QUANTITATIVE HELLY-TYPE THEOREMS VIA SPARSE APPROXIMATION

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Abstract. We prove the following sparse approximation result for polytopes. Assume that $Q$ is a polytope in John’s position. Then there exist at most $2d$ vertices of $Q$ whose convex hull $Q'$ satisfies $Q \subseteq -2d^2 Q'$. As a consequence, we retrieve the best bound for the quantitative Helly-type result for the volume, achieved by Brazitikos, and improve on the strongest bound for the quantitative Helly-type theorem for the diameter, shown by Ivanov and Naszódi: We prove that given a finite family $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$ with intersection $K$, we may select at most $2d$ members of $\mathcal{F}$ such that their intersection has volume at most $(cd)^{3d/2} \text{vol} K$, and it has diameter at most $2d^2 \text{diam} K$, for some absolute constant $c > 0$.

1. History and results

Helly’s theorem, dated from 1923 [H23], is a cornerstone result in convex geometry. Its finitary version states that the intersection of a finite family of convex sets is empty if and only if there exists a subfamily of $d+1$ sets such that its intersection is empty. In 1982, Bárány, Katchalski and Pach [BKP82] introduced the following quantitative versions of Helly’s theorem: there exist positive constants $v(d), \delta(d)$ such that for a finite family $\mathcal{F}$ of convex bodies, one may select $2d$ members such that their intersection has volume at most $v(d) \text{vol}(\bigcap \mathcal{F})$, or has diameter at most $\delta(d) \text{diam}(\bigcap \mathcal{F})$.

The problem of finding the optimal values of $\delta(d)$ and $v(d)$ has enjoyed special interest in recent years (see e.g. the excellent survey article [BK21]). In [BKP82] (see also [BKP84]) the authors proved that $v(d) \leq d^{2d^2}$ and $\delta(d) \leq d^{2d}$, and they conjectured that $v(d) \approx d^{c_1 d}$ and $\delta(d) \approx c_2 d^{1/2}$ for some positive constants $c_1, c_2 > 0$.

For the volume problem, in a breakthrough paper, Naszódi [N16] proved that $v(d) \leq e^{d+1} d^{2d+\frac{3}{2}}$, while $v(d) \geq d^{d/2}$ must hold. Improving upon his ideas, Brazitikos [B17] found the current best upper bound for volume: $v(d) \leq (cd)^{3d/2}$ for a constant $c > 0$.

For the diameter question, Brazitikos [B18] proved the first polynomial bound on $\delta(d)$ by showing that $\delta(d) \leq cd^{11/2}$ for some $c > 0$. In 2021, Ivanov and Naszódi [IN21] found the best known upper bound, $\delta(d) \leq (2d)^3$, and also proved that $\delta(d) \geq cd^{1/2}$. Thus, the value conjectured in [BKP82] for $\delta(d)$ would be asymptotically sharp.

In the present note, we show that given a finite family $\mathcal{F}$ of closed convex sets, one can select at most $2d$ members such that their intersection sits inside a scaled version of $\bigcap \mathcal{F}$ for
a suitable location of the origin. Clearly, it suffices to prove this statement for the special case when \( \mathcal{F} \) consists of closed halfspaces intersecting in a convex body. As an application, we obtain an improvement on the diameter bound, \( \delta(d) \leq 2d^2 \), and retrieve the best known bound for \( v(d) \). The crux of the argument is the following sparse approximation result for polytopes, which is a strengthening of Theorem 2 in [IN21].

**Theorem 1.** Let \( \lambda > 0 \) and \( Q \subset \mathbb{R}^d \) be a convex polytope such that \( Q \subseteq -\lambda Q \). Then there exist at most \( 2d \) vertices of \( Q \) whose convex hull \( Q' \) satisfies

\[
Q \subseteq -(\lambda + 2)dQ'.
\]

We immediately obtain the following corollary.

