Causal boundary of space-times

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A viable modification of *c*-boundary construction is given which yields a satisfactory causal boundary wherever the Geroch-Kronheimer-Penrose construction does and is free of defects at Taub space-time.

The local structure of singularities in general relativity can be described by adjoining boundary points to the space-time, since the singularities themselves cannot be regarded as actually belonging to the manifold.[1,2] One would expect that these boundary points are determined by the geometry of the space-time and that at least the topological structure of the manifold be extendible to the enlarged space-time. According to this an appropriate boundary construction assigns a topological space $\overline{M}$ and an embedding $i:M\rightarrow\overline{M}$ to the space-time (M,g) such that $i[M]$ is an open, dense topological subspace in $\overline{M}$. Then the points of $\overline{M}\setminus i[M]$ represent the set of the "boundary" points while the topology of $\overline{M}$ tells us when a point sequence in $i[M]$ approaches a boundary point.

The various methods that have been put forward for constructing such a boundary, with the exception of the *c* boundary,[3] yield an unsatisfactory topological structure on $\overline{M}$ (Refs. 4–6). However the "singular" portion of the *c* boundary of Taub plane-symmetric static vacuum space-time is a single point and not, as one would expect, a "one-dimensional" set.[7] In fact, the authors in Ref. 7 suggest that it might not be fruitful to describe the structure of singularities using the *c*-boundary construction. Nevertheless we should like to save the *c*-boundary construction because it has the advantage that it is comparatively simple and for causally continuous space-times there is a reasonable, conformally invariant way to attach a boundary to (M,g) (Ref. 8). Since Taub space-time is causally continuous one can hope that there exists a slight modification of the *c*-boundary construction [hereinafter a GKP (Geroch-Kronheimer-Penrose) construction] which has all the advantages of the GKP construction and avoids the difficulties. The source of the problems appears to be that the open sets are too small in $M^{\#}$ (Ref. 3). For example a terminal indecomposable past (TIP) may only be in such open sets which are generated by indecomposable futures (IF's). This is not so in topology, which is defined in Ref. 8, for causally continuous space-times.

In this paper we give a viable modification of the *c*-boundary construction. Our construction yields a satisfactory causal boundary construction wherever the GKP construction does and is free of defects at Taub space-time.

First we collect the standard definitions and results we shall need: In the framework of general relativity the space-time is represented by a pair (M,g), where M is a four-dimensional differentiable manifold and g is a Lorentzian metric on M. The Lorentzian metric determines the causal structure of the space-time up to conformal transformations.[1] Denote by $I^-(p)$ the chronological past of $p\in M$, i.e., the set of points in the space-time which may be reached from p along a timelike curve. To avoid pathological situations it is convenient to require that the past- and future-distinguishing causal conditions hold on the space-time, i.e., $I^-(p)=I^-(q)$ or $I^+(p)=I^+(q)$ implies that $p=q$. The past of a subset S of M, $I^-[S]$, is defined as the union of the pasts of the points in S: formally $I^-[S]\equiv U_{p\in S}I^-(p)$. A set $P\subset M$ is called a past set if $P=I^-[S]$ for some $S\subset M$. A past set P is called indecomposable if it is not empty and it cannot be expressed as a union of two proper subsets which are themselves past sets. Denote $\hat{M}$ the set of indecomposable past sets (IP's). Trivially $I^-(p)$ for an arbitrary $p\in M$ is an IP. We call such an IP "proper" IP (PIP). The other IP's are "terminal" IP's (TIP's). One can similarly define the chronological future of $p\in M$, $I^+(p)$ (or $S\subset M$, $I^+[S]$), the notions of future set, the indecomposable future (IF) set, proper IF, terminal IF, and $\check{M}$ by interchanging "past" and "future" throughout. We shall omit the duals of definitions and statements.

