k,l} \in \mathbb{C}$ and $|t|$, $|t_{k,l}| \ll 1$. Then $\widetilde{D}_{\underline{t}} - \widetilde{D}$ is the divisor $\widetilde{D}' = \mathrm{div}(g)$, where $g := ((v + t + \sum_{k \geq 1, l \geq 0} t_{k,l} u^k v^l)/v)^\ell$, supported in B. In particular, $\mathcal{O}(\widetilde{D}') \in \mathrm{Pic}^0(\widetilde{X}) \subset H^1(\widetilde{X}, \mathcal{O}^*_{\widetilde{X}})$ can be represented by the cocycle $g|_{B^*} \in \mathcal{O}^*(B^*)$, where $B^* = \{\epsilon/2 < |v| < \epsilon, |u| < \epsilon\}$. Therefore, $\log(g|_{B^*})$ is a cocycle in B^* representing its lifting into $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. This paired with $\widetilde{\omega}$ gives for $\langle\langle \widetilde{D}_{\underline{t}}, [\widetilde{\omega}] \rangle\rangle := \langle \exp^{-1} \mathcal{O}_{\widetilde{X}}(\widetilde{D}_t - \widetilde{D}), [\widetilde{\omega}] \rangle$ the expression

$$\ell \cdot \int\limits_{|u|=\epsilon,\ |v|=\epsilon} \log \Big(1 + \frac{t + \sum_{k,l} t_{k,l} u^k v^l}{v} \Big) \cdot \sum_{i \in \mathbb{Z}, j \geq 0} a_{i,j} u^i v^j du \wedge dv. \tag{2.2.4}$$

If $\widetilde{\omega}$ has no pole then $\langle\langle \widetilde{D}_{\underline{t}}, [\widetilde{\omega}] \rangle\rangle = 0$. As an example, assume that $\widetilde{\omega}$ has the form $(h(u, v)/u^o) du \wedge dv$ with h regular and $h(0, 0) \neq 0$, and $o \geq 1$, while $g = (v + tu^{o-1})/v$ and $\ell = 1$, then

$$\langle\langle \widetilde{D}_{\underline{t}}, [\widetilde{\omega}]\rangle\rangle = \int\limits_{|u|=\epsilon,\ |v|=\epsilon} \log\Big(1+\frac{tu^{o-1}}{v}\Big)\cdot\frac{h}{u^o}\, du\wedge dv = c\cdot t+\{\text{higher order terms}\}\ \ (c\in\mathbb{C}^*). \tag{2.2.5}$$

If $Z\gg 0$ then $H^0(\widetilde{X}\setminus E,\Omega^2_{\widetilde{X}})/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})\simeq H^0(\widetilde{X},\Omega^2_{\widetilde{X}}(Z))/H^0(\widetilde{X},\Omega^2_{\widetilde{X}})$. Furthermore, if $\widetilde{\omega}_1,\ldots,\widetilde{\omega}_{p_g}$ are representatives of a basis of this vector space and $\widetilde{D}_{\underline{t}}$ is considered as a path in $\mathrm{ECa}^{-\ell E^*_v}(Z)$, then $\widetilde{D}_{\underline{t}}\mapsto (\langle\langle \widetilde{D}_{\underline{t}}, [\widetilde{\omega}_1]\rangle\rangle,\ldots,\langle\langle \widetilde{D}_{\underline{t}}, [\widetilde{\omega}_{p_g}]\rangle\rangle)$ is the restriction of the Abel map to $\widetilde{D}_{\underline{t}}$ (associated with Z, and shifted by the image of $\widetilde{D}$) (cf. [21]).

3. Resolutions with generic analytic structure

3.1. The setup. We fix a topological type of a normal surface singularity. This means that we fix either the C^∞ oriented diffeomorphism type of the link, or, equivalently, one of the dual graphs of a good resolution (all of them are equivalent up to blowing up/down rational (-1)–vertices). We assume that the link is a rational homology sphere, that is, the graph is a tree of rational vertices.

Any such topological type might support several analytic structures. The moduli space of the possible analytic structures is not described yet in the literature, hence we cannot rely on it. In particular, the 'generic analytic structure', as a 'generic' point of this moduli space, in this way is not well–defined. However, in order to run/prove the concrete properties regarding generic analytic structures, instead of such theoretical definition it would be even much better to consider a definition based on a list of stability properties under certain concrete deformations (whose validity could be expected for the 'generic' analytic structure in the presence of a classifying space). Hence, for us in this note, a generic analytic structure will be a structure, which will satisfy such stability properties. In order to define them it is convenient to fix a resolution graph Γ and treat deformation of analytic structures supported on resolution spaces having dual graph Γ.

The type of stability we wish to have is the following. The topological type (or, the graph Γ) determines a lower bound for the possible values of the geometric genus (which usually depends on the analytic type). Let $\mathrm{MIN}(\Gamma)$ be the unique optimal bound, that is, $\mathrm{MIN}(\Gamma)\leq p_g(X,o)$ for any singularity (X,o) which admits Γ as a resolution graph, and $\mathrm{MIN}(\Gamma)=p_g(X,o)$ for some (X,o). Then one of the requirements for the 'generic analytic structure' (X_{gen},o) is that $p_g(X_{gen},o)=\mathrm{MIN}(\Gamma)$. (In the body of the paper $\mathrm{MIN}(\Gamma)$ will be determined explicitly.) However, we will need several similar stability requirements involving other line bundles as well (besides the trivial one, which provides p_g). For their definition we need a preparation.

3.2. Laufer's results. In this subsection we review some results of Laufer regarding deformations of the analytic structure on a resolution space of a normal surface singularity with fixed resolution graph (and deformations of non–reduced analytic spaces supported on exceptional curves) [14].

First, let us fix a normal surface singularity (X,o) and a good resolution $\phi:(\widetilde{X},E)\to(X,o)$ with reduced exceptional curve $E=\phi^{-1}(o)$, whose irreducible decomposition is

$\cup_{v\in\mathcal{V}}E_v$ and dual graph Γ. Let $\mathcal{I}_v$ be the ideal sheaf of $E_v\subset\widetilde{X}$. Then for arbitrary positive integers $\{r_v\}_{v\in\mathcal{V}}$ one defines two objects, an analytic one and a topological (combinatorial) one. At analytic level, one sets the ideal sheaf $\mathcal{I}(r):=\prod_v\mathcal{I}_v^{r_v}$ and the non–reduces space $Z(r)$ with structure sheaf $\mathcal{O}_{Z(r)}:=\mathcal{O}_{\widetilde{X}}/\mathcal{I}(r)$ supported on E.

The topological object is a graph decorated with multiplicities, denoted by $\Gamma(r)$. As a non–decorated graph $\Gamma(r)$ coincides with the graph Γ without decorations. Additionally each vertex v has a 'multiplicity decoration' r_v, and we put also the self–intersection decoration E_v^2 whenever $r_v>1$. (Hence, the vertex v does not inherit the self–intersection decoration of v if $r_v=1$.) Note that the abstract 1–dimensional analytic space $Z(r)$ determines by its reduced structure the shape of the dual graph Γ, and by its non–reduced structure all the multiplicities $\{r_v\}_{v\in\mathcal{V}}$, and additionally, all the self–intersection numbers E_v^2 for those v's when $r_v>1$ (see [14, Lemma 3.1]).

We say that the space $Z(r)$ has topological type $\Gamma(r)$.

Clearly, the analytic structure of (X,o), hence of $\widetilde{X}$ too, determines each 1–dimensional non–reduced space $Z(r)$. The converse is also true in the following sense.

Theorem 3.2.1. *[12, Th. 6.20], [14, Prop. 3.8] (a) Consider an abstract 1–dimensional space $Z(r)$, whose topological type $\Gamma(r)$ can be completed to a negative definite graph Γ (or, lattice L). Then there exists a 2–dimensional manifold $\widetilde{X}$ in which $Z(r)$ can be embedded with support E such that the intersection matrix inherited from the embedding $E\subset\widetilde{X}$ is the negative definite lattice L. In particular (since by Grauert theorem [7] the exceptional locus E in $\widetilde{X}$ can be contracted to a normal singularity), any such $Z(r)$ is always associated with a normal surface singularity (as above).*

(b) Suppose that we have two singularities (X,o) and (X',o) with good resolutions as above with the same resolution graph Γ. Depending solely on Γ, the integers $\{r_v\}_v$ may be chosen so large that if $\mathcal{O}_{Z(r)}\simeq\mathcal{O}_{Z'(r)}$, then $E\subset\widetilde{X}$ and $E'\subset\widetilde{X}'$ have biholomorphically equivalent neighbourhoods via a map taking E to E'. (For a concrete estimate how large r should be see Theorem 6.20 in [12].)

In particular, in the deformation theory of $\widetilde{X}$ it is enough to consider the deformations of non–reduced spaces of type $Z(r)$.

Fix a non–reduced 1–dimensional space $Z=Z(r)$ with topological type $\Gamma(r)$. Following Laufer and for technical reasons (partly motivated by further applications in the forthcoming continuations of the series of manuscripts) we also choose a closed subspace Y of Z (whose support can be smaller, it can be even empty). More precisely, (Z,Y) locally is isomorphic with $(\mathbb{C}\{x,y\}/(x^ay^b),\mathbb{C}\{x,y\}/(x^cy^d))$, where $a\geq c\geq 0$, $b\geq d\geq 0$, $a>0$. The ideal of Y in $\mathcal{O}_Z$ is denoted by $\mathcal{I}_Y$.

Definition 3.2.2. [14, Def. 2.1] A deformation of Z, fixing Y, consists of the following data:

(i) There exists an analytic space $\mathcal{Z}$ and a proper map $\lambda:\mathcal{Z}\to Q$, where Q is a manifold containing a distinguished point 0.

(ii) Over a point $q \in Q$ the fiber Z_q is the subspace of $\mathcal{Z}$ determined by the ideal sheaf $\lambda^*(\mathfrak{m}_q)$ (where $\mathfrak{m}_q$ is the maximal ideal of q). Z is isomorphic with Z_0, usually they are identified.

(iii) λ is a trivial deformation of Y (that is, there is a closed subspace $\mathcal{Y} \subset \mathcal{Z}$ and the restriction of λ to $\mathcal{Y}$ is a trivial deformation of Y).

(iv) λ is *locally trivial* in a way which extends the trivial deformation $\lambda|_{\mathcal{Y}}$. This means that for ant $q \in Q$ and $z \in \mathcal{Z}$ there exist a neighbourhood W of z in $\mathcal{Z}$, a neighbourhood V of z in Z_q, a neighbourhood U of q in Q, and an isomorphism $\phi : W \to V \times U$ such that $\lambda|_W = pr_2 \circ \phi$ (compatibly with the trivialization of $\mathcal{Y}$ from (iii)), where pr_2 is the second projection; for more see [14].

One verifies that under deformations (with connected base space) the topological type of the fibers Z_q, namely $\Gamma(r)$, remains constant (see [14, Lemma 3.1]).

Definition 3.2.3. [14, Def. 2.4] A deformation $\lambda : \mathcal{Z} \to Q$ of Z, fixing Y, is complete at 0 if, given any deformation $\tau : \mathcal{P} \to R$ of Z fixing Y, there is a neighbourhood R' of 0 in R and a holomorphic map $f : R' \to Q$ such that τ restricted to $\tau^{-1}(R')$ is the deformation $f^*\lambda$. Furthermore, λ is complete if it is complete at each point $q \in Q$.

Laufer proved the following results.

Theorem 3.2.4. *[14, Theorems 2.1, 2.3, 3.4, 3.6] Let $\theta_{Z,Y} = \mathcal{H}om_Z(\Omega^1_Z, \mathcal{I}_Y)$ be the sheaf of germs of vector fields on Z, which vanish on Y, and let $\lambda : \mathcal{Z} \to Q$ be a deformation of Z, fixing Y.*

(a) If the Kodaira–Spencer map $\rho_0 : T_0Q \to H^1(Z, \theta_{Z,Y})$ is surjective then λ is complete at 0.

(b) If ρ_0 is surjective then ρ_q is surjective for all q sufficiently near to 0.

(c) There exists a deformation λ with ρ_0 bijective. In such a case in a neighbourhood U of 0 the deformation is essentially unique, and the fiber above q is isomorphic to Z for only at most countably many q in U.

It is worth to stress that any two analytic types on a fixed topological type can be connected by a path, which can be covered by finitely many deformations of the above type (see [14, Th. 3.2]).

3.2.5. Functoriality. Let Z' be a closed subspace of Z such that $\mathcal{I}_{Z'} \subset \mathcal{I}_Y \subset \mathcal{O}_Z$. Then there is a natural reduction of pairs $(\mathcal{O}_Z, \mathcal{O}_Y) \to (\mathcal{O}_{Z'}, \mathcal{O}_Y)$. Hence, any deformation $\lambda : \mathcal{Z} \to Q$ of Z fixing Y reduces to a deformation $\lambda' : \mathcal{Z}' \to Q$ of Z' fixing Y. Furthermore, if λ is complete then λ' is automatically complete as well (since $H^1(Z, \theta_{Z,Y}) \to H^1(Z', \theta_{Z',Y})$ is onto).

3.3. The '0–generic analytic structure'. We wish to define when is the analytic structure of a fiber Z_q ($q \in Q$) of a deformation 'generic'. We proceed in two steps. The '0–genericity' is the first one (corresponding to the Chern class $l' = 0$), which will be defined in this subsection.

It is rather advantageous to set a definition, which is compatible with respect to all the restrictions $\mathcal{O}_Z \to \mathcal{O}_{Z'}$. In order to do this, let us fix the coefficients $\tilde{r} = \{\tilde{r}_v\}_v$ so large that for them Theorem 3.2.1 is valid. In this way basically we fix a resolution $(\widetilde{X}, E)$ and some large infinitesimal neighbourhood $Z(\tilde{r})$ associated with it. Moreover, let us also fix a *complete* deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ whose fibers have the topological type of $\Gamma(\tilde{r})$. Next, we consider all the other coefficient sets $r := \{r_v\}_v$ such that $0 \leq r_v \leq \tilde{r}_v$ for all v, not all $r_v = 0$. Such a choice, by restriction as in 3.2.5, automatically provides a deformation $\lambda(r) : \mathcal{Z}(r) \to Q$. Then set

$$\Delta(0,r) := \{q \in Q \,:\, h^i(Z(r)_q, \mathcal{O}_{Z(r)_q}) \text{ is not constant in a neighbourhood of } q \text{ for some } i\}. \tag{3.3.1}$$

Then $\Delta(0,r)$ is a closed (reduced) proper subspace of Q, see [34,35] (one can use also an argument similar to Lemma 3.6.1 written for $l' = 0$). Define $\Delta^0(\tilde{r}) := \cup_{r_v \leq \tilde{r}_v} \Delta(0,r)$. Then $\Delta^0(\tilde{r})$ is also closed and $\Delta^0(\tilde{r}) \neq Q$.

Definition 3.3.2. We say that the fiber $Z(\tilde{r})_q$ of $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ is 0–generic if $q \in Q \setminus \Delta^0(\tilde{r})$.

Next, we wish to generalize this definition for all Chern classes $l' \in L'$, or, for all 'natural line bundles', as generalizations of the trivial bundle corresponding to $l' = 0$.

3.4. Natural line bundles. Let us start again with a good resolution $\phi : (\widetilde{X}, E) \to (X, o)$ of a normal surface singularity with rational homology sphere link, and consider the cohomology exact sequence associated with the exponential exact sequence of sheaves

$$0 \to \mathrm{Pic}^0(\widetilde{X}) \xrightarrow{\epsilon} \mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} H^2(\widetilde{X}, \mathbb{Z}) \to 0. \tag{3.4.1}$$

Here $c_1(\mathcal{L}) \in H^2(\widetilde{X}, \mathbb{Z}) = L'$ is the first Chern class of $\mathcal{L}$. Then, see e.g. [30,24], there exists a unique homomorphism (split) $s : L' \to \mathrm{Pic}(\widetilde{X})$ of c_1 such that $c_1 \circ s = id$ and s restricted to L is $l \mapsto \mathcal{O}_{\widetilde{X}}(l)$. The line bundles $s(l')$ are called *natural line bundles* of $\widetilde{X}$, and are denoted by $\mathcal{O}_{\widetilde{X}}(l')$. For several definitions of them see [24]. E.g., $\mathcal{L}$ is natural if and only if one of its power has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some *integral* cycle $l \in L$ supported on E. Here we recall another construction from [30,24], which will be extended later to the deformations space of singularities.