**Corollary 2.** Assume that \( Q = -Q \) is a symmetric convex polytope in \( \mathbb{R}^d \). Then we may select a set of at most \( 2d \) vertices of \( Q \) with convex hull \( Q' \) such that

\[
Q \subseteq 3dQ'.
\]

As usual, let \( B^d \) denote the unit ball of \( \mathbb{R}^d \) and let \( S^{d-1} \) be the unit sphere of \( \mathbb{R}^d \). A standard consequence of Fritz John’s theorem [J48] states that if \( K \subset \mathbb{R}^d \) is a convex body in John’s position, that is, the largest volume ellipsoid inscribed in \( K \) is \( B^d \), then \( B^d \subseteq K \subseteq dB^d \subseteq -dK \) (see e.g. [B97]). Thus, we derive

**Corollary 3.** Assume that \( Q \subset \mathbb{R}^d \) is a convex polytope in John’s position. Then there exists a subset of at most \( 2d \) vertices of \( Q \) whose convex hull \( Q' \) satisfies

\[
Q \subseteq 2d^2 Q'.
\]

For a family of sets \( \{K_1, \ldots, K_n\} \subset \mathbb{R}^d \) and for a subset \( \sigma \subset [n] = \{1, \ldots, n\} \), we will use the notation

\[
K_\sigma = \bigcap_{i \in \sigma} K_i,
\]

as in [IN21]. Also, \( \binom{[n]}{\leq k} \) stands for the set of all subsets of \([n]\) with cardinality at most \( k \). Using this terminology, we are ready to state our quantitative Helly-type result.

**Theorem 4.** Let \( \{K_1, \ldots, K_n\} \) be a family of closed convex sets in \( \mathbb{R}^d \) with \( d \geq 2 \) such that their intersection \( K = K_1 \cap \cdots \cap K_n \) is a convex body. Then there exists a \( \sigma \in \binom{[n]}{\leq 2d} \) such that

\[
\text{vol}_d K_\sigma \leq (cd)^{3d/2} \text{vol}_d K \quad \text{and} \quad \text{diam} K_\sigma \leq 2d^2 \text{diam} K
\]

for some constant \( c > 0 \).

To conclude the section we formulate the following conjecture, which was essentially stated already in [BKPS2]. This would imply the asymptotically sharp bound for \( v(d) \).

**Conjecture 5.** Assume that \( \{u_1, \ldots, u_n\} \subset S^{d-1} \) is a set of unit vectors satisfying the conditions of Fritz John’s theorem. That is, there exist positive numbers \( \alpha_1, \ldots, \alpha_n \) for which \( \sum_{i=1}^{n} \alpha_i u_i = 0 \) and \( \sum_{i=1}^{n} \alpha_i u_i \otimes u_i = I_d \), the identity operator on \( \mathbb{R}^d \). Then there exists a subset \( \sigma \subset [n] \) with cardinality at most \( 2d \) so that

\[
B^d \subset cd \text{conv}\{u_i : i \in \sigma\}
\]

with an absolute constant \( c > 0 \).

That the above estimate would be asymptotically sharp is shown by taking \( n = d + 1 \) and letting \( \{u_1, \ldots, u_n\} \) to be the set of vertices of a regular simplex inscribed in \( S^{d-1} \).
2. Proofs

Proof of Theorem 7. The condition $Q \subseteq -\lambda Q$ ensures that $0 \in \text{int} Q$. Among all simplices with $d$ vertices from the vertices of $Q$ and one vertex at the origin, consider a simplex $S = \text{conv}\{0,v_1,\ldots,v_d\}$ with maximal volume. We write $S$ in the form

\begin{equation}
S = \left\{ x \in \mathbb{R}^d : x = \alpha_1 v_1 + \ldots + \alpha_d v_d \text{ for } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{d} \alpha_i \leq 1 \right\}.
\end{equation}

For every $i = 1, \ldots, d$, let $H_i$ be the hyperplane spanned by $\{0,v_1,\ldots,v_d\} \setminus \{v_i\}$, and let $L_i$ be the strip between the hyperplanes $v_i + H_i$ and $-v_i + H_i$. Define $P = \bigcap_{i\in[d]} L_i$ (see Figure 1).