The IP's (IF's) can be characterized by timelike or causal curves as follow: A subset P of the space-time is an IP if and only if $P=I^+[\gamma]$ for some timelike curve γ (Refs. 1 and 3). (Or) Let γ be a causal curve. Then $I^-[\gamma]$ is a PIP, $I^-(p)$, if and only if γ has a future end point p. Consequently $I^-[\gamma]$ is a TIP if and only if γ is future endless.[3]

The sets $I^-(p)$ and $I^+(p)$ are open in manifold topology of the space-time for an arbitrary $p\in M$, so all of the IP's and IF's are also open sets. Using the chronological past and future of space-time events one can define a more physical topology on M. Stated more precisely, let the Alexandrov topology A of a space-time be defined as the coarsest topology in which all of the PIP's and PIF's are open. Evidently the manifold topology is never coarser than the Alexandrov topology and we recall that the Alexandrov topology on M coincides with the ordinary manifold topology if and only if the strong causality condition holds on M (Ref. 1).

On the basis of singularity theorems one expects that the boundary points are represented as "ideal" end points of b-incomplete, inextendable causal curves.[1] If we use only the conformal-invariant causal structure to construct such "ideal" end points then TIP's or TIF's represent not only the singular portion of the boundary but boundary at "infinity" as well. Nevertheless it is simple to construct $\hat{M}$ and $\check{M}$; furthermore, there exist past and future end points in $\hat{M} \cup \check{M}$ to every causal curve in that sense that $\hat{M}$ ($\check{M}$) is future (past) complete.[3] We have to define some kind of identifications on $\hat{M} \cup \check{M}$. All of the space-time events are represented doubly in $\hat{M} \cup \check{M}$ by their chronological future and past. Moreover some further identifications may also be required between "ideal" points to get an appropriate boundary.[3] The aim is to get a topology on $\overline{M}$ which is an extension of the Alexandrov topology of the space-time.

Now we shall construct $\overline{M}$ as union with identifications of $\check{M}$ and $\hat{M}$; furthermore a Hausdorff topology $\overline{A}$ on $\overline{M}$ such that there exists an embedding $i: M \to \overline{M}$ for which $i[M]$ is an open and dense topological subspace in $\overline{M}$. The fact is that the topology and the identifications are obtained simultaneously. Let us first define the (extended Alexandrov) topology A^* on $\hat{M} \cup \check{M}$. The A^* topology is the coarsest topology in which, for each PIF, F, and PIP, P, the four sets F^{int}, F^{ext}, P^{int}, P^{ext} are all open, where

$$F^{\text{int}} \equiv \{ A \in \hat{M} \cup \check{M} \mid A \in \hat{M} \text{ and } A \cap F \neq \emptyset \text{ or } A \in \check{M} \text{ and } (\text{for all } S \subset M)\ I^+[S] = A \Rightarrow I^-[S] \cap F \neq \emptyset \} ,$$

$$F^{\text{ext}} \equiv \{ A \in \hat{M} \cup \check{M} \mid A \in \check{M} \text{ and } A \not\subset F \text{ or } A \in \hat{M} \text{ and } (\text{for all } S \subset M)\ A = I^-[S] \Rightarrow I^+[S] \not\subset F \} ,$$

P^{int} and P^{ext} being defined similarly, with the roles of past and future interchanged. The sets F^{int} and P^{int} are the analogues in $\hat{M} \cup \check{M}$ of the sets $I^-(p)$ and $I^+(q)$. The sets F^{ext} and P^{ext} are needed to get appropriate neighborhoods for the points of the "null" part of the boundary.[3] The most significant difference between the GKP construction and ours are that we did not use the intermediate space $M^{\#}$ and the open sets may include TIP's and TIF's simultaneously.

Now the following simple lemmas are easily seen to hold.

Lemma 1. F^{int} and F^{ext} are disjoint.

Lemma 2. Let O be an open subset in $\hat{M} \cup \check{M}$. Then $I^-(p) \in O$ if and only if $I^+(p) \in O$.

Lemma 3. The map $I^-: M \to \hat{M} \cup \check{M}$ is a dense topological embedding of (M, A) into $(\hat{M} \cup \check{M}, A^*)$.

