PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special issue dedicated to Mirosław Baran on the occasion of his 60th birthday, Volume 14 · 2021 · Pages 66–73

Bernstein-Markov type inequalities and discretization of norms

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Abstract

In this expository paper we will give a survey of some recent results concerning discretization of uniform and integral norms of polynomials and exponential sums which are based on various new Bernstein-Markov type inequalities.

AMS Subject classification: 41A17, 41A63.

Key words and phrases: multivariate polynomials, Marcinkiewicz-Zygmund, Bernstein and Schur type inequalities, discretization of L^p norm, doubling and Jacobi type weights.

1 Introduction

In the past 15-20 years the problem of discretization of uniform and L^q norms in various finite dimensional spaces has been widely investigated. In case of $L^q, 1 \leq q < \infty$ norms for trigonometric polynomials this problem is usually referred to as the *Marcinkiewicz-Zygmund type problem*, on the other hand when uniform norm and algebraic polynomials are considered then the terms *norming sets* or *optimal meshes* are usually used in the literature. Historically the first discretization result was given by S.N. Bernstein [3] in 1932 who showed that for any trigonometric polynomial t_n of degree $\leq n$ and any $0 = x_0 < x_1 < ... < x_N < 2\pi = x_{N+1}$ with $\max_{0\leq j\leq N}(x_{j+1} - x_j) \leq \frac{2\sqrt{\tau}}{n}$, $0 < \tau < 2$ we have

$$\max_{x\in[0,2\pi]} |t_n(x)| \leq (1+\tau) \max_{0\leq j\leq N} |t_n(x_j)|. \tag{1}$$

The above estimate essentially shows that the uniform norm of trigonometric polynomials of degree $\leq n$ can be discretized with accuracy τ using $N \sim \frac{n}{\sqrt{\tau}}$ properly chosen nodes. A standard substitution $x = \cos t$ leads to an extension of (1) for algebraic polynomials when $\max_{0\leq j\leq N}(\arccos x_{j+1} - \arccos x_j) \leq \frac{2\sqrt{\tau}}{n}$. (See also [7], p. 91-92 for details.)

The first result on the discretization of the $L^q, 1 < q < \infty$ norm is due to Marcinkiewicz and Zygmund [25] who verified in 1937 that for any univariate trigonometric polynomial t_n of degree at most n and every $1 < q < \infty$ we have

$$\int |t_n|^q \sim \frac{1}{n} \sum_{s=0}^{2n} \left| t_n\left(\frac{2\pi s}{2n+1}\right)\right|^q \tag{2}$$

where the constants involved in the above equivalence relation depend only on q. Above relation provides discretization of the $L^q, 1 < q < \infty$ norm of trigonometric polynomials of degree $\leq n$ with $2n+1$ nodes.

Above relations give an effective tool used for the discretization of the L^q norms of univariate trigonometric and algebraic polynomials which is widely applied in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, etc. Various generalizations were given for weighted L^q norms in [26]; multivariate polynomial on sphere and ball and general convex domains [14], [9], [16], [6], [11]; exponential polynomials [10], [32], [20], [21].

In terms of the methods used for the discretization several general approaches can be mentioned:

1. *Functional analytic methods*
2. *Probabilistic methods*
3. *Methods based on Bernstein-Markov type inequalities*

While above approaches complement each other in different ways and make it possible to cover various cases it should be mentioned that in contrast to the functional analytic and probabilistic methods the Bernstein-Markov approach always yields *explicit* discretization nodes. The main goal of the present paper is to give a survey of some recent discretization results based on various classic and new Bernstein-Markov type inequalities. The first part of the paper gives an overview of corresponding Bernstein-Markov type inequalities including some recently established estimates for exponential sums, as well. Then the second part of the paper shows how these Bernstein-Markov type inequalities yield new discretization results.

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2 Bernstein and Markov type inequalities for derivatives of polynomials and exponential sums

2.1 Bernstein-Markov type inequalities for univariate polynomials

Let us first recall some classical Markov and Bernstein type inequalities for univariate polynomials.