Note that

\begin{equation}
P = \{ x \in \mathbb{R}^d : \text{vol}_d(\text{conv}\{0,x,v_1,\ldots,v_d\} \setminus \{v_i\}) \leq \text{vol}_d(S) \text{ for all } i = 1, \ldots, d \}.
\end{equation}

This follows from the volume formula

\[\text{vol}_d(\text{conv}\{0,w_1,\ldots,w_d\}) = \frac{1}{d!} \lvert \det(w_1 w_2 \ldots w_d) \rvert\]

for arbitrary $w_1, \ldots, w_d \in \mathbb{R}^d$, which implies that for all $x \in \mathbb{R}^d$ of the form $x = cv_i + w$ with $w \in H_i$, $i = 1, \ldots, d$,

\[\text{vol}_d(\text{conv}\{0,x,v_1,\ldots,v_d\} \setminus \{v_i\}) = |c| \text{vol}_d(S).
\]

Next, we show that

\begin{equation}
P = \{ x \in \mathbb{R}^d : x = \beta_1 v_1 + \ldots + \beta_d v_d \text{ for } \beta_i \in [-1,1] \}.
\end{equation}

Indeed, since $v_1, \ldots, v_d$ are linearly independent, we may consider the linear transformation $A$ with $A(v_i) = e_i$ for $i = 1, \ldots, d$. Note that

\[A(P) = A\left( \bigcap_{i\in[d]} L_i \right) = \bigcap_{i\in[d]} A(L_i) = \{ x \in \mathbb{R}^d : x = \beta_1 e_1 + \ldots + \beta_d e_d \text{ for } \beta_i \in [-1,1] \}.
\]

Thus, (3) holds.

Since $S$ is chosen maximally, equation (2) shows that for any vertex $w$ of $Q$, $w \in P$. By convexity,

\begin{equation}
Q \subseteq P.
\end{equation}

Let $S' = -2dS + (v_1 + \ldots + v_d)$. By (1),

\begin{equation}
S' = \left\{ x \in \mathbb{R}^d : x = \gamma_1 v_1 + \ldots + \gamma_d v_d \text{ for } \gamma_i \leq 1 \text{ and } \sum_{i=1}^{d} \gamma_i \geq -d \right\}.
\end{equation}

Then, from (3) and (5),

\begin{equation}
P \subseteq S'.
\end{equation}

Let $u = \frac{1}{n}(v_1 + \ldots + v_d)$ be the centroid of the facet $\text{conv}\{v_1,\ldots,v_d\}$ of $S$. Let $y$ be the intersection of the ray from $0$ through $-u$ and the boundary of $Q$. By Carathéodory’s theorem, we can choose $k \leq d$ vertices $\{v'_1,\ldots,v'_k\}$ of $Q$ such that $y \in \text{conv}\{v'_1,\ldots,v'_k\}$. Set $Q' = \text{conv}\{v_1,\ldots,v_d,v'_1,\ldots,v'_k\}$.

Note that $[y,u] \subseteq Q'$, which implies $0 \in Q'$. Thus,

\begin{equation}
S \subseteq Q'.
\end{equation}
Since $Q \subseteq -\lambda Q$, we have that $-u \in \lambda Q$. Since $\lambda y$ is on the boundary of $\lambda Q$, we also have that $-u \in [0, \lambda y]$. We know that $0, \lambda y \in \lambda Q'$, so
\begin{equation}
 u \in -\lambda Q'.
\end{equation}
Combining (4), (6), (7), and (8):
\begin{equation}
 Q \subseteq P \subseteq S' = -2dS + du \subseteq -2d Q' - \lambda d Q' = -(\lambda + 2)d Q'.
\end{equation}