However our aim is to construct a Hausdorff topological space $\overline{M}$ so we must carry out some identifications in $\hat{M} \cup \check{M}$. The minimal requirement for an appropriate identification is that $P = I^-(p)$ and $F \in \check{M}$ be identified if and only if $F = I^+(p)$. Let R be an equivalence relation on $\hat{M} \cup \check{M}$ which satisfies this requirement. Denote by $\hat{M} \cup \check{M}/R$ the quotient space and A^*/R the quotient topology on $\hat{M} \cup \check{M}/R$ (i.e., a set is open in A^*/R if and only if its inverse image under the identification map $\pi_R: \hat{M} \cup \check{M} \to \hat{M} \cup \check{M}/R$ is open in the original space). Using lemmas 2 and 3 one can prove the following.

Proposition 1. Let $\pi_R: \hat{M} \cup \check{M} \to \hat{M} \cup \check{M}/R$ be the natural projection generated by the equivalence relation R. Then the map $\pi_R \circ I^-$ is a dense embedding of (M, A) into $(\hat{M} \cup \check{M}/R, A^*/R)$.

Now the question is if there exists an equivalence relation R with the additional property that the topology A^*/R is Hausdorff.

Lemma 4. Let $A = I^-(p)$ and $B \in \hat{M} \cup \check{M} \setminus \{I^-(p), I^+(p)\}$. Then A and B are T_2 related in A^* if for each $p \in M$ there exist $a, b \in M$ such that $a \in I^-(p)$, $b \in I^+(p)$ and there is no set S, satisfying both $I^+[S] \subset I^+(a)$ and $I^-[S] \subset I^-(b)$, for which $I^+[S]$ is a TIF or $I^-[S]$ is a TIP.

Using lemmas 2 and 4 it is evident that there exists an equivalence relation on $\hat{M} \cup \check{M}$ which has the required property when the causality condition in lemma 4 holds on M, which was already stipulated in Ref. 3. For example, the equivalence relation R can be such that it simultaneously identifies all of the TIP's and TIF's; moreover R should identify each PIP and PIF generated by the same point of M. But we need not, in general, do all of these identifications to produce the "largest viable" Hausdorff space. Denote by R_H the intersection of all of the equivalence relations R on $\hat{M} \cup \check{M}$ such that the topological space $(\hat{M} \cup \check{M}/R, A^*/R)$ is Hausdorff. Then R_H is required "smallest" equivalence relation. Let $\overline{M} \equiv \hat{M} \cup \check{M}/R_H$ and $\overline{A} \equiv A^*/R_H$ then using proposition 1, lemma 2, and lemma 4 one can prove the following.

Corrolary 1. When the causal condition of lemma 4 holds on M there exists a Hausdorff topological space $(\overline{M}, \overline{A})$ such that the map $(\pi_{R_H} \circ I^-): M \to \overline{M}$ is a dense embedding of (M, A) into $(\overline{M}, \overline{A})$, where π_{R_H} is the natural projection from $\hat{M} \cup \check{M}$ to $\overline{M}$ generated by R_H.

As an application of these ideas, we now show that the "singular" portion of the causal boundary of Taub plane-symmetric static vacuum space-time is a one dimensional set. It was proved in Ref. 7 that $I^+(\eta^c_-)^* \in M^{\#}$ and $I^-(\eta^{\tilde{c}}_+)^* \in M^{\#}$ are non-T_2 related in $M^{\#}$ provided that $\tilde{c} < c$. Now it is easy to show that $I^+(\eta^c_-) \in [I^+(p)]^{\text{int}}$ and $I^-(\eta^{\tilde{c}}_+) \in [I^-(p)]^{\text{int}}$ if (for example) $p \in I^-(\eta^c_+) \cap I^+(\eta^{\tilde{c}}_-)$ hold. According to this $I^+(\eta^c_-) \in \check{M}$ and $I^-(\eta^{\tilde{c}}_+) \in \hat{M}$ are T_2 related in A^* whenever $\tilde{c} \neq c$. The sets $I^+(\eta^c_-)$ and $I^-(\eta^{\tilde{c}}_+)$ are non-T_2 related when $\tilde{c} = c$, which is just the one-dimensional property of the "singular" portion of the causal boundary of Taub space-time.

It is difficult to foresee whether or not our construction gives a satisfactory boundary structure for an arbitrary space-time, since the identifications on the set TIP's and TIF's are given (as in GKP construction) in an implicit manner. It would be worth finding an explicit identification rule (in that way it was given in Ref. 8) for any causally "well-behaved" space-times.

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