It is well known that for any algebraic polynomial p of degree $\le n$ we have a Markov type inequality

$$\|p'\|_{L^q[-1,1]} \le c_q n^2 \|p\|_{L^q[-1,1]}, q > 0 \tag{3}$$

with the constant c_q in the above estimate depending only on q.

On the other hand for trigonometric polynomials t of degree $\le n$ the Bernstein type inequality

$$\|t'\|_{L^q[-\pi,\pi]} \le n\|t\|_{L^q[-\pi,\pi]}, \quad q > 0 \tag{4}$$

is known to hold. Above sharp upper bound can be found in [1]. In addition, with a constant factor on the right hand side this inequality can be found in [26] and [13] for the weighted L^q norms with the so called *doubling weights*.

It is remarkable, that the order n^2 of derivatives in (3) in algebraic case reduces to n in trigonometric case (4). It should be noted that this fact makes Bernstein inequalities much more efficient for obtaining discretization nodes of asymptotically optimal cardinality. Since the standard trigonometric substitution $x = \cos t$ transforms algebraic polynomials into trigonometric polynomials we can rewrite (4) as

$$\|\sqrt{1-x^2}p'\|_{L^q[-1,1]} \le n\|p\|_{L^q[-1,1]}, \quad q > 0 \tag{5}$$

with p being an algebraic polynomials of degree n. Thus introduction of a weight $\sqrt{1-x^2}$ into the derivative norms reduces their size by a factor of n. This phenomena and its numerous extensions play a significant role in various discretization results. Clearly, (3) and (5) can be combined into a single inequality

$$\|(\frac{a}{n} + \sqrt{a^2-x^2})p'\|_{L^q[-a,a]} \le c_q n\|p\|_{L^q[-a,a]}, \quad q > 0. \tag{6}$$

It should be mentioned that as verified by Lubinsky [24] for $q \ge 1$ inequality (6) also holds for any trigonometric polynomial $p(t)$ of degree at most n if $0 < a < \frac{1}{2}$.

2.2 Bernstein-Markov type inequalities for multivariate polynomials

Now we turn our attention to the multivariate case and the space P_n^d of real algebraic polynomials of d variables and degree at most n. Let $K \subset \mathbb{R}^d$ be a compact *star like set* with respect to the origin, that is $\mathbf{0} \in \mathrm{Int}K$ and for every $\mathbf{x} \in K$ we have that $[\mathbf{0}, \mathbf{x}) \subset \mathrm{Int}K$. Furthermore, let

$$\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\} \tag{7}$$

denote the usual Minkowski functional of K.

In case when $K \subset \mathbb{R}^d$ is a $\mathbf{0}$-symmetric convex body Sarantopoulos [31] established a complete analogue of the univariate Bernstein inequality (5) for $q = \infty$ showing that for every $\mathbf{x} \in \mathrm{Int}K$ and each $\mathbf{u} \in \mathbb{R}^d$ normalized by $\varphi_K(\mathbf{u}) = 1$ we have

$$|D_{\mathbf{u}}p|(\mathbf{x}) \le \frac{n}{\sqrt{1-\varphi_K(\mathbf{x})^2}} \|p\|_{L^\infty(K)}, \tag{8}$$

where $D_{\mathbf{u}}p$ stands for the derivative in direction $\mathbf{u}$. Note that the above inequality was also independently verified by Baran [2].

The L^q Bernstein-Markov type inequalities of the previous section also admit an extension to the multivariate case for any $q \ge 1$ and every convex domain, or more generally domains $K \subset \mathbb{R}^d, d > 1$ with Lip1 boundary. The quantity $\sqrt{a^2-x^2}$ in (6) which measures the distance to the boundary of the interval in case of a convex body or domain $K \subset \mathbb{R}^d$ with Lip 1 boundary can be replaced by the Hausdorff distance to the boundary $h_K(\mathbf{x}) := \inf_{\mathbf{y} \in BdK} |\mathbf{x} - \mathbf{y}|$, with Bd$K$ being the boundary of the set. This leads to the estimate

$$\|\left(\frac{1}{n} + \sqrt{h_K(\mathbf{x})}\right) \partial p\|_{L^q(K)} \le cn\|p\|_{L^q(K)}, \ \ p \in P_n^d, \ q \ge 1, \tag{9}$$

where ∂p stands for the gradient of p and $c = c(K, d, q)$, see e.g. [33], [15], $q = \infty$ or [28], [18], $1 \le q < \infty$.