Fix some $l' \in L'$ and let n be the order of its class in L'/L. Then nl' is an integral cycle; its reinterpretation as a divisor supported on E will be denoted by $\mathrm{div}(nl')$. We claim that there exists a divisor $D = D(l')$ in $\widetilde{X}$ such that one has a linear equivalence $nD \sim \mathrm{div}(nl')$ and $c_1(\mathcal{O}_{\widetilde{X}}(D)) = l'$. Furthermore, $D(l')$ is unique up to linear equivalence, hence $l' \mapsto \mathcal{O}_{\widetilde{X}}(D(l'))$ is the wished split of (3.4.1). Indeed, since c_1 is onto, there exists a divisor D_1 such that $c_1(\mathcal{O}_{\widetilde{X}}(D_1)) = l'$. Hence $\mathcal{O}_{\widetilde{X}}(nD_1 - \mathrm{div}(nl'))$ has the form $\epsilon(\mathcal{L})$ for some $\mathcal{L} \in \mathrm{Pic}^0(\widetilde{X}) = H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \mathbb{C}^{p_g}$. Define D_2 such that $\mathcal{O}_{\widetilde{X}}(D_2) = \frac{1}{n}\mathcal{L}$ in $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Then $D_1 - D_2$ works. The uniqueness follows from the fact that $\mathrm{Pic}^0(\widetilde{X})$ is torsion free.

The following warning is appropriate. Note that if $\widetilde{X}_1$ is a connected small convenient neighbourhood of the union of some of the exceptional divisors (hence $\widetilde{X}_1$ also stands as the resolution of the singularity obtained by contraction of that union of exceptional curves) then one can repeat the definition of natural line bundles at the level of $\widetilde{X}_1$ as well. However, the restriction to $\widetilde{X}_1$ of a natural line bundle of $\widetilde{X}$ (even of type $\mathcal{O}_{\widetilde{X}}(l)$ with l integral cycle supported on E) usually is not natural on $\widetilde{X}_1$: $\mathcal{O}_{\widetilde{X}}(l')|_{\widetilde{X}_1} \neq \mathcal{O}_{\widetilde{X}_1}(R(l'))$ (where $R : H^2(\widetilde{X}, \mathbb{Z}) \to H^2(\widetilde{X}_1, \mathbb{Z})$ is the natural cohomological restriction), though their Chern classes coincide.

In the sequel we will deal with the family of 'restricted natural line bundles' obtained by *restrictions of* $\mathcal{O}_{\widetilde{X}}(l')$. Even if we need to descend to a 'lower level' $\widetilde{X}_1$ with smaller exceptional curve, or to any cycle Z with support included in E (but not necessarily E) our 'restricted natural line bundles' will be associated with Chern classes $l' \in L' = L'(\widetilde{X})$ via the restrictions $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(\widetilde{X}_1)$ or $\mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(Z)$ of bundles of type $\mathcal{O}_{\widetilde{X}}(l') \in \mathrm{Pic}(\widetilde{X})$. This basically means that we fix a tower of resolution of singularities $\{\widetilde{X}_1\}_{\widetilde{X}_1 \subset \widetilde{X}}$, or $\{\mathcal{O}_Z\}_{|Z| \subset E}$, determined by the 'top level' $\widetilde{X}$, and all the restricted natural line bundles, even at intermediate levels, are restrictions from the top level.

We use the notations $\mathcal{O}_{\widetilde{X}_1}(l') := \mathcal{O}_{\widetilde{X}}(l')|_{\widetilde{X}_1}$ and $\mathcal{O}_Z(l') := \mathcal{O}_{\widetilde{X}}(l')|_Z$ respectively.

3.5. The universal family of natural line bundles. Next, we wish to extend the definition of the line bundles $\mathcal{O}_Z(l')$ to the total space of a deformation (at least locally, over small balls in the complement of $\Delta^0(\tilde{r})$).

We fix some $Z = Z(\tilde{r})$ with all $\tilde{r}_v \gg 0$, supported on E, such that Theorem 3.2.1 is valid (similarly as in 3.3). Fix also some $Y \subset Z$, and a complete deformation $\lambda : \mathcal{Z}(\tilde{r}) \to Q$ of (Z, Y) as in Definition 3.2.2 such that all the fibers have the same fixed topological type $\Gamma(\tilde{r})$. We consider the discriminant $\Delta^0(\tilde{r}) \subset Q$, and we fix some $q_0 \in Q \setminus \Delta^0(\tilde{r})$, and a small ball U, $q_0 \in U \subset Q \setminus \Delta^0(\tilde{r})$. Above U the topologically trivial family of irreducible exceptional curves form the irreducible divisors $\{\mathcal{E}_v\}_v$, such that $\mathcal{E}_v$ above any point $q \in U$ is the corresponding irreducible exceptional curve $E_{v,q}$ of $\widetilde{X}_q$. With the notations of the previous paragraph, if nl' has the form $\sum_v n_v E_v$ write $\mathrm{div}_\lambda(nl') := \sum_v n_v \mathcal{E}_v$ for the corresponding divisor in $\lambda^{-1}(U)$. Since U is contractible, one has $H^2(\lambda^{-1}(U), \mathbb{Z}) = L'$ and $H^1(\lambda^{-1}(U), \mathbb{Z}) = 0$, hence the exponential exact sequence on $\lambda^{-1}(U)$ gives

$$0 \to \mathrm{Pic}^0(\lambda^{-1}(U)) \longrightarrow \mathrm{Pic}(\lambda^{-1}(U)) \xrightarrow{c_1} L' \to H^2(\lambda^{-1}(U), \mathcal{O}_{\lambda^{-1}(U)}). \tag{3.5.1}$$

Lemma 3.5.2. *$H^2(\lambda^{-1}(U), \mathcal{O}_{\lambda^{-1}(U)}) = 0$ and the first Chern class morphism c_1 in (3.5.1) is onto.*

Proof. We use the Leray spectral sequence. Recall, see e.g. EGA III.2 §7, or [32], that if $q \mapsto h^i(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ is constant over some open set U (and all i) then $R^i\lambda(\tilde{r})_*\mathcal{O}_{\mathcal{Z}(\tilde{r})}$ is locally free over U and $R^i\lambda(\tilde{r})_*\mathcal{O}_{\mathcal{Z}(\tilde{r})} \otimes_{\mathcal{O}_U} \mathbb{C}(q) \to H^i(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ is an isomorphism for $q \in U$.

Hence, since $R^i\lambda_*\mathcal{O}_{\lambda^{-1}(U)}$ is locally free, $H^i(U, R^{2-i}\lambda_*\mathcal{O}_{\lambda^{-1}(U)}) = 0$ for $i > 0$. On the other hand, $R^2\lambda_*\mathcal{O}_{\lambda^{-1}(U)} = 0$ since $R^2\lambda_*\mathcal{O}_{\lambda^{-1}(U)} \otimes_{\mathcal{O}_U} \mathbb{C}(q) \to H^2(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ is an isomorphism and $H^2(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q}) = 0$ by dimension argument. $\square$

Then, if in the above construction of the split of c_1 in (3.4.1) we replace $\widetilde{X}$ by $\lambda^{-1}(U)$ and $\mathrm{div}(nl')$ by $\mathrm{div}_\lambda(nl')$, we get the following statement.

Lemma 3.5.3. *For any $l' \in L'$ there exists a divisor $D_\lambda(l')$ in $\lambda^{-1}(U)$ such that one has a linear equivalence $nD_\lambda(l') \sim \mathrm{div}_\lambda(nl')$ in $\lambda^{-1}(U)$ and $c_1(\mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l')) = l'$. Furthermore, $D_\lambda(l')$ is unique up to linear equivalence, hence $l' \mapsto \mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))$ is a split of (3.5.1) which extends the natural split $L \ni \sum_v m_vE_v \mapsto \mathcal{O}_{\lambda^{-1}(U)}(\sum_v m_v\mathcal{E}_v)$ over L. Since $\mathrm{Pic}^0(\lambda^{-1}(U)) = H^1(\lambda^{-1}(U), \mathcal{O}_{\lambda^{-1}(U)})$ is torsion free, there exists a unique split over L' with this extension property.*

Let us summarize what we obtained: For any $q_0 \in Q \setminus \Delta^0(\tilde{r})$, and small ball U with $q_0 \in U \subset Q \setminus \Delta^0(\tilde{r})$, we have defined for each $l' \in L'$ a line bundle $\mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))$ in $\mathrm{Pic}(\lambda^{-1}(U))$, such that its restriction to each fiber $Z(\tilde{r})_q$ is the line bundle $\mathcal{O}_{Z(\tilde{r})_q}(l')$. Let us denote it by $\mathcal{O}_{\lambda^{-1}(U)}(l')$.

3.6. The semicontinuity of $q \mapsto h^1(Z_q, \mathcal{O}_{Z_q}(l'))$. We fix a complete deformation $\lambda : \mathcal{Z}(\tilde{r}) \to Q$, and we consider the set of multiplicities $r_v \leq \tilde{r}_v$, not all zero, as in 3.3. Then, for each r, we have a restricted deformation $\lambda(r) : \mathcal{Z}(r) \to Q$ of $Z(r)$ as in 3.5.

Lemma 3.6.1. *For any restricted natural line bundle the map $q \mapsto h^i(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$ is semicontinuous over $Q \setminus \Delta^0(\tilde{r})$, for $i = 0, 1$.*

(Note that if each $r_v > 1$ then the intersection form on $\Gamma(r)$ is well–defined. In particular, the semicontinuities of h^0 and h^1 are equivalent, since $h^0 - h^1 = (Z(r), l') + \chi(Z(r))$ by Riemann–Roch.)

Proof. We fix a small ball U in $Q \setminus \Delta^0(\tilde{r})$ as in subsection 3.5, and we run $q \in U$.

Let us denote (as above) the exceptional curves in the fiber $\lambda(r)^{-1}(q)$ by $\{E_{v,q}\}_v$, hence the cycle $Z(r)_q$ is $\sum_v r_vE_{v,q}$. Then one has the short exact sequence of sheaves

$$0 \to \mathcal{O}_{Z(r)_q} \otimes \mathcal{O}_{\lambda^{-1}(U)}(l') \to \oplus_v\mathcal{O}_{r_vE_{v,q}} \otimes \mathcal{O}_{\lambda^{-1}(U)}(l') \to \oplus_{(v,w)}\mathbb{C}\{x,y\}/(x^{r_v}, y^{r_w}) \to 0,$$

where the sum in the last term runs over the edges (v, w) of $\Gamma(r)$. This gives the Mayer–Vietoris exact sequence

$$0 \to H^0(Z(r)_q, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{Z(r)_q}) \to \oplus_v H^0(r_v E_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}})$$
$$\xrightarrow{\delta} \oplus_{(v,w)} \mathbb{C}\{x,y\}/(x^{r_v}, y^{r_w}) \to \dots$$

Next, we analyse the vector space $H^0(r_v E_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}})$ for any v. Let us fix an arbitrary $q_0 \in U$. Note that a singularity with a resolution consisting only one rational irreducible divisor is taut, see [15], hence the analytic family $\{Z(\tilde{r})_q\}_q$ restricted to $\{r_v E_{v,q}\}_v$ over a small neighbourhood $U' \subset U$ of q_0 can be trivialized. Furthermore, $\mathrm{Pic}^0(r_v E_{v,q}) = 0$, hence the line bundle $\mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}}$ is uniquely determined topologically by l' and r. Hence, $\mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}}$ also can be trivialised over a small U'. In particular, by these trivializations, $H^0(r_v E_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}})$ can be replaced by the fixed $H^0(r_v E_{v,q_0}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q_0}})$, and the q–dependence is codified in the restriction morphism δ. Hence, there exists a morphism

$$\oplus_v H^0(r_v E_{v,q_0}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q_0}}) \xrightarrow{\delta(q)} \oplus_{(v,w)} \mathbb{C}\{x,y\}/(x^{r_v}, y^{r_w}) \tag{3.6.2}$$

whose kernel is $H^0(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$. Since the rank of $\delta(q)$ is semicontinuous, the statement follows for h^0. But $h^1(Z(r)_q, \mathcal{O}_{Z(r)_q}(l')) = \dim \mathrm{coker}(\delta(q)) + h^1(r_v E_{v,q}, \mathcal{O}_{\lambda^{-1}(U)}(l')|_{r_v E_{v,q}})$, and the second term in this last sum is also topological and constant (by the same argument as above), hence semicontinuity for h^1 follows as well. $\square$

3.7. The 'generic analytic structure'. Now we are ready to give the definition of the 'generic structure'. Let us fix a *complete* deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ as in 3.3 (with $\tilde{r}_v$ large) whose fibers have the topological type of $\Gamma(\tilde{r})$. Similarly as there, we consider all the other coefficient sets $r := \{r_v\}_v$ such that $r_v \leq \tilde{r}_v$ for all v, not all zero, and the induced deformations $\lambda(r) : \mathcal{Z}(r) \to Q$. Then for any $l' \in L'$ consider

$$\mathrm{MIN}(l', r) := \min_{q \in Q \setminus \Delta^0(\tilde{r})} \{h^1(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))\} \tag{3.7.1}$$

and

$$\Delta(l', r) := \text{closure of } \{q \in Q \setminus \Delta^0(\tilde{r}) \,:\, h^1(Z(r)_q, \mathcal{O}_{Z(r)_q}(l')) > \mathrm{MIN}(l', r)\}. \tag{3.7.2}$$

Then $\Delta(l', r)$ is a closed (reduced) proper subspace of Q (for this use e.g. an argument as in the proof of Lemma 3.6.1, or [34,35]). Then set the countable union of closed proper subspaces $\Delta(\tilde{r}) := (\cup_{l' \in L'} \ \cup_{r_v \leq \tilde{r}_v} \Delta(l', r)) \cup \Delta^0(\tilde{r})$. Clearly, $\Delta(\tilde{r}) \subsetneq Q$.

Definition 3.7.3. (a) For a fixed $\Gamma(\tilde{r})$ and for any complete deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ (with all $\tilde{r}_v \gg 0$) we say that the fiber $Z(\tilde{r})_q$ of $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ is generic if $q \in Q \setminus \Delta(\tilde{r})$.

(b) Consider a singularity (X, o) and one of its resolutions $\widetilde{X}$ with dual graph Γ. We say that the analytic type on $\widetilde{X}$ is generic if there exists $\tilde{r} \gg 0$, and a complete deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ with fibers of topological type $\Gamma(\tilde{r})$, and $q \in Q \setminus \Delta(\tilde{r})$ such that $\lambda(\tilde{r})^{-1}(q)$ is the space with structure sheaf $\mathcal{O}_{\widetilde{X}}|_{\sum_v \tilde{r}_v E_v}$.

Remark 3.7.4. (a) Fix any 1–dimensional space Z with fixed topology $\Gamma(\tilde{r})$ with all $\tilde{r}_v \gg 0$. Then in any complete deformation λ of Z there exists a generic structure arbitrary close to Z.

(b) Though the above construction does not automatically imply that $Q \setminus \Delta(\tilde{r})$ is open, for any $q_0 \in Q \setminus \Delta(\tilde{r})$ and for any *finite* set $FL' \subset L'$ there exists a small neighbourhood U of q_0 such that $h^1(\mathcal{O}_{Z(r)_q}, \mathcal{O}_{Z(r)_q}(l')) = \mathrm{MIN}(l', r)$ for any r (as above), $l' \in FL'$, and $q \in U$.

(c) Fix a complete deformation $\lambda : \mathcal{Z}(\tilde{r}) \to Q$ of some (Z, Y) with some fixed $\tilde{r}_v \gg 0$ as above. Then, by Theorem 3.2.1*(b)* for any $q \in Q$ the fiber $Z(\tilde{r})_q$ determines uniquely a holomorphic neighbourhood $\widetilde{X}_q$ of E. (Some $\{\tilde{r}_v\}_v$ very large works uniformly for all fibers, since a convenient $\{\tilde{r}_v\}_v$ can be chosen topologically.) Furthermore, $h^1(\widetilde{X}_q, \mathcal{O}_{\widetilde{X}_q})$ can be recovered from λ as $h^1(Z(\tilde{r})_q, \mathcal{O}_{Z(\tilde{r})_q})$ by the formal function theorem. This is the geometric genus of the singularity (X_q, o) obtained by contracting E in this $\widetilde{X}_q$. Since $\Delta(0, \tilde{r}) = \{q \in Q \ : \ p_g(X_q, o) \neq \mathrm{MIN}(\Gamma)\}$ is part of the discriminant $\Delta(\tilde{r})$ (and it is closed), for any 'generic' $q \in Q \setminus \Delta(\tilde{r})$ there is a ball $q \in U \subset Q \setminus \Delta(0, \tilde{r})$ such that λ simultaneously blows down to a flat family $\mathcal{X} \to U$. This follows from [34,35,38] by the constancy of Γ and p_g.