Proof of Theorem 4. As shown in [BKP82], we may assume that \{K_1, \ldots, K_n\} consists of closed halfspaces such that $K = \bigcap K_i$ is a $d$-dimensional polytope. Let $T$ be the affine transformation sending $K$ to John’s position. Let $\tilde{K}_i = TK_i$ for $i \in [n]$, $\tilde{K} = TK$, and for some $\sigma \subset [n]$, let $\tilde{K}_\sigma = \bigcap_{i \in \sigma} \tilde{K}_i$. We claim that there exists $\sigma \in \binom{[n]}{\leq 2d}$ such that the following two properties hold:
\begin{align}
 \tilde{K}_\sigma &\subseteq -2d^2 \tilde{K}, \\
 \text{vol}_d \tilde{K}_\sigma &\leq (cd)^{3d/2} \text{vol}_d \tilde{K}
\end{align}
for some constant $c > 0$. Statements (10) and (11) are affine invariant, so this will suffice to prove Theorem 4.

Recall that since $\tilde{K}$ is in John’s position, $B^d \subseteq \tilde{K} \subseteq dB^d$ (see [B97] or [GLMP04, Theorem 5.1]). Setting $Q = (\tilde{K})^\circ$, this yields that $\frac{1}{2} B^d \subseteq Q \subseteq B^d$. In particular, $Q \subseteq -dQ$. Hence, we may apply Theorem 1 to $Q$ with $\lambda = d$, we obtain a subset of at most $2d$ vertices of $Q$ such that their convex hull $Q'$ satisfies
\begin{equation}
 Q \subseteq -(d + 2)dQ' \subseteq -2d^2 Q'.
\end{equation}
The family of closed halfspaces supporting the facets of $(Q')^\circ$ is a subset of $\{\tilde{K}_1, \ldots, \tilde{K}_n\}$ with at most $2d$ elements. Thus, we can choose $\sigma \in \binom{[n]}{\leq 2d}$ such that $\tilde{K}_\sigma = (Q')^\circ$. Taking the polar of (12), we obtain
\begin{align}
 \tilde{K}_\sigma &\subseteq -(d + 2)d\tilde{K} \subseteq -2d^2 \tilde{K},
\end{align}
which shows (10).
Let $P$ be the parallelotope enclosing $Q$ from the proof of Theorem 1 and set $P' = -\frac{1}{2\pi} P$.

Statement (9) implies

\[ Q' \supseteq P'. \]

Since $S$ is chosen maximally, the volume of $S$ is at least the volume of the simplex obtained from the Dvoretzky-Rogers lemma [DR50] (see also [N16, Lemma 1.4]):

\[ \text{vol}_d(S) \geq \frac{1}{\sqrt{d!d^{d/2}}}. \]

Using (13),

\[ \text{vol}_d(P') = (2d^2)^{-d} \cdot \text{vol}_d(P) \geq d^{-5d/2}(d!)^{1/2}. \]

Note that $P'$ is centrally symmetric, so we can use the Blaschke-Santaló inequality (see [AGM15, Theorem 1.5.10]) for $P'$:

\[ \text{vol}_d(P') \cdot \text{vol}_d((P')^c) \leq \text{vol}_d(B_2^d)^2. \]

Using the inclusions $\tilde{K} \supseteq B_2^d$ and $\tilde{K}_\sigma = (Q')^c \subseteq (P')^c$, as well as (14) and (15):

\[ \frac{\text{vol}_d(\tilde{K})}{\text{vol}_d(\tilde{K}_\sigma)} \leq \frac{\text{vol}_d((P')^c)}{\text{vol}_d(B_2^d)} \leq \frac{\text{vol}_d(B_2^d)}{\text{vol}_d(P')} \leq \frac{\pi^{d/2}d^{d/2}(d!)^{-1/2}}{\Gamma((d/2)+1)} \leq (cd)^{3d/2} \]

for some absolute constant $c > 0$. This shows (11) and concludes the proof. \qed

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References


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