In addition to the above estimates the boundary properties of derivatives of polynomials play a crucial role in deriving discretization meshes of asymptotically optimal cardinality. Therefore special attention has to be given to the study of *tangential* Bernstein-Markov type inequalities for multivariate polynomials.

Let $K \subset \mathbb{R}^d$ be a compact star like set with respect to the origin and assume that its Minkowski functional $\varphi_K(\mathbf{x}) := \inf\{\alpha > 0 : \mathbf{x}/\alpha \in K\}$ is continuously differentiable on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. For any $\mathbf{x} \in BdK$ denote by $T_K(\mathbf{x})$ the set of all tangent unit vectors of BdK at $\mathbf{x}$. Then given $1 < \alpha \le 2$, we will say that the star like domain $K \subset \mathbb{R}^d$ is C^α if $\partial\varphi_K \in \mathrm{Lip}(\alpha - 1)$, that is for some $M > 0$ depending on K we have

$$|\partial\varphi_K(\mathbf{x}) - \partial\varphi_K(\mathbf{x}+\mathbf{h})| \le M|\mathbf{h}|^{\alpha-1}, \ \ \mathbf{x} \in S^{d-1}, |\mathbf{h}| \le 1.$$

Then as shown in [17] whenever $K \subset \mathbb{R}^d$ is a C^α star like domain with some $1 \le \alpha \le 2$ then for any $\mathbf{u} \in T_K(\mathbf{x})$ we have the next *tangential Bernstein type inequality*

$$\|(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}} D_{\mathbf{u}} q\|_{L^\infty(K)} \le c_K n\|q\|_{L^\infty(K)}, \ \ q \in P_n^d. \tag{10}$$

Here $c_K > 0$ depends only on K, and $D_{\mathbf{u}}$ stands for the derivative in direction $\mathbf{u}$. If $\alpha = 1$, i.e. K is a C^1 domain then we have $(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}} = \sqrt{1-\varphi_K(\mathbf{x})}$ in (10) which up to a constant has the same size as the quantity $\sqrt{h_K(\mathbf{x})}$ in (9). On the other hand when $\alpha > 1$ the quantity $(1-\varphi_K(\mathbf{x}))^{\frac{1}{\alpha}-\frac{1}{2}}$ which measures the distance to the boundary in (10) gives a slower than "square root" order of decrease to 0 at the boundary. We will see below that this phenomena has a significant effect on decreasing the cardinality of discretization meshes.

The above tangential Bernstein type inequality relies on certain smoothness property of the domain. Now we will present another important tangential Bernstein type inequality which holds for any *convex body* in $K \subset \mathbb{R}^2$. Denote by $D_T p(\mathbf{x})$ the maximal tangential derivative of p at $\mathbf{x} \in BdK$. Then as shown in [19] for any convex body $K \subset \mathbb{R}^2$ we have

$$\|D_T p\|_{L^1(BdK)} \leq c_K n \|p\|_{L^\infty(K)}, \ \ p \in P_n^2. \tag{11}$$

It should be noted that the size of the tangential derivative of $p \in P_n^2$ in (11) is measured in the L^1 norm along the boundary of the convex body while on the right hand side of (11) we have the $L^\infty(K)$ norm of p. This means that the $L^1(BdK) \to L^\infty(K)$ norm of the tangential derivative operator has norm $\sim n$. The above estimate is a considerable improvement compared with the $L^\infty(BdK) \to L^\infty(K)$ norm of the same operator which is known to be of order $\sim n^2$, in general. As shown below this decrease of the magnitude of the norm of derivative operator will result in discretization meshes of optimal cardinality.