3.8. Extension of sections. Consider a complete deformation $\lambda(\tilde{r}) : \mathcal{Z}(\tilde{r}) \to Q$ as above, and *let $Z(\tilde{r})_q$ be a generic fiber* as in Definition 3.7.3. Let U be a small neighbourhood of q such that $U \subset Q \setminus \Delta^0(\tilde{r})$. For any $l' \in L'$ fixed consider the universal family of line bundles $\mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))$ constructed in subsection 3.5. Fix also some $r := \{r_v\}_v$ ($0 \leq r_v \leq \tilde{r}_v$ for all v, not all $r_v = 0$, as above). Assume that $\mathcal{O}_{Z(r)_q}(l') = \mathcal{O}_{\lambda^{-1}(U)}(D_\lambda(l'))|_{Z(r)_q}$ admits a global section $s \in H^0(Z(r)_q, \mathcal{O}_{Z(r)_q}(l'))$ without fixed components.

Lemma 3.8.1. *After decreasing U if it necessary, the following facts hold:*

(a) the section s has an extension $\mathfrak{s} \in H^0(\lambda(r)^{-1}(U), \mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l'))$ with $\mathfrak{s}_q = s$.

(b) $\mathfrak{s}_{q'}$ ($q' \in U$, $q' \neq q$) has no fixed components either.

Proof. (a) Since $Z(\tilde{r})_q$ is generic, q does not sit in the union of the discriminant spaces considered in 3.7. In that subsection we considered all the discriminants associated with all the Chern classes and the 'r–tower', hence, in particular, we had countably many discriminant obstructions. By assumption, q is not contained in any of these. In this proof we have to concentrate on the Chern class l' and the tower level $Z(r)$, hence only one discriminant. In particular, $q \in Q$ has a small neighbourhood which does not intersect it. Therefore, decreasing the representative of (Q, q) we get the stability of the corresponding h^1–cohomology sheaves. Furthermore, λ is proper, $\mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l'))$ is coherent, and $q' \mapsto h^1(Z(r)_{q'}, \mathcal{O}_{Z(r)_{q'}}(l'))$ is constant. Hence by EGA III.2 §7 (or, see e.g. [32]), $R^0\lambda_*(\mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l')))$ is locally free and $R^0\lambda_*(\mathcal{O}_{\lambda(r)^{-1}(U)}(D_\lambda(l')))\otimes_{\mathcal{O}_{(Q,q)}}\mathbb{C}(q)\to H^0(Z(r)_q,\mathcal{O}_{Z(r)_q}(l'))$ is an isomorphism. $\square$

4. A special 1–parameter deformation

4.1. Next, we describe a special 1–parameter deformation of a fixed resolution of a normal surface singularity (X,o), which will play a crucial role in the proof of the main Theorem 5.1.1.

We choose any good resolution $\phi:(\widetilde{X},E)\to(X,o)$, and write $\cup_v E_v=E=\phi^{-1}(o)$ as above. Since each E_v is rational, a small tubular neighbourhood of E_v in $\widetilde{X}$ can be identified with the disc-bundle associated with the total space $T(e_v)$ of $\mathcal{O}_{\mathbb{P}^1}(e_v)$, where $e_v=E_v^2$. (We will abridge $e:=e_v$.) Recall that $T(e)$ is obtained by gluing $\mathbb{C}_{u_0}\times\mathbb{C}_{v_0}$ with $\mathbb{C}_{u_1}\times\mathbb{C}_{v_1}$ via identification $\mathbb{C}^*_{u_0}\times\mathbb{C}_{v_0}\sim\mathbb{C}^*_{u_1}\times\mathbb{C}_{v_1}$, $u_1=u_0^{-1}$, $v_1=v_0u_0^{-e}$, where $\mathbb{C}_w$ is the affine line with coordinate w, and $\mathbb{C}^*_w=\mathbb{C}_w\setminus\{0\}$.

Next, fix any curve E_w of $\phi^{-1}(o)$ and also a *generic* point $P_w\in E_w$. There exists an identification of the tubular neighbourhood of E_w via $T(e)$ such that $u_1=v_1=0$ is P_w. By blowing up $P_w\in\widetilde{X}$ we get a second resolution $\psi:\widetilde{X}'\to\widetilde{X}$; the strict transforms of $\{E_v\}$'s will be denoted by E'_v, and the new exceptional (-1) curve by E_{new}. If we contract $E'_w\cup E_{new}$ we get a cyclic quotient singularity, which is taut, hence the tubular neighbourhood of $E'_w\cup E_{new}$ can be identified with the tubular neighbourhood of the union of the zero sections in $T(e-1)\cup T(-1)$. Here we represent $T(e-1)$ as the gluing of $\mathbb{C}_{u'_0}\times\mathbb{C}_{v'_0}$ with $\mathbb{C}_{u'_1}\times\mathbb{C}_{v'_1}$ by $u'_1=u_0'^{-1}$, $v'_1=v'_0u_0'^{-e+1}$. Similarly, $T(-1)$ as $\mathbb{C}_\beta\times\mathbb{C}_\alpha$ with $\mathbb{C}_\delta\times\mathbb{C}_\gamma$ by $\delta=\beta^{-1}$, $\gamma=\alpha\beta$. Then $T(e-1)$ and $T(-1)$ are glued along $\mathbb{C}_{u'_1}\times\mathbb{C}_{v'_1}\sim\mathbb{C}_\beta\times\mathbb{C}_\alpha$ by $u'_1=\alpha$, $v'_1=\beta$ providing a neighbourhood of $E'_w\cup E_{new}$ in $\widetilde{X}'$. Then the neighbourhood $\widetilde{X}'$ of $\cup_v E'_v\cup E_{new}$ will be modified by the following 1–parameter family of spaces: the neighbourhood of $\cup_v E'_v$ will stay unmodified, however $T(-1)$, the neighbourhood of E_{new} will be glued along $\mathbb{C}_{u'_1}\times\mathbb{C}_{v'_1}\sim\mathbb{C}_\beta\times\mathbb{C}_\alpha$ by $u'_1+t=\alpha$, $v'_1=\beta$, where $t\in(\mathbb{C},0)$ is a small holomorphic parameter. The smooth complex surface obtained in this way will be denoted by $\widetilde{X}'_t$, and the 'moved' (-1)–curve in $\widetilde{X}'_t$ by $E_{new,t}$. If we blow down $E_{new,t}$ we obtain the surface $\widetilde{X}_t$.

By construction, the family of spaces $\{\widetilde{X}'_t\}_{t\in(\mathbb{C},0)}$ form a smooth 3–fold $\widetilde{\mathcal{X}}'$, together with a flat map $\lambda':(\widetilde{\mathcal{X}}',\widetilde{X}')\to(\mathbb{C},0)$, a C^∞ trivial fibration, such that $\lambda'^{-1}(t)=\widetilde{X}'_t$. Similarly, the family $\{\widetilde{X}_t\}_{t\in(\mathbb{C},0)}$ form a smooth 3–fold $\widetilde{\mathcal{X}}$, together with a flat map $\lambda:(\widetilde{\mathcal{X}},\widetilde{X})\to(\mathbb{C},0)$, a C^∞ trivial fibration, such that $\lambda^{-1}(t)=\widetilde{X}_t$.

Remark 4.1.1. Such a deformation $\lambda:(\widetilde{\mathcal{X}},\widetilde{X})\to(\mathbb{C},0)$, reduced to some $\Gamma(\tilde r)$, say with $\tilde r\gg 0$, is always the pullback of a complete deformation of $\mathcal{O}_{\widetilde{X}}|Z(\tilde r)$. Hence, if $\widetilde{X}$ is generic, then the base point q_0 corresponding to the fiber $\mathcal{O}_{\widetilde{X}}|Z(\tilde r)$ is in $Q\setminus\Delta(\tilde r)$. Since for such q_0 there is a ball $q\in U\subset Q\setminus\Delta(0,\tilde r)$ such that λ simultaneously blows down to a flat family $\mathcal{X}\to U$ (cf. 3.7.4(c)), the deformation $\lambda:(\widetilde{\mathcal{X}},\widetilde{X})\to(\mathbb{C},0)$ also blows down to a deformation $\mathcal{X}\to(\mathbb{C},0)$ of (X,o). In fact, λ is a weak simultaneous resolution of the (topological constant) deformation $\mathcal{X}\to(\mathbb{C},0)$, cf. [17,10]. The point is that along the deformation λ automatically we will have the h^1–stabilities for *any* other finitely many restricted natural line bundles as well, cf. Remark 3.7.4(b) (that is, for the very same $\widetilde{X}$ and its deformation λ, the finitely many Chern classes — whose h^1–stability we wish — can be chosen arbitrarily, depending on the geometrical situation we treat).

5. The cohomology of restricted natural line bundles

5.1. The setup. We fix a normal surface singularity (X, o) and one of its good resolutions $\widetilde{X}$ with exceptional divisor E and dual graph Γ. For any integral effective cycle $Z = Z(r)$ whose support $|Z|$ is included in E (not necessarily the same as E) write $\mathcal{V}(|Z|)$ for the set of vertices $\{v : \ E_v \subset |Z|\}$ and $\mathcal{S}'(|Z|) \subset L'(|Z|)$ for the Lipman cone associated with the induced lattice $L(|Z|)$. As above, for any $l' \in L'$ we denote the restriction of the natural line bundle $\mathcal{O}_{\widetilde{X}}(l')$ to Z by $\mathcal{O}_Z(l')$. Denote also by $\tilde{l}$ the cohomological restriction $R(l')$ of $l' \in L'$ to $L'(|Z|)$. Recall also that for any $-\tilde{l} \in \mathcal{S}'(|Z|)$ one has the Abel map $c^{\tilde{l}} : \mathrm{ECa}^{\tilde{l}}(Z) \to \mathrm{Pic}^{\tilde{l}}(Z)$.

Theorem 5.1.1. *Assume that $\widetilde{X}$ is generic in the sense of Definition 3.7.3. Fix also some $Z = Z(r)$ as above. Choose $l' = \sum_{v\in\mathcal{V}} l'_v E_v \in L'$ such that $l'_v < 0$ for any $v \in \mathcal{V}(|Z|)$. Then the following facts hold.*

(I) Assume additionally that $-\tilde{l} \in \mathcal{S}'(|Z|) \setminus \{0\}$. Then the following facts are equivalent:
(a) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, that is, $H^0(Z, \mathcal{O}_Z(l'))_{reg} \neq \emptyset$;
(b) $c^{\tilde{l}}$ is dominant, or equivalently, for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$ one has $\mathcal{L}_{gen} \in \mathrm{im}(c^{\tilde{l}})$ (that is, $H^0(Z, \mathcal{L}_{gen})_{reg} \neq \emptyset$).
(c) $\mathcal{O}_Z(l') \in \mathrm{im}(c^{\tilde{l}})$, and for any $D \in (c^{\tilde{l}})^{-1}(\mathcal{O}_Z(l'))$ the tangent map $T_D c^{\tilde{l}} : T_D\mathrm{ECa}^{\tilde{l}}(Z) \to T_{\mathcal{O}_Z(l')}\mathrm{Pic}^{\tilde{l}}(Z)$ is surjective.

(II) $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$ and $i = 0, 1$.

(For a remark regarding the assumptions of the theorem see 6.1.1(c).)

Remark 5.1.2. The theorem shows that if we fix $\Gamma(r)$ then the restrictions of natural line bundles of generic singularities cohomologically behave similarly as the generic line bundles. This is the main guiding principle of the present article. This principle, in general, can be formulated as follows. Fix some invariant associated with line bundles of resolutions with fixed graph and fixed Chern class. Then one expects that the invariant evaluated on the restricted natural line bundle in the context of the generic singularity agrees with the value of the invariant evaluated on the generic bundle with the same topological data (associated with an arbitrary fixed analytic type).

Note that by [21, Theorem 5.3.1] the cohomology of the generic line bundles depends only on the combinatorics of Γ (for the formula see e.g. the introduction or (6.1.2)).

5.1.3. Starting the proof of Theorem 5.1.1. We use double induction over the cardinality of the subset $\mathcal{V}(|Z|) \subset \mathcal{V}$ and $\sum_v r_v$.

If $|\mathcal{V}(|Z|)| = 1$ then $\mathrm{Pic}^0(Z) = 0$ and all line bundles with the same Chern class are isomorphic, hence all the statements are trivially true for any Z and any l'. Hence let us fix some virtual support $|Z|$ and assume that all the statements are valid for any cycle with support smaller than $|Z|$ and for any l' with the corresponding restrictions.

Next, we run induction over $\sum_{v\in\mathcal{V}(|Z|)} r_v$. Assume that $r_v \leq 1$ for all v. Then $\mathrm{Pic}^0(Z) = 0$ again and both (I) and (II) hold. Hence, we assume that (I) and (II) hold for all cycles with $\sum_v r_v < N$ (and any l' with the required restrictions) and we consider some $Z = Z(r)$ with $\sum_v r_v = N$.

5.1.4. The first part of the proof of Theorem 5.1.1(I). First we verify the 'easy' implications.

$(c) \Rightarrow (b)$ Since $\mathrm{ECa}^{\tilde{l}}(Z)$ is smooth (cf. [21, Th. 3.1.10]), by local submersion theorem, if $T_D c^{\tilde{l}}$ is surjective then the germ $c^{\tilde{l}} : (\mathrm{ECa}^{\tilde{l}}(Z), D) \to (\mathrm{Pic}^{\tilde{l}}(Z), \mathcal{O}_Z(l'))$ is surjective too. Since $c^{\tilde{l}}$ is an algebraic morphism and its image contains a small analytic ball of top dimension, $c^{\tilde{l}}$ is dominant.

$(b) \Rightarrow (a)$ Since $H^0(Z, \mathcal{L}_{gen})_{reg} \neq \emptyset$, one has $h^0(Z, \mathcal{L}_{gen}) \neq 0$, hence by the semicontinuity of $\mathcal{L} \mapsto h^0(Z, \mathcal{L})$ (cf. [21, Lemma 5.2.1]) $h^0(Z, \mathcal{O}_Z(l')) \neq 0$ too. Next, assume that $h^0(Z, \mathcal{O}_Z(l'))_{reg} = \emptyset$, that is, there exists $v \in \mathcal{V}(|Z|)$ such that $h^0(Z, \mathcal{O}_Z(l')) = h^0(Z - E_v, \mathcal{O}_Z(l')(-E_v))$. Note that $\mathcal{O}_Z(l')(-E_v)|_{Z-E_v}$ is also a restricted natural line bundle, it is $\mathcal{O}_{Z-E_v}(l' - E_v)$. Furthermore, from $l'_u < 0$ for $u \in \mathcal{V}(|Z|)$ we obtain $(l' - E_v)_u < 0$ too. Therefore, by the inductive step (part II) $h^0(Z - E_v, \mathcal{O}_Z(l' - E_v)) = h^0(Z - E_v, \mathcal{L}_{gen}(-E_v))$ and by the assumption $h^0(Z - E_v, \mathcal{L}_{gen}(-E_v)) < h^0(Z, \mathcal{L}_{gen})$. Thus $h^0(Z, \mathcal{O}_Z(l')) < h^0(Z, \mathcal{L}_{gen})$, a fact, which contradicts the semicontinuity of $\mathcal{L} \mapsto h^0(Z, \mathcal{L})$.

The proof of $(a) \Rightarrow (c)$ in (I) is much harder and longer, and it is the core of the present theorem.

5.2. The proof of $(a) \Rightarrow (c)$ in short. The detailed proof is presented in 5.3; in this subsection we summarize the main steps in order to help the reading of the complete proof, though in this way inevitably some repetitions will occur. (Since the idea of the proof – based on the construction of the 1–parameter family – is quite fruitful, it will be used several times in forthcoming manuscripts as well, hence in the future work we will refer to these paragraphs as the basic prototype.)

First we identify $\mathrm{Pic}^{\tilde{l}}(Z)$ with $\mathrm{Pic}^0(Z)$ by $\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_Z(-l')$, and $\mathrm{Pic}^0(Z)$ with $H^1(Z, \mathcal{O}_Z)$, and we replace $c^{\tilde{l}}(Z)$ with $\widetilde{c}^{l'}(Z) : \mathrm{ECa}^{\tilde{l}}(Z) \to H^1(\mathcal{O}_Z)$. Therefore, we wish to show that for any $D \in (\widetilde{c}^{l'})^{-1}(0)$ the tangent map $T_D\widetilde{c}^{l'} : T_D\mathrm{ECa}^{\tilde{l}}(Z) \to T_0H^1(\mathcal{O}_Z)$ is surjective.

Assume that this is not happening. Then there exists a linear functional $\varsigma \in H^1(\mathcal{O}_Z)^*$, $\varsigma \neq 0$, such that $\varsigma|_{\mathrm{im}(T_D\widetilde{c}^{l'})} = 0$. This lifts to a nonzero functional $\widetilde{\varsigma}$ of $H^1(\mathcal{O}_{\widetilde{X}})$, which necessarily has the form $\widetilde{\varsigma} = \langle \cdot, [\widetilde{\omega}] \rangle$ for some $\widetilde{\omega} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$, which necessarily must have a pole along some E_w. Using [21] one shows that in fact we can choose $E_w \subset |Z|$.

Next, we modify $\widetilde{X}$ by a sequence of blow ups. First we blow up $\widetilde{X}$ at generic point of E_w creating the new exceptional divisor F_1, then we blow up a generic point of F_1 creating F_2, etc. The sequence of n such blow ups will be denoted by $b_n : \widetilde{X}_n \to \widetilde{X}$, which has exceptional divisors $\cup_{i=1}^n F_i$. We define ς_n by the composition $H^1(\mathcal{O}_{b_n^*(Z)}) \to H^1(\mathcal{O}_Z) \xrightarrow{\varsigma} \mathbb{C}$ (where the first arrow is an isomorphism by Leray spectral sequence); and similarly we set $\widetilde{\varsigma}_n$ associated with some $\widetilde{Z} \gg 0$ (instead of Z). Note that $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z}))$ corresponds to an integration of the 2–form $b_n^*(\widetilde{\omega})$ paired with divisors supported on F_n. Since the pole order along F_n of $b_n^*(\widetilde{\omega})$ decreases by one after each blow up, after some steps n it will have no pole along F_n, hence $\varsigma_n \circ \widetilde{c}^{-F_n^*}(b_n^*(Z)) : \mathrm{ECa}^{-F_n^*}(b_n^*(Z)) \to H^1(\mathcal{O}_{b_n^*(Z)}) \to \mathbb{C}$ is constant. Let k be the smallest integer such that this map is constant. Then $b_k^*(\widetilde{\omega})$ has a pole of order one along F_{k-1}.

Next, let $U \subset \widetilde{X}_k$ be a small tubular neighbourhood of the exceptional curve $E_U := E \cup (\cup_{i=1}^{k-1} F_i)$. Let Γ_U be the dual graph of E_U. One considers the homological projection $\pi_U : L(\Gamma) \to L(\Gamma_U)$ and the cohomological restriction $R_U : L'(\Gamma) \to L'(\Gamma_U)$ (dual to the natural homological injection of cycles). Then first one identifies the germs in the corresponding spaces of effective Cartier divisors $(\mathrm{ECa}^{\tilde{l}}(Z), D) \simeq (\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z)), D) \simeq (\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z))), D)$, then one shows that $(\mathrm{ECa}^{\tilde{l}}(Z), D) \xrightarrow{\widetilde{c}^{l'}} H^1(\mathcal{O}_Z) \xrightarrow{\varsigma} \mathbb{C}$ factorizes through $\big(\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z))), D\big) \xrightarrow{\widetilde{c}^{R_U b_k^*(l')}} H^1(\mathcal{O}_{\pi_U(b_k^*(Z))}) \xrightarrow{\varsigma_k^U} \mathbb{C}$. This, and the choice of ς show that

$$(\dagger) \qquad \varsigma_k^U \circ T_D\big(\widetilde{c}^{R_U(b_k^*(l'))}(\pi_U(b_k^*(Z))\big) = 0.$$

Now we continue with the key construction of the proof. Using the exceptional divisors F_{k-1} and F_k we construct the 1–parameter family of deformation $\{\widetilde{X}_{k,t}\}_t$ of $\widetilde{X}_k$ (by moving the intersection point of $F_{k,t}$ along F_{k-1}), as in section 4. In this deformation one considers the universal family of natural line bundles. Since in the central fiber D is the divisor of a section of the corresponding natural line bundle, and along the deformation the cohomology groups of the bundles are stable (here we use the genericity), by Lemma 3.8.1 this extends to a family of sections. In this way we construct a path in $\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}\big(\pi_U(b_k^*(Z))\big)$ at D, $t \mapsto \gamma(t)$ (or, $\{D_t\}_t$ with $D_0 = D$). By the choice of ς and $(\dagger)$ and the chain rule, $\varsigma \circ \widetilde{c} \circ \gamma$ must have zero derivative at $t = 0$. This is valid even for any common multiple of the divisors $\{D_t\}_t$. On the other hand, this derivative can be computed differently by Laufer integration. Indeed, by taking a convenient multiple, the corresponding powers of the members of the family of natural line bundles restricted on U have the form $\mathcal{O}_{\pi_U(b_k^*(Z))}(\sum_v N l'_v E_v + \ell \sum_{i=1}^{k-1} F_i + \ell F_{k,t})$ with $\ell \neq 0$. Here $\ell F_{k,t} \cap F_{k-1}$ is moving divisor along F_{k-1}. It paired with the differential form of pole one by Laufer pairing has a non-trivial linear part, cf. (2.2.5). Hence its derivative at $t = 0$ is nonzero, a fact which contradicts the previous statement.

5.3. The detailed proof of $(a) \Rightarrow (c)$. Fix any $l^* \in L'$ and write $\bar{l} \in L'(|Z|)$ for its restriction. Then there is a canonical identification of $\mathrm{Pic}^{\bar{l}}(Z)$ with $\mathrm{Pic}^0(Z)$ by

$\mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_Z(-l^*)$. Also, $\mathrm{Pic}^0(Z)$ identifies with $H^1(Z, \mathcal{O}_Z)$ by the inverse of the exponential map such that $\mathcal{O}_Z$ is identified with 0. In particular, $c^{\bar{l}}(Z) : \mathrm{ECa}^{\bar{l}}(Z) \to \mathrm{Pic}^{\bar{l}}(Z)$ can be identified with its composition with the above two maps, namely with $\widetilde{c}^{l^*}(Z) : \mathrm{ECa}^{\bar{l}}(Z) \to H^1(\mathcal{O}_Z)$. In the sequel l^* stands either for l' or for different cycles of type E_u^* with $E_u \in |Z|$. In this latter case, the restriction of $E_u^* \in L'$ is $E_u^*(|Z|)$, where this second dual is considered in $L'(|Z|)$. We use sometimes the same notation E_u^* for both of them, from the context will be clear which one is considered.

Therefore, the wished statement $(a) \Rightarrow (c)$ transforms into the following: If $D \in (\widetilde{c}^{l'})^{-1}(0)$ then the tangent map $T_D\widetilde{c}^{l'} : T_D\mathrm{ECa}^{\tilde{l}}(Z) \to T_0H^1(\mathcal{O}_Z)$ is surjective (under the assumptions of part (I)).

Assume that this is not the case for some D. Then there exists a linear functional $\varsigma \in H^1(\mathcal{O}_Z)^*$, $\varsigma \neq 0$, such that $\varsigma|_{\mathrm{im}(T_D\widetilde{c}^{l'})} = 0$. During the proof we fix such a $D \in (\widetilde{c}^{l'})^{-1}(0)$ and ς.

First, we concentrate on ς.

Lemma 5.3.1. *For any $\varsigma \in H^1(\mathcal{O}_Z)^*$, $\varsigma \neq 0$, there exists $E_w \subset |Z|$ such that $\varsigma \circ \widetilde{c}^{-E_w^*} : \mathrm{ECa}^{-E_w^*}(Z) \to \mathbb{C}$ is not constant.*

Proof. Let $\widetilde{Z} = \sum_v \tilde{r}_v E_v$ be a large cycle with all $\tilde{r}_v \gg 0$ ($v \in \mathcal{V}$) so that $h^1(\mathcal{O}_{\widetilde{Z}}) = h^1(\mathcal{O}_{\widetilde{X}})$. Define $\widetilde{\varsigma}$ by the composition $H^1(\mathcal{O}_{\widetilde{Z}}) \xrightarrow{\rho} H^1(\mathcal{O}_Z) \xrightarrow{\varsigma} \mathbb{C}$. Since ρ is onto, $\widetilde{\varsigma} \neq 0$ too. Recall that any functional on $H^1(\mathcal{O}_{\widetilde{X}})$ has the form $\widetilde{\varsigma} = \langle \cdot, [\widetilde{\omega}]\rangle$, cf. (2.2.3), for some $\widetilde{\omega} \in H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})$. Since $\widetilde{\varsigma} \neq 0$ the form necessarily must have a pole along some E_w. By combination of Theorems 6.1.9(d) and 8.1.3 of [21] we know that the kernel of ρ is dual with the subspace of forms which have no pole along $|Z|$. Therefore, $\widetilde{\omega}$ must have a pole along some $E_w \subset |Z|$. Since $\mathrm{ECa}^{-E_w^*}(Z)$ is the space of effective Cartier divisors of $\widetilde{X}$ (up to the equation of Z), which intersect (transversally) only E_w, again by local nature of the integration formula, $\widetilde{\varsigma} \circ \widetilde{c}^{-E_w^*}(\widetilde{Z}) : \mathrm{ECa}^{-E_w^*}(\widetilde{Z}) \to \mathbb{C}$ is nonconstant, cf. (2.2.5). But $\varsigma \circ \widetilde{c}^{-E_w^*}(Z)$ composed with $R : \mathrm{ECa}^{-E_w^*}(\widetilde{Z}) \to \mathrm{ECa}^{-E_w^*}(Z)$ is exactly this map $\widetilde{\varsigma} \circ \widetilde{c}^{-E_w^*}(\widetilde{Z})$. Since R is surjective (cf. [21, Theorem 3.1.10]), $\varsigma \circ \widetilde{c}^{-E_w^*}(Z)$ is nonconstant too. $\square$

5.3.2. Let Z, ς and $E_w \subset |Z|$ be as in Lemma 5.3.1, and $\widetilde{\omega}$ as in its proof, $\widetilde{\varsigma} = \langle \cdot, [\widetilde{\omega}]\rangle$. We wish to modify the resolution $\widetilde{X}$ (and the space Z) dictated by a certain property of $\widetilde{\omega}$. For this we blow up $\widetilde{X}$ at generic point of E_w creating the new exceptional divisor F_1, then we blow up a generic point of F_1 creating the new exceptional divisor F_2, etc. The sequence of n such blow ups will be denoted by $b_n : \widetilde{X}_n \to \widetilde{X}$, which has exceptional divisors $\cup_{i=1}^n F_i$. Note also that $H^1(\mathcal{O}_{b_n^*(Z)}) \to H^1(\mathcal{O}_Z)$ is an isomorphism (use Leray spectral sequence). We define ς_n by the composition $H^1(\mathcal{O}_{b_n^*(Z)}) \to H^1(\mathcal{O}_Z) \xrightarrow{\varsigma} \mathbb{C}$.

Lemma 5.3.3. *For n sufficiently large the next morphism is constant:*

$$\varsigma_n \circ \widetilde{c}^{-F_n^*}(b_n^*(Z)) : \mathrm{ECa}^{-F_n^*}(b_n^*(Z)) \to H^1(\mathcal{O}_{b_n^*(Z)}) \to \mathbb{C}. \tag{5.3.4}$$

Proof. Consider $\widetilde{Z}$ and the notations of the proof of Lemma 5.3.1, and the composition $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z}))$, similar to (5.3.4), but with $\widetilde{Z}$ instead of Z. This for any n gives the diagram

$$\begin{array}{ccccc} \mathrm{ECa}^{-F_n^*}(b_n^*(\widetilde{Z})) & \stackrel{\widetilde{c}^{-F_n^*}}{\longrightarrow} & H^1(\mathcal{O}_{b_n^*(\widetilde{Z})}) & \stackrel{\widetilde{\varsigma}_n}{\longrightarrow} & \mathbb{C} \\ \downarrow R_n & & \downarrow & & \downarrow \simeq \\ \mathrm{ECa}^{-F_n^*}(b_n^*(Z)) & \stackrel{\widetilde{c}^{-F_n^*}}{\longrightarrow} & H^1(\mathcal{O}_{b_n^*(Z)}) & \stackrel{\varsigma_n}{\longrightarrow} & \mathbb{C} \end{array} \tag{5.3.5}$$

Note that $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z}))$ corresponds to an integration of the 2–form $b_n^*(\widetilde{\omega})$ paired with a divisor supported on F_n (cf. 2.2). Since the pole order along F_n of $b_n^*(\widetilde{\omega})$ decreases by one after each blow up, after some steps n it will have no pole along F_n, hence $\widetilde{\varsigma}_n \circ \widetilde{c}^{-F_n^*}(b_n^*(\widetilde{Z})) = \varsigma_n \circ \widetilde{c}^{-F_n^*}(b_n^*(Z)) \circ R_n$ is constant. Since R_n is surjective (see e.g. [21, Theorem 3.1.10]), the statement follows. □

5.3.6. In the sequel, let $k \geq 1$ be the smallest integer such that $\varsigma_k \circ \widetilde{c}^{-F_k^*}(b_k^*(Z))$ is constant. Consider again $\widetilde{Z}$ as in the proof of Lemmas 5.3.1 and 5.3.3. The functionals ς_{k-1}, ς_k, $\widetilde{\varsigma}_{k-1}$ and $\widetilde{\varsigma}_k$ (as in 5.3.2 and (5.3.5)) form the following commutative diagram:

$$\begin{tikzcd} H^1(\mathcal{O}_{b_k^*(\widetilde{Z})}) \arrow[r, "\simeq"] \arrow[d] \arrow[rr, bend left, "\widetilde{\varsigma}_k"] & H^1(\mathcal{O}_{b_{k-1}^*(\widetilde{Z})}) \arrow[r, "\widetilde{\varsigma}_{k-1}"] \arrow[d] & \mathbb{C} \arrow[d, "\simeq"] \\ H^1(\mathcal{O}_{b_k^*(Z)}) \arrow[r, "\simeq"] \arrow[rr, bend right, "\varsigma_k"'] & H^1(\mathcal{O}_{b_{k-1}^*(Z)}) \arrow[r, "\varsigma_{k-1}"] & \mathbb{C} \end{tikzcd} \tag{5.3.7}$$

By the choice of k and by the diagrams (5.3.5)–(5.3.7) $\widetilde{\varsigma}_{k-1} \circ \widetilde{c}^{-F_{k-1}^*}(b_k^*(\widetilde{Z}))$ is nonconstant, while $\widetilde{\varsigma}_k \circ \widetilde{c}^{-F_k^*}(b_k^*(\widetilde{Z}))$ is constant. Therefore, $b_k^*(\widetilde{\omega})$ has a pole of order one along F_{k-1}. In particular, the maps $\mathrm{ECa}^{-F_{k-1}^*}(b_k^*(V)) \to H^1(\mathcal{O}_{b_k^*(V)}) \to \mathbb{C}$ (where V is either $\widetilde{Z}$ or Z) depend only on the reduced structure of $b_k^*(V)$ along F_{k-1}, and they all can be identified with the map represented by Laufer's integration pairing. (For this check the integrals from 2.2 for a 2–form with pole of order one.)

5.3.8. In Lemma 5.3.3 and in the discussion from 5.3.6 one can replace in $\mathrm{ECa}^{-F_{k-1}^*}$ and in $\mathrm{ECa}^{-F_k^*}$ the cycles F_{k-1}^* and F_k^* by any multiple of them: NF_{k-1}^* and NF_k^* respectively, for any $N \in \mathbb{Z}_{>0}$. Indeed, the space of divisors has a natural 'additive' structure, namely a dominant map $s^{l'_1,l'_2}(V) : \mathrm{ECa}^{l'_1}(V) \times \mathrm{ECa}^{l'_2}(V) \to \mathrm{ECa}^{l'_1+l'_2}(V)$ which satisfies $\widetilde{c}^{l'_1+l'_2} \circ s^{l'_1,l'_2} = \widetilde{c}^{l'_1} + \widetilde{c}^{l'_2}$. Therefore, if for $n = k-1$ or $n = k$ the image $\mathrm{im}(\widetilde{c}^{-F_n^*})$ belongs to an affine subspace A of $H^1(\mathcal{O}_{b_n^*(Z)})$, then $\mathrm{im}(\widetilde{c}^{-NF_n^*})$ belongs to $NA := A + \cdots + A$ too. In particular, $\varsigma_{k-1} \circ \widetilde{c}^{-NF_{k-1}^*}(b_k^*(Z))$ is nonconstant, while $\varsigma_k \circ \widetilde{c}^{-NF_k^*}(b_k^*(Z))$ is constant. (Compare also with the ℓ–dependence in (2.2.4).) Furthermore, the discussion from 5.3.6 can be repeated for any N, the composed maps depend only on the reduced structure of $b_k^*(Z)$, hence Z can be replaced by any large $\widetilde{Z}$, in which case the composition can be computed by Laufer's integration duality formula.