2.3 Bernstein-Markov type inequalities for exponential sums

In this section we will present some new Bernstein- Markov type inequalities for general exponential sums. Our starting point is an elegant estimate given in [5], p. 293, E.4.d according to which for any $q(t) = \sum_{0\leq j\leq n} c_j e^{\mu_j t}, c_j \in \mathbb{R}$ with arbitrary $\mu_j \in \mathbb{R}$ we have

$$\|(1-x^2)q'\|_{L^\infty[-1,1]} \leq (4n-2)\|q\|_{L^\infty[-1,1]}. \tag{12}$$

The surprising feature of the above Bernstein type inequality consists in the fact that it is independent of the choice of exponents $\mu_j \in \mathbb{R}$ and their *degree*

$$\mu_n^* := \max_{0\leq j\leq n} |\mu_n|. \tag{13}$$

It is crucial on the other hand that the norm of the derivative is measured with the weight $1-x^2$ which is smaller compared to $\sqrt{1-x^2}$ used in the classical case (5).

The Bernstein type estimate (12) was used in [20] in order to verify the following Markov type bounds for general exponential sums:

For any $[\alpha,\beta] \subset \mathbb{R}, 0 < \delta \leq 1$ and every exponential sum $g(t) = \sum_{1\leq j\leq n} c_j e^{\mu_j t}, c_j \in \mathbb{R}$ with $\mu_j \in \mathbb{R}, 0 \leq j \leq n$ satisfying $\mu_{j+1} - \mu_j \geq \frac{\delta}{\beta-\alpha}$ we have with some absolute constant $c > 0$

$$\|g'\|_{L^\infty[\alpha,\beta]} \leq \frac{cn\mu_n^*}{\delta}\|g\|_{L^\infty[\alpha,\beta]}, \tag{14}$$

where μ_n^* is the degree of the exponential sums given by (13).

In addition, as shown in [20] the last upper bound can be used to derive the next general Markov type estimate for the derivatives of multivariate exponential sums on arbitrary convex bodies in $\mathbb{R}^d$:

Consider an arbitrary convex body $K \subset \mathbb{R}^d, d \geq 1$ with r_K being the radius of its largest inscribed ball. Then for every exponential sum

$$g(\boldsymbol{w}) = \sum_{1\leq j\leq n} c_j e^{\langle \mu_j, \, w\rangle}, \ \ \boldsymbol{w} \in \mathbb{R}^d$$

with exponents $\mu_j \in \mathbb{R}^d$ such that $|\mu_k - \mu_j| \geq \frac{\delta}{r_K}, j \neq k$ with a given $0 < \delta \leq 1$ we have

$$\|\partial g\|_{L^\infty(K)} \leq \frac{cd^3 n^3 \mu_n^*}{\delta}\|g\|_{L^\infty(K)}, \tag{15}$$

where μ_n^* is the degree of the exponential sums given by (13) and $c > 0$ is an absolute constant.

Again similarly to the univariate case the above estimate for the derivatives of multivariate exponential sum $g(\mathbf{w}) = \sum_{1\leq j\leq n} c_j e^{\langle \mu_j, \, \mathbf{w}\rangle}$ is essentially independent of the exponents $\mu_j \in \mathbb{R}^d$ with only their degree μ_n^*, dimension n and the separation parameter δ effecting the upper bound. This important fact will lead to exponent independent discretization results presented below.

Estimates (12) and (15) provide the needed L^∞ Bernstein- Markov type inequalities for exponential sums which can be applied to derive corresponding results for the discretization of their uniform norm.