This shows that one has a factorization (where $V = \widetilde{Z}$ or Z, and $\varsigma_{V,k} = \widetilde{\varsigma}_k$ or ς_k respectively)

$$\begin{tikzcd}
\mathrm{ECa}^{-NF_{k-1}^*}(b_k^*(V)) \arrow[r, "\widetilde{c}^{-NF_{k-1}^*}"] \arrow[d, two heads] & H^1(\mathcal{O}_{b_k^*(V)}) \arrow[r, "\varsigma_{V,k}"] & \mathbb{C} \\
\mathrm{ECa}^{-NF_{k-1}^*}(F_{k-1}) \arrow[urr, bend right] & &
\end{tikzcd} \tag{5.3.9}$$

Though in (5.3.9) this factorization through $\mathrm{ECa}^{-NF_{k-1}^*}(F_{k-1})$ exists (and it is nonconstant), a factorization through $\mathrm{ECa}^{-NF_{k-1}^*}(F_{k-1}) \to H^1(\mathcal{O}_{F_{k-1}})$ definitely does not exist (because, e.g., $H^1(\mathcal{O}_{F_{k-1}}) = 0$). On the other hand, a factorization through a nontrivial quotient of $H^1(\mathcal{O}_{b_k^*(V)}) = H^1(\mathcal{O}_V)$ does exist, a fact which will be crucial later. This is what we explain next.

5.3.10. In the space of resolution $\widetilde{X}_k$ let $U \subset \widetilde{X}_k$ be a small tubular neighbourhood of the exceptional curve $E_U := E \cup (\cup_{i=1}^{k-1} F_i)$. Let Γ_U be the dual graph of E_U. (Note that contracting E_U in U provides a singularity with different topological type than Γ, one of its dual graphs is Γ_U.) One can restrict sheaves/bundles from $\widetilde{X}_k$ to U. At cycle level one has the homological projection $\pi_U(\sum_v n_v E_v + \sum_{i=1}^k m_i F_i) := \sum_v n_v E_v + \sum_{i=1}^{k-1} m_i F_i$. One also has the cohomological restriction $R_U : L'(\Gamma) \to L'(\Gamma_U)$ (dual to the natural homological injection of cycles); e.g. the restriction $R_U(F_{k-1}^*)$ of F_{k-1}^* is the antidual rational cycle $F_{k-1}^*(\Gamma_U)$ associated with F_{k-1} in the lattice of Γ_U. Then, for both $V = \widetilde{Z}$ or Z, one has the natural injection (which, for $V = \widetilde{Z}$ and Z fit in a commutative diagram): $\mathrm{ECa}^{-NF_{k-1}^*}(b_k^*(V))$ is a Zariski open set in $\mathrm{ECa}^{-NR_U(F_{k-1}^*)}\big(\pi_U(b_k^*(V))\big)$. Indeed, both of them depend only on the multiplicity m_{k-1} of F_{k-1} in $b_k^*(V)$ and $\pi_U(b_k^*(V))$ (which are equal), the second set contains divisors up to the equation of $m_{k-1}F_{k-1}$ supported on $F_{k-1} \setminus F_{k-2}$ with total multiplicity N, while in the first set consists of those divisors of the second set whose support does not contain $F_{k-1} \cap F_k$.

On the other hand, the natural epimorphism $\rho_V : H^1(\mathcal{O}_{b_k^*(V)}) \to H^1(\mathcal{O}_{\pi_U(b_k^*(V))})$ usually is not a monomorphism. However, one has the following fact.

Lemma 5.3.11. *$\varsigma_{V,k} : H^1(\mathcal{O}_{b_k^*(V)}) \to \mathbb{C}$ factors through $\rho_V : H^1(\mathcal{O}_{b_k^*(V)}) \to H^1(\mathcal{O}_{\pi_U(b_k^*(V))})$.*

Proof. First, we concentrate on the map $\widetilde{c}^{-F_k^*} : \mathrm{ECa}^{-F_k^*}(b_k^*(V)) \to H^1(\mathcal{O}_{b_k^*(V)})$. Let A be the smallest affine subspace of $H^1(\mathcal{O}_{b_k^*(V)})$ which contains $\mathrm{im}(\widetilde{c}^{-F_k^*})$, and let A_0 be the parallel linear subspace of the same dimension. As above, we denote the sum $A + \cdots + A$ (m times) by mA, clearly all of these affine subspaces have the same dimension, and are parallel to each other. Next, consider also the 'multiples' $\widetilde{c}^{-mF_k^*} : \mathrm{ECa}^{-mF_k^*}(b_k^*(V)) \to H^1(\mathcal{O}_{b_k^*(V)})$ (cf. [21, §6], or see 5.3.8). Therefore, $\mathrm{im}(\widetilde{c}^{-mF_k^*}) \subset mA$, and in fact, by [21, Theorem 6.1.9], for $m \gg 0$, they agree. Furthermore, by the same theorem, $A_0 = \ker(\rho_V)$.

By the choice of k, $\varsigma_{V,k}$ restricted on the image of $\widetilde{c}^{-F_k^*}$ is constant, which means that $\varsigma_{V,k}|A$ is constant, or $A_0 \subset \ker(\varsigma_{V,k})$. Hence $\ker(\rho_V) \subset \ker(\varsigma_{V,k})$, and $\varsigma_{V,k}^U$ with $\varsigma_{V,k}^U \circ \rho_V = \varsigma_{V,k}$ exists. $\square$

This lemma has the following geometric interpretation. If $\varsigma_{V,k} = \langle \cdot, [b_k^*\widetilde{\omega}] \rangle$ (at the level of V or $\widetilde{X}_k$), then $\varsigma_{V,k}^U = \langle \cdot, [b_k^*\widetilde{\omega}|_U] \rangle$ at the level of U. The form $b_k^*\widetilde{\omega}|_U$ again has order one along F_{k-1} and all the local integration formulas along E_U are the same.

5.3.12. Next, we concentrate on the divisor $D \in \mathrm{ECa}^{\tilde{l}}(Z)$ and on the line bundle $\mathcal{O}_Z(l') = \mathcal{O}_Z(D)$. As the center of blow up of b_1 is generic on E_w, we can assume that it is not in the support of D. This guarantees that the divisor D lifts canonically into any of the spaces $\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z))$ (still denoted by D), and the germs $(\mathrm{ECa}^{\tilde{l}}(Z), D)$ and $(\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z)), D)$ are canonically isomorphic.

Furthermore, this germ is preserved under the restriction to U (see also the argument from 5.3.10), hence all these facts together with the existence of factorization from Lemma 5.3.11 can be inserted in the following commutative diagram:

$$\begin{array}{ccccc} \big(\mathrm{ECa}^{\tilde{l}}(Z), D\big) & \xrightarrow{\widetilde{c}^{l'}} & H^1(\mathcal{O}_Z) & \xrightarrow{\varsigma} & \mathbb{C} \\ \uparrow\simeq & & b_n' \uparrow\simeq & & \uparrow\simeq \\ \big(\mathrm{ECa}^{b_k^*(\tilde{l})}(b_k^*(Z)), D\big) & \xrightarrow{\widetilde{c}^{b_k^*(l')}} & H^1(\mathcal{O}_{b_k^*(Z)}) & \xrightarrow{\varsigma_k} & \mathbb{C} \\ \downarrow\simeq & & \rho_Z \downarrow & & \downarrow\simeq \\ \big(\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z))), D\big) & \xrightarrow{\widetilde{c}^{R_U b_k^*(l')}} & H^1(\mathcal{O}_{\pi_U(b_k^*(Z))}) & \xrightarrow{\varsigma_k^U} & \mathbb{C} \end{array} \tag{5.3.13}$$

This diagram shows that $\varsigma_k \circ T_D(\widetilde{c}^{b_k^*(l')}(b_k^*(Z))) = 0$ and also

$$\varsigma_k^U \circ T_D\big(\widetilde{c}^{R_U(b_k^*(l'))}(\pi_U(b_k^*(Z))\big) = 0. \tag{5.3.14}$$

5.3.15. On $b_k^*(Z)$ now we have the pullback line bundle $b_k^*(\mathcal{O}_Z(l')) = b_k^*(\mathcal{O}_Z(D)) = \mathcal{O}_{b_k^*(Z)}(D)$.

Lemma 5.3.16. $b_k^*(\mathcal{O}_{\widetilde{X}}(l')) = \mathcal{O}_{\widetilde{X}_k}(b_k^*(l'))$, *that is, the pullback of the natural line bundle* $\mathcal{O}_{\widetilde{X}}(l')$ *is the natural line bundle associated with the Chern class* $b_k^*(l')$*. Therefore,* $b_k^*(\mathcal{O}_Z(l')) = \mathcal{O}_{\widetilde{X}_k}(b_k^*(l')|_{b_k^*(Z)})$ *(which will be denoted by* $\mathcal{O}_{b_k^*(Z)}(b_k^*(l'))$*).*

Proof. A bundle is natural if one of its power has the form $\mathcal{O}(l)$ for some integral cycle l. In this case the Chern classes of the two bundles agree. Furthermore, if nl' is integral for certain $n \in \mathbb{Z}_{>0}$, then $b_k^*(\mathcal{O}_{\widetilde{X}}(l')^{\otimes n}) = \mathcal{O}_{\widetilde{X}_k}(b_k^*(nl'))$, hence $b_k^*(\mathcal{O}_{\widetilde{X}}(l'))$ is natural with Chern class $b_k^*(l')$. $\square$

After all these preparations, we start with the key construction of the proof. We will construct a path in $\mathrm{ECa}^{R_U(b_k^*(\tilde{l}))}(\pi_U(b_k^*(Z)))$ at D, $t \mapsto \gamma(t)$ (or, $\{D_t\}_t$ with $D_0 = D$) with the following properties. Firstly, by the choice of ς and (5.3.14) $\varsigma \circ \widetilde{c} \circ \gamma$ must have zero derivative at $t = 0$. On the other hand, we will compute by integration explicitly $\varsigma \circ \widetilde{c} \circ \gamma$ and we will show that its linear part is nontrivial, hence its derivative at $t = 0$ is nonzero, a fact which leads to a contradiction.

The local path of divisors will be constructed via a deformation, based on section 4.

5.3.17. A special deformation of the analytic structure of $\mathcal{O}_{\widetilde{X}_k}$. Let $(\widetilde{X}_k, E \cup \cup_{i=1}^k F_i)$ be the resolution as in 5.3.2, with the choice of k as in 5.3.6. Here we concentrate on the exceptional components F_{k-1} and F_k, where F_k is obtained by blowing up a generic point P. (If $k = 1$ then $F_{k-1} = E_w$.) Then for the pair (F_{k-1}, F_k) we apply the construction of section 4, that is, we move F_k and its intersection point with F_{k-1} locally along F_{k-1}. In this way we obtain a 1–parameter family of deformations of the resolution $\widetilde{X}_k$, denoted by $\lambda_k : (\widetilde{\mathcal{X}}_k, \widetilde{X}_k) \to (\mathbb{C}, 0)$, with fibers $\widetilde{X}_{k,t}$. In $\widetilde{X}_{k,t}$ the exceptional curve has components $E \cup \cup_{i=1}^{k-1} F_i \cup F_{k,t}$. If we blow down the F–type curves in $\widetilde{X}_{k,t}$ we get a resolution $\widetilde{X}_t$, they form a family $(\widetilde{\mathcal{X}}, \widetilde{X})$. If we contract all the exceptional curves we get a family of singularities $\{(X_t, o)\}_t$. Since the analytic structure we started with is generic, the geometric genus $h^1(\mathcal{O}_{\widetilde{X}_{k,t}})$ stays constant and the deformation blows down to a deformation $(\mathcal{X}, X) \to (\mathbb{C}, 0)$ with fibers X_t (cf. 4). We denote the contraction $\widetilde{\mathcal{X}}_k \to \widetilde{\mathcal{X}}$ by the same symbol b_k.

We assume that the base space of λ is so small that the universal map $(\mathbb{C}, 0) \to Q$ to the base space of a complete deformation omits the discriminant $\Delta(\tilde{r})$; this fact is guaranteed by the choice of the generic structure of the singularity.

Therefore, for the very same $l' \in L'$ (which provides the bundle $\mathcal{O}_Z(l')$) we can consider the universal line bundles constructed in Lemma 3.5.3, namely $\mathcal{O}_{\widetilde{\mathcal{X}}_k}(b_k^*(l')) \in \mathrm{Pic}(\widetilde{\mathcal{X}}_k)$ and $\mathcal{O}_{\widetilde{\mathcal{X}}}(l') \in \mathrm{Pic}(\widetilde{\mathcal{X}})$. By similar argument as in Lemma 5.3.16 we have $b_k^*(\mathcal{O}_{\widetilde{\mathcal{X}}}(l')) = \mathcal{O}_{\widetilde{\mathcal{X}}_k}(b_k^*(l'))$. The restriction to the fibers of the deformations are the natural line bundles of the fibers.

Corresponding to the irreducible exceptional curves $\{E_v\}_v$ and $\{F_i\}_{i=1}^k$ in $\widetilde{X}_k$ we have the irreducible exceptional surfaces $\{\mathcal{E}_v\}_v$ and $\{\mathcal{F}_i\}_{i=1}^k$ in $\widetilde{\mathcal{X}}_k$. (Here $(\mathcal{F}_n)_t = F_n$ for $n < k$ but $(\mathcal{F}_k)_t = F_{k,t}$.) If $Z = \sum_v r_v E_v$ then $b_k^*(Z) = \sum_v r_v E_v + r_w \sum_{i=1}^k F_i$. Let us set $b_k^*(\mathcal{Z}) = \sum_{v \in \mathcal{V}} r_v \mathcal{E}_v + r_w \sum_{i=1}^k \mathcal{F}_i$. Then we restrict $\mathcal{O}_{\widetilde{\mathcal{X}}_k}(b_k^*(l'))$ to $b_k^*(\mathcal{Z})$ and we get $\mathcal{O}_{b_k^*(\mathcal{Z})}(b_k^*(l')) \in \mathrm{Pic}(b_k^*(\mathcal{Z}))$.

Let $\lambda : b_k^*(\mathcal{Z}) \to (\mathbb{C}, 0)$ be the projection of the deformation. The central fiber is $\mathcal{O}_{b_k^*(Z)}(b_k^*(l'))$. In particular, over $t = 0$ the bundle $\mathcal{O}_{b_k^*(Z)}(b_k^*(l'))$ has a global section s

whose divisor is D (by the definition of D from 5.3 and identification (5.3.13)). Then Lemma 3.8.1 implies the following fact.

Lemma 5.3.18. *There exists an extension* $\mathfrak{s} \in H^0\big(b_k^*(\mathcal{Z}), \mathcal{O}_{b_k^*(\mathcal{Z})}(b_k^*(l'))\big)$ *of* $s \in H^0\big(b_k^*(Z), \mathcal{O}_{b_k^*(Z)}(b_k^*(l'))\big)$ *such that* $\mathfrak{s}_0 = s$. *Furthermore,* $\mathfrak{s}_t$ *has no fixed component either.*

Let D_t be the restriction of the divisor of $\mathfrak{s}$ to the fiber over t.

Since the support of $D = D_0$ is disjoint with the center of b_1, the same is true for each D_t (for $|t| \ll 1$). Hence, in this way we get a path germ $\{\gamma_t\}_t = \{D_t\}_t$, each D_t being a divisor of the bundle $\mathcal{O}_{b_k^*(\mathcal{Z})_t}(D_t) = \mathcal{O}_{b_{k,t}^*(Z)}(D_t) = \mathcal{O}_{b_{k,t}^*(Z)}(b_{k,t}^*(l'))$, where $b_{k,t}$ is the contraction/blow up $\widetilde{X}_{k,t} \to \widetilde{X}_t$.

Note also that in the cycles $b_{k,t}^*(Z)$ the curve $F_{k,t}$ (with its stable multiplicity) is 'moving' along the deformation, the other components with their multiplicities are stable, and the divisors D_t are supported by this stable part (but they might move). More precisely, by the construction from 5.3.17 we obtain that $\pi_U(b_k^*(\mathcal{Z})_t)$ is t–independent, and it equals $\pi_U(b_k^*(Z))$. (It is worth to mention that $\pi_U(b_k^*(Z))$ is not the same as $b_{k-1}^*(Z)$, they differ even topologically at Euler number level.)

Then, by the choice of ς and D and the chain rule (compare also with (5.3.13) and (5.3.14)):

$$\begin{gathered}
\frac{d}{dt}\Big|_{t=0}\Big(\varsigma_k \circ \widetilde{c}^{\,b_{k,t}^*(l')}(b_{k,t}^*(Z))(\gamma(t))\Big) \overset{(1)}{=\!=} \frac{d}{dt}\Big|_{t=0}\Big(\varsigma_k^U \circ \widetilde{c}^{\,R_U(b_{k,t}^*(l'))}(\pi_U(b_{k,t}^*(Z)))(\gamma(t))\Big) \overset{(2)}{=\!=} \\
T_D\Big(\varsigma_k^U \circ \widetilde{c}^{\,R_U(b_k^*(l'))}(\pi_U(b_k^*(Z)))\Big)\Big(\frac{d\gamma}{dt}\Big|_{t=0}\Big) \overset{(3)}{=\!=} \\
\varsigma_k^U \circ T_D\Big(\widetilde{c}^{\,R_U(b_k^*(l'))}(\pi_U(b_k^*(Z)))\Big)\Big(\frac{d\gamma}{dt}\Big|_{t=0}\Big) \overset{(4)}{=\!=} 0.
\end{gathered} \tag{5.3.19}$$

Above, (1) follows from the fact that the support of each D_t is in U, (2) from the definition of T_D, (3) from the chain rule and from the fact that ς_k^U is linear, and (4) from (5.3.14).

The same is valid if we replace the family D_t by any of its multiple $N \cdot D_t$.

5.3.20. Let us summarize what we have. On each $b_{k,t}^*(Z)$ we can consider the restricted natural line bundle $\mathcal{O}_{b_{k,t}^*(Z)}(b_{k,t}^*(l'))$. Then, if we take its restriction to U, namely $\mathcal{O}_{b_{k,t}^*(Z)}(b_{k,t}^*(l'))|_U \in \mathrm{Pic}(\pi_U(b_k^*(Z)))$ and we shift it back with the natural line bundle $\mathcal{O}_{\pi_U(b_k^*(Z))}(R_U(b_k^*(l')))^{-1}$ we get a path in $\mathrm{Pic}^0(\pi_U(b_k^*(Z))) = H^1(\mathcal{O}_{\pi_U(b_k^*(Z))})$, whose differential at $t = 0$ is in the kernel of ς_k^U.

Now, let us compute these objects directly, in fact, for a certain N–multiple of the corresponding bundles. Let N be an integer so that $Nl' = \sum_v Nl'_v E_v$ is an integral cycle and write $\ell := Nl'_w$. Then, $Nb_k^*(l') = \sum_v Nl'_v E_v + \ell \sum_{i=1}^k F_i$. Furthermore, $(\mathcal{O}_{b_{k,t}^*(Z)}(b_{k,t}^*(l')))^N$, being natural with integral Chern class, should equal $\mathcal{O}_{b_{k,t}^*(Z)}(\sum_v Nl'_v E_v + \ell \sum_{i=1}^k F_{i,t})$ and its restriction to U is $\mathcal{O}_{\pi_U(b_k^*(Z))}(\sum_v Nl'_v E_v +$

$\ell\sum_{i=1}^{k-1}F_i+\ell F_{k,t})$. By the same reason, $\mathcal{O}_{\pi_U(b_k^*(Z))}(R_U(b_k^*(l')))^{-N}$ is $\mathcal{O}_{\pi_U(b_k^*(Z))}(\sum_v Nl'_vE_v + \ell\sum_{i=1}^k F_i)$. Hence, the N–multiple of the path is $\mathcal{O}_{\pi_U(b_k^*(Z))}(\ell(P_t - P))$, where $P_t = F_{k,t}\cap F_{k-1}$, $P = F_k\cap F_{k-1}$ as above. By assumption on l'_w we have $\ell\neq 0$.

That is, $\mathcal{O}_{\pi_U(b_k^*(Z))}(\ell P_t - \ell P)$ is a path in $H^1(\mathcal{O}_{\pi_U(b_k^*(Z))})$ and (5.3.19) reads as

$$\frac{d}{dt}\Big|_{t=0}\big(\varsigma_k^U(\mathcal{O}_{\pi_U(b_k^*(Z))}(\ell P_t - \ell P)\big) = 0. \tag{5.3.21}$$

Next we compute the left hand side of (5.3.21) in a different way.

By Lemma 5.3.11 (and comment after it) $\varsigma_k^U = \langle\cdot,[b_k^*\widetilde{\omega}|_U\rangle$, and the form $b_k^*\widetilde{\omega}|_U$ has a pole of order one along F_{k-1}. Moreover, P is a generic point of F_{k-1} and in a local neighbourhood B of P in local coordinates (u,v) one has $F_{k-1}\cap B = \{u=0\}$, $P_t = \{v+t=0\}$. Hence (2.2.5) with $o=1$ reads as

$$\varsigma_k^U\big(\mathcal{O}_{\pi_U(b_k^*(Z))}(\ell P_t - \ell P)\big) = t\ell c + \{\text{higher order terms}\} \quad (c\in\mathbb{C}^*), \tag{5.3.22}$$

whose derivative at $t=0$ is non–zero. This contradicts (5.3.21).

5.4. The proof of part (II). Note that the equalities for $i=0$ and $i=1$ are equivalent by Riemann–Roch. We will prove (II) in three steps.

5.4.1. The proof of part (II), case 1. Assume that $l'_v<0$ for any $v\in\mathcal{V}(|Z|)$ and $-\tilde{l}\in\mathcal{S}'(|Z|)\setminus\{0\}$.

Then part (I) — already proved — can be applied.

First assume that the equivalent assumptions *(a)-(b)-(c)* of (I) are satisfied. Then by [21, Th. 4.1.1] $h^1(Z,\mathcal{L}_{gen})=0$. Hence we have to show that $h^1(Z,\mathcal{O}_Z(l'))=0$ too. Choose an element $s\in H^0(Z,\mathcal{O}_Z(l'))_{reg}$ with divisor D and consider the exact sequence of sheaves $0\to\mathcal{O}_Z\xrightarrow{\times s}\mathcal{O}_Z(l')\to\mathcal{O}_D(D)\to 0$ (where the second morphism is multiplication by s).

Then one has the cohomology exact sequence

$$H^0(Z,\mathcal{O}_Z(l'))\to\mathcal{O}_D(D)\xrightarrow{\delta}H^1(\mathcal{O}_Z)\to H^1(Z,\mathcal{O}_Z(l'))\to 0.$$

Then δ can be identified with $T_D(c^{\tilde{l}})$ (see [21, Prop. 3.2.2], or [20, p. 164], [8, Remark 5.18], [9, §5]). Since $T_D(c^{\tilde{l}})$ is onto by (I)(c), $h^1(Z,\mathcal{O}_Z(l'))=0$ follows.

Next, assume that the equivalent assumptions of (I) are not satisfied. That is, $H^0(Z,\mathcal{O}_Z(l'))_{reg} = H^0(Z,\mathcal{L}_{gen})_{reg} = \emptyset$. These facts read as $h^0(Z,\mathcal{O}_Z(l')) = \max_v\{h^0(Z-E_v,\mathcal{O}_Z(l'-E_v))\}$ and $h^0(Z,\mathcal{L}_{gen}) = \max_v\{h^0(Z-E_v,\mathcal{L}_{gen}(-E_v))\}$. But, by induction (applied for part (II) similarly as in the proof of case $(b)\Rightarrow(c)$ in 5.1.4, see also 5.1.3) $\max_v\{h^0(Z-E_v,\mathcal{O}_Z(l'-E_v))\} = \max_v\{h^0(Z-E_v,\mathcal{L}_{gen}(-E_v))\}$, hence $h^0(Z,\mathcal{O}_Z(l')) = h^0(Z,\mathcal{L}_{gen})$ follows too.

5.4.2. The proof of part (II), case 2. Assume that $l'_v < 0$ for any $v \in \mathcal{V}(|Z|)$ and $\tilde{l} = 0$. (If this happens then necessarily $|Z| < E$. Recall also that $\mathcal{O}_Z(l')$ is the restriction of the natural line bundle $\mathcal{O}_{\widetilde{X}}(l')$ to Z.)

If $h^1(\mathcal{O}_Z) = 0$ then $\mathcal{L}_{gen} = \mathcal{O}_Z(l')$, hence the statement follows. If $h^0(\mathcal{O}_Z(l')) = 0$ then by the semicontinuity of $\mathcal{L} \mapsto h^0(Z, \mathcal{L})$ (cf. [21, Lemma 5.2.1]) $h^0(\mathcal{L}_{gen}) = 0$ too.

In the sequel we assume that $h^1(\mathcal{O}_Z) \neq 0$ and $h^0(\mathcal{O}_Z(l')) \neq 0$.

Assume that $H^0(Z, \mathcal{O}_Z(l'))_{reg} \neq \emptyset$, that is, $\mathcal{O}_Z(l')$ has a section without fixed components. But, then by Chern class computation, this section has no zeros, hence $\mathcal{O}_Z(l') = \mathcal{O}_Z$, see also (2.2.1).

We claim that this identity $\mathcal{O}_Z(l') = \mathcal{O}_Z$ cannot happen for generic (X, o).

The argument runs similarly as the proof of $(a) \Rightarrow (c)$ in (I).

Since $h^1(\mathcal{O}_Z) \neq 0$ we can choose a nonzero functional $\omega \in H^1(\mathcal{O})^*$ for which we can repeat the arguments from 5.3. In particular, there exists $E_w \subset |Z|$ which satisfies Lemma 5.3.1, we can consider the sequence of blow ups as in 5.3.2, and we can choose k as in 5.3.6. Finally we consider the deformation of singularities as in 5.3.17. In this way we get a family of restricted line bundles $\mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))$, so that for $t = 0$ the corresponding bundle is the trivial one. We wish to show that for generic t the corresponding term cannot be the trivial bundle. Indeed, as in (5.3.22) we get that $t \mapsto \mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))|_U \in \mathrm{Pic}(\pi_U(b^*_k(Z)))$ is not constant. This implies that the path $t \mapsto b^*_{k,t}(\mathcal{O}_Z(l')) = \mathcal{O}_{b^*_{k,t}(Z)}(b^*_{k,t}(l'))$ cannot give for all t the trivial bundle either since otherwise its restriction to $\pi_U(b^*_k(Z))$ would be constant (since the restriction of the structures sheaf is the t–independent constant structure sheave). In particular, for generic t we have $\mathcal{O}_{Z_t}(l') \neq \mathcal{O}_{Z_t}$.

However, we can prove that in this situation necessarily $h^1(\mathcal{O}_{Z_t}(l')) < h^1(\mathcal{O}_{Z_t})$ for generic t (though the Chern classes agree), hence $t = 0$ is a jumping discriminant point of $l' \mapsto h^1(\mathcal{O}_{Z_t}(l'))$, a fact which contradict the genericity.

Indeed, since $\mathcal{O}_{Z_t}(l') \neq \mathcal{O}_{Z_t}$ for generic t (and $H^1(\mathcal{O}_{Z_t})$ is constant nonzero), $\mathcal{O}_{Z_t}(l')$ must have fix components (use $c_1(\mathcal{O}_{Z_t}(l')) = 0$ and (2.2.1)). Let $E_u \in |Z|$ be a fix component. Then $H^0(Z_t, \mathcal{O}_{Z_t}) \to H^0(E_u, \mathcal{O}_{Z_t}) = \mathbb{C}$ is surjective, while $H^0(Z_t, \mathcal{O}_{Z_t}(l')) \to H^0(E_u, \mathcal{O}_{Z_t}(l')) = \mathbb{C}$ is zero. Since their kernels have the same h^0 by the inductive step, $h^0(\mathcal{O}_{Z_t}(l')) < h^0(\mathcal{O}_{Z_t})$, hence the inequality follows by Riemann–Roch. This proves the claim.

After this discussion we can assume that $h^1(\mathcal{O}_Z) \neq 0$, $h^0(\mathcal{O}_Z(l')) \neq 0$, but $H^0(Z, \mathcal{O}_Z(l'))_{reg} = \emptyset$. By (2.2.1) $\mathcal{L}_{gen} \neq \mathcal{O}_Z$ (since $\mathrm{Pic}^0(\mathcal{O}_Z) \neq 0$), hence $H^0(Z, \mathcal{L}_{gen})_{reg} = \emptyset$ too. Then we proceed as in the last paragraph of 5.4.1, induction shows that $h^0(Z, \mathcal{O}_Z(l')) = h^0(Z, \mathcal{L}_{gen})$.

5.4.3. The proof of part (II), case 3. Finally, assume that $l'_v < 0$ for all $v \in \mathcal{V}(|Z|)$, and $-\tilde{l} \notin \mathcal{S}'(|Z|)$. Then there exists E_v in the support of Z such that $(l', E_v) = (\tilde{l}, E_v) < 0$. Hence for any $\mathcal{L} \in \mathrm{Pic}^{\tilde{l}}(Z)$ the exact sequence $0 \to \mathcal{L}(-E_v)|_{Z-E_v} \to \mathcal{L} \to \mathcal{L}|_{E_v} \to 0$ and vanishing $H^0(\mathcal{L}|_{E_v}) = 0$ give $h^0(Z - E_v, \mathcal{L}(-E_v)) = h^0(Z, \mathcal{L})$. By this step we replaced the Chern class $\tilde{l}$ by $\tilde{l} - E_v$. After finitely many such steps we necessarily get a new

Chern class in the corresponding Lipman cone (see e.g. [24, Prop. 4.3.3]). Hence, in this way we reduced this third case to the first two cases.

6. Applications. Analytic invariants

6.1. In this section we will fix a resolution graph Γ (hence, the lattice L associated with it as well), and we treat singularities (X,o), together with their resolution $\widetilde{X}$ whose dual graph is Γ. The goal is to list some consequences of Theorem 5.1.1: hence we will assume that $\widetilde{X}$ is generic, and we will provide combinatorial expressions for several analytic invariants in terms of L. We will use the notations from the setup of 5.1.

The first group of results provides topological formulae for the **cohomology of** certain **natural line bundles** over an arbitrary $Z>0$.

Remark 6.1.1. (a) By [21, Theorem 5.3.1] for any $l'\in L'$ and $\mathcal{L}_{gen}$ generic in $\mathrm{Pic}^{R(l')}(Z)$

$$h^1(Z,\mathcal{L}_{gen})=\chi(-l')-\min_{0\leq l\leq Z,l\in L}\{\chi(-l'+l)\}. \tag{6.1.2}$$

In particular, if $l'=\sum_{v\in\mathcal{V}}l'_vE_v\in L'$ satisfies $l'_v<0$ for any $v\in\mathcal{V}(|Z|)$ and $\widetilde{X}$ is generic then Theorem 5.1.1 gives the following topological characterization for the cohomology of $\mathcal{O}_Z(l')$

$$h^1(Z,\mathcal{O}_Z(l'))=\chi(-l')-\min_{0\leq l\leq Z,l\in L}\{\chi(-l'+l)\}. \tag{6.1.3}$$

This will be extended in Theorem 6.1.5 for a larger family of l'–values.

(b) Note that the identity $h^1(Z,\mathcal{O}_Z(l'))=h^1(Z,\mathcal{L}_{gen})$ (hence (6.1.3) too), in general, does not hold for an arbitrary l' (that is, without some negativity condition regarding the coefficients of l'). Indeed, assume e.g. that $|Z|=E$ and all the coefficients of Z are very large, and $l'=0$. Then using the quadratic form of χ one has $\min_{0\leq l\leq Z,l\in L}\{\chi(l)\}=\min_{l\in L_{\geq 0}}\{\chi(l)\}$, hence $h^1(Z,\mathcal{L}_{gen})=-\min_{l\in L_{\geq 0}}\{\chi(l)\}$ by (6.1.2). But $h^1(Z,\mathcal{O}_Z)=1-\min_{l\in L_{\geq 0}}\{\chi(l)\}$ whenever (X,o) is not rational, see Corollary 6.2.4.