Next we present some L_q Bernstein- Markov type inequalities for univariate exponential sums. First we recall the following $L_q, 1 \le q < \infty$ Bernstein type inequality for derivatives of univariate exponential sums $f_n(x) = \sum_{1\le j\le n} c_j e^{\lambda_j x}$ given in [12], Theorem 3.4

$$\|f_n'\|_{L_q[-1+\delta,1-\delta]} \le \frac{2n-1}{\delta}\|f_n\|_{L_q[-1,1]}, \quad 0<\delta<1. \tag{16}$$

This elegant result provides exponent independent upper bounds for L_q norms of derivatives of exponential sums inside the interval. The drawback of the above estimate is the appearance of the term $\frac{1}{\delta}$ in the upper estimate which leads to larger than required discretization sets in L_q Marcinkiewicz-Zygmund type inequalities. The next estimate which is verified in [21], Lemma 1 shows that introducing a weight $1-x^2$ into the L_q norms of derivatives of exponential sums allows to replace $\frac{1}{\delta}$ by a substantially smaller term $\ln\frac{2}{\delta}$. This improvement can be subsequently used in order to verify near optimal discretization meshes.

Let $1 \le q < \infty, 0 < \delta < 1, n \in \mathbb{N}$. Then for any distinct real numbers $\lambda_1, ..., \lambda_n \in \mathbb{R}$ and any exponential sum $f_n(x) = \sum_{1\le j\le n} c_j e^{\lambda_j x}, \forall c_j \in \mathbb{R}$ we have

$$\|(1-x^2)f_n'(x)\|_{L_q[-1+\delta,1-\delta]} \le 9n\ln^{\frac{1}{q}}\frac{2}{\delta}\|f_n\|_{L_q[-1,1]}. \tag{17}$$

Finally, we would like to point out that for exponential sums $g(x) = \sum_{1\le j\le n} a_j e^{\lambda_j x}$, $x \in \mathbb{R}$ having *nonnegative* coefficients $a_j \ge 0$ a stronger Bernstein type upper bound independent of n and λ_j-s was verified in [21], Lemma 5.

For any distinct real numbers $\lambda_j \in \mathbb{R}, 1 \le j \le n$ and arbitrary exponential sum $g(x) = \sum_{1\le j\le n} a_j e^{\lambda_j x}, a_j \ge 0$ with nonnegative coefficients we have

$$\|(1-x^2)g'\|_{L^q[-1,1]} \le 4\|g\|_{L^q[-1,1]}, \quad \forall q, n \in \mathbb{N}. \tag{18}$$

Note that the last upper bound for exponential sums with nonnegative coefficients for L_q norms with integer $q \in \mathbb{N}$ is much stronger that upper bound (12) in the sense that it provides even *dimension independent* estimate of the derivatives. This in turn will be shown to result in much stronger discretization results for exponential sums with nonnegative coefficients.

3 Discretization of the uniform norm of polynomials and exponential sums

In this section we will discuss application of Bernstein-Markov type inequalities in discretization of uniform norm of polynomials and exponential sums.

3.1 Discretization of uniform norms of polynomials

We consider the problem of finding *norming sets* $Y_N \subset K$ of cardinality $\mathrm{Card}Y_N = N$ in a compact set $K \subset \mathbb{R}^d$ for which there exists $c_K > 0$ depending only on the domain so that

$$\|p\|_{L^\infty(K)} \le c_K\|p\|_{L^\infty(Y_N)}, \quad \forall p \in P_n^d. \tag{19}$$

The main goal here is to find discrete sets of possibly smallest cardinality N. Since $\dim P_n^d = \binom{n+d}{n}$ we clearly must have $N > \binom{n+d}{n} \sim n^d$ in order for (19) to be possible. This naturally leads to the notion of *optimal meshes* which are defined as discrete sets of $\mathrm{Card}Y_N \sim n^d$ satisfying (19).