(c) Recall that if $-l'\in\mathcal{S}'\setminus\{0\}$ then all the coefficients l'_v of l' are strictly negative. However, if the support of $|Z|$ is strict smaller than E, then $-R(l')\in\mathcal{S}'(|Z|)\setminus\{0\}$ does not necessarily imply that $l'_v<0$ for $v\in\mathcal{V}(|Z|)$. (Take e.g. $Z=E_v$ a (-2)–curve, choose E_u an adjacent vertex with it and set $l'=E_v+3E_u$. Then $-R(l')\in\mathcal{S}'(E_v)\setminus\{0\}$ however $l'_v=1$.)

6.1.4. The setup for generalization. We construct the following 'Laufer type computation sequence' (see e.g. [13] or [24, Prop. 4.3.3]). We start with a class $l'\in L'$ and an effective cycle Z with $|Z|\subset E$. Let $\tilde{l}\in L'(|Z|)$ be the restriction of l' as in Theorem 5.1.1.

Assume that $-\tilde{l}\notin\mathcal{S}'(|Z|)$. Then there exists $E_w\subset|Z|$ so that $(l',E_w)<0$. Then, for both line bundles $\mathcal{L}=\mathcal{L}_{gen}$ and $\mathcal{L}=\mathcal{O}_Z(l')$ of $\mathrm{Pic}^{\tilde{l}}(Z)$ one can consider the exact sequence $0\to\mathcal{L}(-E_w)|_{Z-E_w}\to\mathcal{L}\to\mathcal{L}|_{E_w}\to 0$, hence $h^0(\mathcal{L}(-E_w)|_{Z-E_w})=h^0(\mathcal{L})$. Hence whenever $h^0(\mathcal{O}_Z(l'-E_w)|_{Z-E_w}) = h^0(\mathcal{L}_{gen}(-E_w)|_{Z-E_w})$ one also has $h^0(\mathcal{O}_Z(l')) = h^0(\mathcal{L}_{gen})$.

Let us construct the following sequence of pairs $(l'_k, Z_k)_{k=0}^t$. By definition, $(l'_0, Z_0) = (l', Z)$ the objects we started with. If $-\tilde{l} = -R(l') \notin \mathcal{S}'(|Z|)$, then define $(l'_1, Z_1) := (l' - E_w, Z - E_w)$ for some $E_w \subset |Z|$ with $(E_w, l') < 0$. If $-\tilde{l}_1 := -R(l'_1) \notin \mathcal{S}'(|Z_1|)$ we repeat the procedure, otherwise we stop. After finitely many steps necessarily $-\tilde{l}_t := -R(l'_t) \in \mathcal{S}'(|Z_t|)$ (here $Z_t = 0$ is also possible). (The choice of the sequence is not unique, however by similar argument as in [13] or [24, Prop. 4.3.3]) one can show that the last term (l'_t, Z_t) of the sequence is independent of all the choices: it is the unique $(l'-D, Z-D)$ with D minimal such that $Z \geq D \geq 0$, $D \in L$, and $-(l'-D) \in \mathcal{S}'(|Z-D|)$.)

Theorem 6.1.5. *Assume that $\widetilde{X}$ is generic with fixed dual graph Γ, and we choose an effective cycle Z and $l' \in L'$. Assume that the last term (l'_t, Z_t) of the Laufer type computation sequence $\{(l'_k, Z_k)\}_{k=0}^t$ has the following property: if $l'_t = \sum_v l'_{t,v} E_v$, then $l'_{t,v} < 0$ for any $v \in \mathcal{V}(|Z_t|)$. Then $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ for a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{\tilde{l}}(Z)$ ($i = 0, 1$), i.e. (6.1.3) holds.*

Proof. Use Theorem 5.1.1(II) and the discussion from 6.1.4. □

Example 6.1.6. Let $\widetilde{X}$ be generic, Z an effective cycle and $l' \in L'$. Assume that $l'_v \leq 0$ for all $v \in \mathcal{V}(|Z|)$ and for any connected component Z_{con} of Z there exists $v \in \mathcal{V}$ adjacent with Z_{con} with $l'_v < 0$. (The adjacent condition is $|Z_{con}| \cap E_v \neq \emptyset$.) Then the conditions from Theorem 6.1.5 are satisfied, hence $h^i(Z, \mathcal{O}_Z(l')) = h^i(Z, \mathcal{L}_{gen})$ and (6.1.3) holds.

Indeed, first note that if for some vertex with $l'_v = 0$ one has $(l', E_v) \geq 0$ then $l'_u = 0$ for all adjacent vertices u of v. Hence, $(l', E_v) \geq 0$ for all vertices v with $l'_v = 0$ contradicts the assumption. That is, there exists $v \in \mathcal{V}(|Z|)$ so that $l'_v = 0$ and $(l', E_v) < 0$.

Then we construct the computation sequence as follows. At the first part of the computation sequence, at step (l'_k, Z_k) we choose $E_{w(k)}$ so that $E_{w(k)} \subset |Z_k|$, the $E_{w(k)}$–coefficient of l'_k is zero, and $(E_{w(k)}, l'_k) < 0$. After finitely many such steps we arrive to the situation when along the support of $Z_{k'}$ all the coefficients of $l'_{k'}$ will be strict negative. Then we can continue the algorithm arbitrarily.

Corollary 6.1.7. *If $\widetilde{X}$ is generic with dual graph Γ and $|Z|$ is connected then*

$$h^1(\mathcal{O}_Z) = 1 - \min_{0<l\leq Z, l\in L} \{\chi(l)\} = 1 - \min_{|Z|\leq l\leq Z, l\in L} \{\chi(l)\}. \tag{6.1.8}$$

Proof. For $D = |Z|$ or $D = E_v$ for any $E_v \subset |Z|$ one has

$$0 \to H^0(Z-D, \mathcal{O}_Z(-D)) \to H^0(\mathcal{O}_Z) \xrightarrow{\delta} H^0(\mathcal{O}_D) \to H^1(Z-D, \mathcal{O}_Z(-D)) \xrightarrow{\iota} H^1(\mathcal{O}_Z) \\ \to 0. \tag{6.1.9}$$

Since δ is onto ι is an isomorphism. But for $h^1(Z-D,\mathcal{O}_Z(-D))$ Example 6.1.6 and (6.1.3) hold. $\square$

6.2. The cohomology of natural line bundles over $\widetilde{X}$. Next we apply the results of the previous subsection for a cycle Z with all its coefficients very large. Recall that by Artin's Criterion $p_g=0$ (that is, (X,o) is rational) if and only if $\min_{l\in L_{>0}}\{\chi(l)\}=1$ [2,3]. Furthermore, for any singularity $\min_{l\in L_{\geq 0}}\{\chi(l)\}=\min_{l\in L}\{\chi(l)\}$, see e.g. [24, Prop. 4.3.3].

Corollary 6.2.1.

$$p_g(X,o)=1-\min_{l\in L_{>0}}\{\chi(l)\}=-\min_{l\in L}\{\chi(l)\}+\begin{cases}1 & \text{if } (X,o) \text{ is not rational,}\\ 0 & \text{else.}\end{cases} \tag{6.2.2}$$

Proof. For the first identity use (6.1.8), for the second one use Artin's Criterion for rationality. $\square$

Remark 6.2.3. (a) For *any* non–rational analytic structure (X,o) one has $p_g(X,o)\geq 1-\min_{l\in L}\{\chi(l)\}$ [37,29]. The above corollary shows that this topological bound in fact is optimal.

(b) If (X,o) is elliptic then $\min_{l\in L_{>0}}\{\chi(l)\}=0$. Hence, if the analytic structure is generic then $p_g=1-\min_{l\in L_{>0}}\{\chi(l)\}=1$. This was proved (even without the assumption that the link is a rational homology sphere) by Laufer in [16].

Corollary 6.2.4. *Assume that $\widetilde{X}$ is generic with dual graph Γ. Choose any $l'\in L'$ and consider $\mathcal{O}_{\widetilde{X}}(l')$, the natural line bundle on $\widetilde{X}$. Then*

$$h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(l'))=\chi(-l')-\min_{l\in L_{\geq 0}}\{\chi(-l'+l)\}+\epsilon(l'), \tag{6.2.5}$$

where

$$\epsilon(l')=\begin{cases}1 & \text{if } l'\in L,\ l'\geq 0, \text{ and } (X,o) \text{ is not rational,}\\ 0 & \text{else.}\end{cases}$$

Proof. For any effective cycle Z (with $|Z|=E$) and $l'\in L'$ let us write $\Delta(Z,l'):=h^1(Z,\mathcal{O}_Z(l'))-\chi(-l')+\min_{0\leq l\leq Z, l\in L}\{\chi(-l'+l)\}$. In order to compute $h^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}(l'))$ let us fix some Z with all its coefficients very large. Then, if we start with the pair (l',Z), the Laufer sequence from 6.1.4 ends with some (l'_t,Z_t) with $Z_t\geq E$ (still with large coefficients), and $-l'_t\in\mathcal{S}'$. We claim that $\Delta(Z_k,l'_k)$ is constant along the computation sequence. Indeed, from the cohomological exact sequence used in 6.1.4 (for $k=0$) $h^1(Z,\mathcal{O}(l'))=h^1(Z-E_w,\mathcal{O}(l'-E_w))-1-(E_w,l')$. Then, we compare $\min_{0\leq l\leq Z}\chi(-l'+l)$ and $\min_{0\leq l\leq Z-E_w}\chi(-l'+E_w+l)$. Since for any $x\geq 0$ with $E_w\notin|x|$ we have $\chi(-l'+E_w+x)\leq\chi(-l'+x)$, these two minima agree. Hence the claim follows.

Now, for the pair (l'_t, Z_t), with $-l'_t \in \mathcal{S}'$, we distinguish two cases. The case $l'_t = 0$ occurs exactly when $l' \in L_{\geq 0}$ (because l'_t is the largest element of $(-\mathcal{S}') \cap (l' - L_{\geq 0})$, cf. [24, Prop. 4.3.3]). In this case $\Delta(Z_t, l'_t)$ can be computed from (6.2.2). Or, $l'_t \neq 0$. In this case all the coefficients of l'_t are strict negative (use e.g. Remark 6.1.1(c)), and $\Delta(Z_t, l'_t) = 0$ by (6.1.3). $\square$

Example 6.2.6. For any $h \in H$ define $k_h := -Z_K + 2r_h$ and

$$\chi_{k_h}(x) := -(x, x + k_h)/2 = \chi(x) - (x, r_h) = \chi(x + r_h) - \chi(r_h).$$

(For the definition of r_h see 2.1.) It is known (use e.g. the algorithm from [24, Prop. 4.3.3]) that for any $h \in H$ one has $\min_{l \in L_{\geq 0}} \chi(r_h + l) = \min_{l \in L} \chi(r_h + l)$. Therefore, for $h \neq 0$ one has

$$h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) = \chi(r_h) - \min_{l \in L} \chi(r_h + l) = -\min_{l \in L}\{\chi_{k_h}(l)\} = -\min_{l \in L_{\geq 0}}\{\chi_{k_h}(l)\}. \quad (6.2.7)$$

Remark 6.2.8. (a) Let (X_{ab}, o) be the **universal abelian covering** of (X, o). Then

$$p_g(X_{ab}, 0) = \sum_{h \in H} h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)),$$

see e.g. [24]. Hence $p_g(X_{ab}, 0)$ is topologically (and explicitly) computable by (6.2.2) and (6.2.7).

(b) For a conjectural identity which connects $\min_{l \in L} \chi(r_h + l)$ with the Heegaard Floer d–invariant associated with the link of the singularity and the *spin*^{c}–structure attached to the characteristic element k_h see [26, §5.2].

6.3. The cohomological cycle of $\widetilde{X}$. For any non–rational germ and fixed resolution the set $\{Z \in L_{>0} : h^1(\mathcal{O}_Z) = p_g(X, o)\}$ has a unique minimal element Z_{coh}, called the cohomological cycle. It also satisfies the next property: $h^1(\mathcal{O}_Z) < p_g$ for any $Z \not\geq Z_{coh}$, $Z > 0$ (see e.g. [33, 4.8]).

In parallel, let us mention the following topological statement. For any fixed non–rational resolution graph, $\mathcal{M} := \{Z \in L_{>0} : \chi(Z) = \min_{l \in L} \chi(l)\}$ has a unique minimal and a unique maximal element. Indeed, if $l_1, l_2 \in \mathcal{M}$, then for $m := \min\{l_1, l_2\}$ and $M := \max\{l_1, l_2\}$ one has $\chi(M) + \chi(m) = \chi(l_1) + \chi(l_2) - (l_1 - m, l_2 - m) \leq 2 \min \chi$, hence $\chi(m) = \chi(M) = \min \chi$. Hence, $M \in \mathcal{M}$ always, and $m \in \mathcal{M}$ whenever $m \neq 0$. However, if $m = 0$ then the germ is elliptic and $\mathcal{M}$ admits a minimal element, namely the minimally elliptic cycle [16,22,23].

Corollary 6.3.1. *Assume that $\widetilde{X}$ is generic with a non–rational dual graph Γ. Then the cohomological cycle $Z_{coh} := \min\{Z \in L_{>0} : h^1(\mathcal{O}_Z) = p_g(X, o)\}$, is $\min\{Z \in L_{>0} : \chi(Z) = \min_{l \in L} \chi(l)\}$.*

6.4. The cohomological cycle of a line bundle. For any $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ with $h^1(\widetilde{X}, \mathcal{L}) > 0$ the set $L_{\mathcal{L}} := \{l \in L_{>0} \,:\, h^1(l, \mathcal{L}) = h^1(\widetilde{X}, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{coh}(\mathcal{L})$, called the cohomological cycle of $\mathcal{L}$ (and of ϕ). Similarly, for any $Z > 0$ and $\mathcal{L} \in \mathrm{Pic}(Z)$ with $h^1(Z, \mathcal{L}) > 0$ the set $L_{Z,\mathcal{L}} := \{l \in L,\ 0 < l \leq Z \,:\, h^1(l, \mathcal{L}) = h^1(Z, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{coh}(Z, \mathcal{L})$, called the cohomological cycle of $(Z, \mathcal{L})$. (For detail see e.g. [21, 5.5].)

Corollary 6.4.1. *Assume that $\widetilde{X}$ is generic.*

(a) Fix any $l' \in L'$ with $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(l')) \neq 0$. Then the set

$$L_{l'} := \{l_{min} \in L_{\geq 0} \mid \chi(-l' + l_{min}) = \min_{l \in L_{\geq 0}} \chi(-l' + l)\}$$

has a unique minimal element $Z_{coh}(l')$, which coincides with the cohomological cycle of $\mathcal{O}_{\widetilde{X}}(l')$.

(b) For any $Z > 0$ and $l' \in L'$ with $h^1(Z, \mathcal{O}_{\widetilde{X}}(l')) \neq 0$ the set

$$L_{Z,l'} := \{l_{min} \in L,\ 0 \leq l_{min} \leq Z,\ \mid \chi(-l' + l_{min}) = \min_{0 \leq l \leq Z,\, l \in L} \chi(-l' + l)\},$$

has a unique minimal element $Z_{coh}(Z, l')$, which coincides with the cohomological cycle of $\mathcal{O}_{\widetilde{X}}(l')|_Z$.

Remark 6.4.2. [21, 5.5] For any analytic structure (X, o) supported on the fixed topological type and for any resolution ϕ, fix l' such that for the generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(\widetilde{X})$ one has $h^1(\widetilde{X}, \mathcal{L}_{gen}) \neq 0$. Then the cohomology cycle of $\mathcal{L}_{gen}$ is $Z_{coh}(l')$ (independently of the analytic structure). Similarly, if $h^1(Z, \mathcal{L}_{gen}) \neq 0$ for the generic $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ then the cohomological cycle of the pair $(Z, \mathcal{L}_{gen})$ is $Z_{coh}(Z, l')$.