Finding exact geometric properties characterizing sets which possess optimal meshes appears to be a rather difficult problem. It was shown in [16] using multivariate Bernstein-Markov type inequalities that any C^2 star like domains and arbitrary convex polytopes in $\mathbb{R}^d$ possess optimal meshes. (In [29] a similar statement is proved with somewhat broader interpretation of the C^2 property.) It was also conjectured in [16] that *any convex body in $\mathbb{R}^d$ possess an optimal mesh*. It turned out that tangential Bernstein type inequalities are especially useful in the study of optimal meshes. In particular as shown in [17], the tangential Bernstein type inequality (10) can be used to verify the existence of optimal meshes in C^α star like domains with $2-\frac{2}{d} < \alpha < 2$. This is a substantial decrease in required smoothness of the star like domain in comparison to the C^2 property, especially in case of low dimensions d. Moreover, using the tangential Bernstein type inequality (11) the *existence of optimal meshes in any convex body on the plane* $\mathbb{R}^2$ was verified in [19]. It should be noted that in a recent paper Prymak [30] gave a different proof of the existence of optimal meshes on convex sets in the 2-dimensional plane. The proof in [30] is based on a promising approach initiated by Bos and Vianello [6] which relates the existence of optimal meshes to asymptotic properties of the Christoffel functions.

Finally let us also mention that it was proved in [4] that any compact set $K \subset \mathbb{R}^d$ possesses a *near optimal* norming set $Y_N \subset K$ of cardinality $N = O((n\log n)^d)$ satisfying (19). However, contrary to previous results the proof of existence of near optimal meshes given in [4] is nonconstructive, it is based on Fekete points which in general cannot be found explicitly.

3.2 Discretization of uniform norm of exponential sums

Now we turn our attention to some new results on discretization of the uniform norms of exponential sums

$$g(\mathbf{w}) = \sum_{1\le j\le n} c_j e^{\langle \mu_j, \mathbf{w}\rangle}, \quad \mu_j, \mathbf{w} \in \mathbb{R}^d. \tag{20}$$

In contrast with the trigonometric exponential sums when the exponents $\mu_j \in \mathbb{R}^d$ in (20) are arbitrary the basis functions $e^{\langle \mu_j, \mathbf{w}\rangle}$ are in general not pair wise orthogonal, and hence this crucial Fourier analytic tool is not available here. Instead we will rely again on Bernstein-Markov type inequalities of Section 2.3.

For a given $n \in \mathbb{N}, \delta, M > 0$ let us introduce the following set of *n term exponential sums* in $\mathbb{R}^d$ with exponents separated by δ and bounded by M

$$\Omega^d(n,\delta,M) := \{\sum_{1\le j\le n} c_j e^{\langle \mu_j, \mathbf{w}\rangle}, \quad c_j \in \mathbb{R}, \ \mu_j, \mathbf{w} \in \mathbb{R}^d, |\mu_{j+1} - \mu_j| \ge \delta, |\mu_j| \le M\}.$$

It is important to note that $\Omega^d(n,\delta,M)$ is *not a linear subspace*. First let us present the next discretization result for univariate exponential sums verified in [20].

Given any $n \in \mathbb{N}, 0 < \delta, \tau \le 1, M > 1$ there exist discrete points sets $Y_N \subset [\alpha,\beta] \subset \mathbb{R}$ of cardinality

$$N \le \frac{cn}{\sqrt{\tau}} \ln \frac{M}{\delta\sqrt{\tau}} \tag{21}$$

with an absolute constant $c > 0$, so that for every exponential sum $g \in \Omega^1(n,\delta,M)$ we have

$$\|g\|_{L^\infty[\alpha,\beta]} \le (1+\tau)\|g\|_{L^\infty(Y_N)}.$$

The upper bound for the cardinality of the discrete meshes turns out to be *near optimal* in the sense that (21) is sharp with respect to both dimension n and accuracy τ up to the logarithmic term. The degree M and separation parameter δ of the exponential sums appearing only in the logarithmic term has a limited effect on the bound. Furthermore, an *explicit construction* of nodes used for the discretization is given in [20]. It is based on equidistribution with respect to the measure

$$\mu_1(E) := \int_E \frac{dx}{1-x^2}, \quad E \subset (-1,1) \tag{22}$$

appearing in the Bernstein type inequality (12). In addition, the discrete set is **universal** in the sense that it depends only on dimension n, degree M and separation parameter δ of the exponential sums.