6.5. The Hilbert series. Fix $\widetilde{X}$ generic and let $H(\mathbf{t})$ be the multivariable (equivariant) Hilbert series associated with the divisorial filtration of the local algebra of the universal abelian covering of (X, o) associated with divisors supported on all irreducible exceptional divisors of $\widetilde{X}$; for details see e.g. [4,5,28]. Write $H(\mathbf{t}) = \sum_{l' \in L'} \mathfrak{h}(l') \mathbf{t}^{l'}$. (Here if $l' = \sum_v l'_v E_v$ then $\mathbf{t}^{l'} = \prod_v t_v^{l'_v}$.) It is known that for any l' there exists a unique $s(l') \in \mathcal{S}'$ such that $s(l') - l' \in L_{\geq 0}$, and $s(l')$ is minimal with these properties. Furthermore, for any $l' \in L'$ one has $\mathfrak{h}(l') = \mathfrak{h}(s(l'))$. Hence it is enough to determine $\mathfrak{h}(l')$ for the (closed) first quadrant (because $\mathcal{S}' \subset L'_{\geq 0}$).

Write l' as $r_h + l_0$ for some $l_0 \in L_{\geq 0}$ (and $h = [l']$). Recall that $\mathfrak{h}(l')$ is the dimension of $H^0(\mathcal{O}_{\widetilde{X}}(-r_h))/H^0(\mathcal{O}_{\widetilde{X}}(-l_0 - r_h))$, see e.g. [28, (2.3.3)]. Therefore, for $l_0 = 0$ we get $\mathfrak{h}(r_h) = 0$.

Proposition 6.5.1. *Assume that $l' = r_h + l_0$ with $l_0 > 0$. Then for $h \neq 0$*

$$\mathfrak{h}(l') = \min_{l \in L_{\geq 0}} \{\chi(l' + l)\} - \min_{l \in L_{\geq 0}} \{\chi(r_h + l)\} = \min_{l \in L_{\geq 0}} \{\chi_{k_h}(l_0 + l)\} - \min_{l \in L_{\geq 0}} \{\chi_{k_h}(l)\}. \quad (6.5.2)$$

For $h = 0$ (i.e. when $r_h = 0$ and $l' = l_0 > 0$)

$$\mathfrak{h}(l_0) = \min_{l \in L_{\geq 0}} \{\chi(l_0 + l)\} - \min_{l \in L_{\geq 0}} \{\chi(l)\} + \begin{cases} 1 & \textit{if } (X, o) \textit{ is not rational,} \\ 0 & \textit{else.} \end{cases} \tag{6.5.3}$$

Proof. Use the exact sequence $0 \to \mathcal{O}(-r_h - l_0) \to \mathcal{O}(-r_h) \to \mathcal{O}_{l_0}(-r_h) \to 0$ and Corollary 6.2.4. $\square$

Remark 6.5.4. Proposition 6.5.1 via (6.2.7) and Corollary 6.2.1 can be written h–uniformly:

$$\mathfrak{h}(r_h + l_0) = \min_{l \in L_{\geq 0}} \{\chi_{k_h}(l_0 + l)\} + h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-r_h)) \quad (\forall\ h \in H,\ l_0 \in L_{>0}).$$

6.6. The Poincaré series. Let $P(\mathbf{t})$ be the multivariable equivariant Poincaré series associated with (X, o) and its fixed resolution, cf. [4,5,28]. It is defined as $P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_{v \in \mathcal{V}} (1 - t_v^{-1})$. It is known that it is supported on $\mathcal{S}'$. Proposition 6.5.1 implies the following.

Corollary 6.6.1. *Write $P(\mathbf{t}) = \sum_{l' \in \mathcal{S}'} \mathfrak{p}(l')\mathbf{t}^{l'}$. Then $\mathfrak{p}(0) = 1$ and for $l' > 0$ one has*

$$\mathfrak{p}(l') = \sum_{I \subset \mathcal{V}} (-1)^{|I|+1} \min_{l \in L_{\geq 0}} \chi(l' + l + E_I).$$

6.7. The analytic semigroup. The analytic semigroup is defined as

$$\mathcal{S}'_{an} := \{l' \,:\, H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-l'))_{reg} \neq \emptyset\} = \{l' \,:\, \mathfrak{h}(l') < \mathfrak{h}(l' + E_v) \text{ for any } v \in \mathcal{V}\}.$$

Corollary 6.7.1. *If (X, o) is generic then $\mathcal{S}'_{an} = \{l' \,:\, \chi(l') < \chi(l' + l)$ for any $l \in L_{>0}\} \cup \{0\}$ and $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-l')) = 0$ for any $l' \in \mathcal{S}'_{an} \setminus \{0\}$.*

Proof. Use Corollary 6.2.4 and Proposition 6.5.1. $\square$

Remark 6.7.2. (a) This formula emphasizes once more the parallelism between generic line bundles (associated with an arbitrary analytic structure) and the natural line bundles associated with a generic analytic structure, cf. 5.1.2 and 6.4.2. To explain this in the present situation, consider first an arbitrary analytic structure, a resolution with fixed graph Γ, and an effective cycle $|Z|$ as usual. By [21, §4] the fact that the Abel map $c^{l'} : \mathrm{ECa}^{l'}(Z) \to \mathrm{Pic}^{l'}(Z)$ is dominant is independent of the analytic structure, and it has a purely combinatorial description: $\chi(-l') < \chi(-l' + l)$ for any $l \in L$, $0 < l \leq Z\}$. Assume that $Z \gg 0$ and $l' \neq 0$. Then a generic line bundle $\mathcal{L}_{gen} \in \mathrm{Pic}^{l'}(Z)$ is in $\mathrm{im}(c^{l'})$ if and only if $-l' \in \mathcal{S}'_{dom} := \{-l' \,:\, \chi(-l') < \chi(-l' + l)$ for any $l \in L_{>0}\}$. On the other hand, by Corollary 6.7.1, in the context of a generic analytic type, this happens exactly when the natural line $\mathcal{O}_Z(l')$ is in the image of $\mathrm{im}(c^{l'})$ (that is, $\mathcal{O}_Z(l')$ behaves as a generic line bundle). In particular, for generic $\widetilde{X}$, $\mathcal{S}'_{an} = \mathcal{S}'_{dom} \cup \{0\}$.

(b) In [21, §4] several combinatorial properties of $\mathcal{S}'_{dom}$ are listed.

(c) Corollary 6.7.1 can be compared with the definition of $\mathcal{S}' = \{l' : \chi(l') < \chi(l' + E_v)$ for any $v \in \mathcal{V}\}$.

6.7.3. $\mathcal{S}_{an} := \mathcal{S}'_{an} \cap L$ is the semigroup of divisors (restricted to E) of functions $\phi^*\mathcal{O}_{(X,o)}$. Let Z_{max} be the **maximal ideal cycle** (of S. S.-T. Yau [39]), that is, the divisorial part of $\phi^*(\mathfrak{m}_{(X,o)})$ (here $\mathfrak{m}_{(X,o)}$ is the maximal ideal of $\mathcal{O}_{(X,o)}$). It is the unique smallest nonzero element of $\mathcal{S}_{an}$.

Corollary 6.7.4. *Assume that $\widetilde{X}$ is generic with non–rational graph Γ. Then $\mathcal{M} = \{Z \in L_{>0} : \chi(Z) = \min_{l \in L} \chi(l)\}$ has a unique maximal element and $Z_{max} = \max \mathcal{M}$.*

Proof. For the first part see the second paragraph of 6.3. $\max \mathcal{M} \in \mathcal{S}_{an}$ by the right hand side of 6.7.1, but $\min \mathcal{S}_{an}$ cannot be smaller than $\max \mathcal{M}$ by the very same identity. $\square$

Remark 6.7.5. Recall that the fundamental (or minimal, or Artin) cycle $Z_{min} := \min\{\mathcal{S}' \cap L_{>0}\}$ has the property $h^0(\mathcal{O}_{Z_{min}}) = 1$, hence $h^1(\mathcal{O}_{Z_{min}}) = 1 - \chi(Z_{min})$ (see e.g. [23]). For $\widetilde{X}$ generic and (X,o) non–rational any cycle $Z \in \mathcal{M}$ (in particular Z_{max} too) has this property. Indeed, $h^1(\mathcal{O}_Z) = 1 - \min_{0<l\leq Z} \chi(l) = 1 - \chi(Z)$, hence $h^0(\mathcal{O}_Z) = 1$ too.

Corollary 6.7.6. *For (X,o) generic one has $Z_{max} \geq Z_{coh}$. If additionally (X,o) is numerically Gorenstein then $Z_{coh} + Z_{max} = Z_K$.*

6.8. The $\mathcal{O}_{(X,o)}$–multiplication on $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Assume that $p_g > 0$. On $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ the $\mathcal{O}_{(X,o)}$–module multiplication transforms on the dual vector space $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^* = H^0(\widetilde{X} \setminus E, \Omega^2_{\widetilde{X}})/H^0(\widetilde{X}, \Omega^2_{\widetilde{X}})$ into the multiplication of forms by functions. The filtration on $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ induced by the powers of the maximal ideal agrees with the filtration associated by the nilpotent operator determined by multiplication by a generic element of $\mathfrak{m}_{(X,o)}$. For details see e.g. [36].

The poles of forms are bounded by Z_{coh}. Indeed, by the exact sequence $0 \to \Omega^2 \to \Omega^2(Z_{coh}) \to \mathcal{O}_{Z_{coh}}(Z_{coh} + K_{\widetilde{X}}) \to 0$ and from the vanishing $h^1(\Omega^2) = 0$ (and from Serre duality) we have $\dim H^0(\Omega^2(Z_{coh}))/H^0(\Omega^2) = h^0(\mathcal{O}_{Z_{coh}}(Z_{coh} + K_{\widetilde{X}})) = h^1(\mathcal{O}_{Z_{coh}}) = p_g$. Hence the subspace $H^0(\Omega^2(Z_{coh}))/H^0(\Omega^2) \subset H^0(\widetilde{X} \setminus E, \Omega^2)/H^0(\Omega^2)$ has codimension zero, hence the spaces agree.

Corollary 6.8.1. *If $\widetilde{X}$ is generic then $\mathfrak{m}_{(X,o)} \cdot H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$. In particular, the $\mathcal{O}_{(X,o)}$–module multiplication factorizes to the $\mathbb{C} = \mathcal{O}_{(X,o)}/\mathfrak{m}_{(X,o)}$–vector space structure.*

Proof. Since $Z_{max} \geq Z_{coh}$, cf. 6.7.6, $\mathfrak{m}_{(X,o)} \cdot H^0(\Omega^2(Z_{coh})) \subset H^0(\Omega^2(-Z_{max} + Z_{coh})) \subset H^0(\Omega^2)$. □

6.9. Generic $\mathbb{Q}$–Gorenstein singularities. Recall that a singularity (X, o) is Gorenstein if the anticanonical cycle Z_K is integral, and $\Omega^2_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ equals $\mathcal{O}_{\widetilde{X}}(-Z_K)$. Hence in this case $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is natural. Recall, that more generally, a line bundle $\mathcal{L}$ is natural if and only if one of its powers has the form $\mathcal{O}_{\widetilde{X}}(l)$ for some $l \in L$, or equivalently, if and only if its restriction $\mathcal{L}|_{\widetilde{X}\setminus E} \in \mathrm{Pic}(\widetilde{X} \setminus E) = \mathrm{Cl}(X, o)$ has finite order (that is, it is $\mathbb{Q}$–Cartier). In particular, (X, o) is $\mathbb{Q}$–Gorenstein if and only if $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})$ is a natural line bundle, which automatically should agree with $\mathcal{O}_{\widetilde{X}}(-Z_K)$.

Proposition 6.9.1. *If a $\mathbb{Q}$–Gorenstein singularity (X, o) admits a resolution $\widetilde{X}$ with generic analytic structure, then (X, o) is either rational of minimally elliptic.*

Proof. **Step 1.** Let us fix a resolution $\widetilde{X}$ of a normal surface singularity (X, o). We claim that if (X, o) is neither rational nor minimally elliptic then there exists an effective cycle $Z > 0$, $|Z| \subset E$, with $Z \not\geq Z_K$ and with $h^1(\mathcal{O}_Z) > 0$.

Assume first that $\widetilde{X} = \widetilde{X}_{min}$ is a minimal resolution. Then $Z_K \geq 0$ (by adjunction formulae, see also [18]). By vanishing $h^1(\mathcal{O}_{\widetilde{X}}(-\lfloor Z_K \rfloor)) = 0$ we get that $h^1(\mathcal{O}_{\lfloor Z_K \rfloor}) = p_g$. Since (X, o) is not rational, necessarily $\lfloor Z_K \rfloor > 0$. Hence, if $\lfloor Z_K \rfloor < Z_K$ then $Z = \lfloor Z_K \rfloor$ works.

Assume that $\lfloor Z_K \rfloor = Z_K$. Then $Z_K \in L$ and $Z_K > 0$ (since $p_g > 0$) hence necessarily $Z_K \geq E$ (see [18]). For any $v \in \mathcal{V}$ consider the exacts sequence $0 \to \mathcal{O}_{E_v}(-Z_K + E_v) \to \mathcal{O}_{Z_K} \to \mathcal{O}_{Z_K - E_v} \to 0$. If $h^1(\mathcal{O}_{Z_K - E_v}) > 0$ for some v then we take $Z = Z_K - E_v$. Otherwise, $h^1(\mathcal{O}_{Z_K - E_v}) = 0$ for every v. Since $h^1(\mathcal{O}_{E_v}(-Z_K + E_v)) = 1$ we get that $p_g = 1$ and $Z_K = Z_{coh}$. Then the geometric genus of the singularities obtained by contracting any $E \setminus E_v$ is rational, hence (X, o) is minimally elliptic (for details see [16] or [33]).

Finally, let $\widetilde{X}$ be arbitrary and let $\pi : \widetilde{X} \to \widetilde{X}_{min}$ be the corresponding modification of the minimal one. Let $0 < Z < Z_K$ be the cycle obtained previously for $\widetilde{X}_{min}$. Then $\pi^*(Z)$ works in $\widetilde{X}$.

Step 2. Fix the generic resolution $\widetilde{X}$. Assume that (X, o) is neither rational nor minimally elliptic. Chose a cycle Z as in Step 1. Using $0 \to \Omega^2_{\widetilde{X}} \to \Omega^2_{\widetilde{X}}(Z) \to \mathcal{O}_Z(Z + K_{\widetilde{X}}) \to 0$, we get that $h^1(\Omega^2_{\widetilde{X}}(Z)) = h^1(\mathcal{O}_Z(Z + K_{\widetilde{X}})) = h^0(\mathcal{O}_Z)$. Since (X, o) is $\mathbb{Q}$–Gorenstein, $\Omega^2_{\widetilde{X}}(Z) = \mathcal{O}_{\widetilde{X}}(Z - Z_K)$, hence $h^1(\mathcal{O}_{\widetilde{X}}(Z - Z_K)) = h^0(\mathcal{O}_Z) = \chi(Z) + h^1(\mathcal{O}_Z)$. Now we apply (6.2.5) and (6.1.8), and we get

$$\chi(Z_K - Z) - \min_{l \geq 0}\{\chi(Z_K - Z + l)\} = \chi(Z) + 1 - \min_{0<l\leq Z}\{\chi(l)\}.$$

Since $\chi(D) = \chi(Z_K - D)$ this transforms into $-\min_{l\leq Z}\{\chi(l)\} = 1 - \min_{0<l\leq Z}\{\chi(l)\}$. Next we claim that $\min_{l\leq Z}\{\chi(l)\} = \min_{0\leq l\leq Z}\{\chi(l)\}$. Indeed, if $l = l_+ - l_-$ with $l_+, l_- \geq 0$ and with different supports, then there exists $E_v \in |l_-|$ such that $(E_v, l_-) < 0$; then by a computation $\chi(l + E_v) \leq \chi(l)$. Hence inductively $\chi(l_+) \leq \chi(l)$. Therefore,

$$-\min_{0\leq l\leq Z}\{\chi(l)\}=1-\min_{0<l\leq Z}\{\chi(l)\}.$$

This means that $\min_{0\leq l\leq Z}\{\chi(l)\}$ cannot be realized by an element $l>0$, hence $0=\chi(0)<\min_{0<l\leq Z}\{\chi(l)\}$. But this implies $h^1(\mathcal{O}_Z)=0$ (see [21, Example 4.1.3]), a contradiction. $\square$

Remark 6.9.2. Proposition 6.9.1 generalizes the following result of Laufer [16, Th. 4.3] whenever the link is rational homology sphere (with a different proof): if the generic analytic structure of a numerically Gorenstein topological type is Gorenstein then the topological type is either Klein or minimally elliptic. (Recall that the Klein — or ADE — singularities are exactly the Gorenstein rational singularities.) Laufer's proof works without the $\mathbb{Q}HS^3$–assumption.

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