The sharpness of the $\frac{n}{\sqrt{\tau}}$ term in the upper bound (21) for cardinality follows from the following general statement which can be found in [22].

Let $K \subset \mathbb{R}^d$ be any compact set and assume that it possesses a discrete subset $Y_N \subset K$ of cardinality N so that

$$||p||_{L^\infty(K)} \le (1+\tau)||p||_{L^\infty(Y_N)}, \quad \forall p \in P_n^d.$$

Then we have with some $c_K > 0$ depending only on the domain K

$$N \ge c_K \left(\frac{n}{\sqrt{\tau}}\right)^d.$$

Clearly, in particular case when $d = 1$ and exponents of the univariate exponential sums are chosen to be integers $\mu_j := j, 0 \le j \le n$ the last lower bound shows the sharpness of the $\frac{n}{\sqrt{\tau}}$ term in (21).

The above discretization result was also extended in [20] to convex polytopes in $\mathbb{R}^d, d \ge 2$.

For any convex polytope $K \subset \mathbb{R}^d, d \ge 2$ we can explicitly give discrete points sets $Y_N \subset K$ of cardinality

$$N \le c(K,d)\left(\frac{n}{\sqrt{\tau}} \ln \frac{M}{\delta\tau}\right)^d$$

such that for every exponential sum $g \in \Omega^d(n,\delta,M)$ we have

$$\|g\|_{L^\infty(K)} \le (1+\tau)\|g\|_{L^\infty(Y_N)}.$$

4 Discretization of the integral norm of polynomials and exponential sums

We start this section with a refinement of the classical Marcinkiewicz-Zygmund result given in [22] which is similar to Bernstein's estimate (1).

For any $-\pi = x_0 < x_1 < \ldots < x_m = \pi$ *with*

$$\max_{0 \le j \le m-1} (x_{j+1} - x_j) < \frac{\sqrt{\tau}}{qn},$$

and for every $t_n \in T_n$ *we have*

$$(1-\tau) \sum_{j=0}^{m-1} \frac{x_{j+1} - x_{j-1}}{2} |t_n(x_j)|^q \le \int_{-\pi}^{\pi} |t_n(x)|^q dx \le (1+\tau) \sum_{j=0}^{m-1} \frac{x_{j+1} - x_{j-1}}{2} |t_n(x_j)|^q, \quad q \ge 2. \tag{23}$$

This is a Marcinkiewicz-Zygmund type estimate of precision τ similar to Bernstein's uniform bound (1). In particular, choosing equidistant nodes $x_j := \frac{2\pi(j-1)}{m+1}, 1 \le j \le m+1$ with $m = \left[\frac{2\pi qn}{\sqrt{\tau}}\right] + 2$ we obtain

$$\frac{1-\tau}{m} \sum_{j=1}^{m} |t_n(x_j)|^q \le \frac{1}{2\pi} \int_0^{2\pi} |t_n(x)|^q dx \le \frac{1+\tau}{m} \sum_{j=1}^{m} |t_n(x_j)|^q.$$

It should be noted that the spacing needed above can be achieved with discrete meshes of cardinality $m \sim \frac{n}{\sqrt{\tau}}$. This upper bound for cardinality turned out to be sharp with respect to τ, as well. Furthermore, let us mention that in the forthcoming paper [23] an extension of (23) for every $1 \le q < \infty$ is given. In addition certain new Marcinkiewicz-Zygmund results are proved therein for $0 < q < 1$.

Various generalizations of the Marcinkiewicz-Zygmund type results to the multivariate setting can be found in the literature. For instance, in [27] the Marcinkiewicz-Zygmund type problem based on scattered data on the unit sphere is studied. Feng Dai [9] gave some analogues of Marcinkiewicz-Zygmund type inequalities for multivariate algebraic polynomials on the sphere and ball in $\mathbb{R}^d$. In a recent paper [11] using Bernstein-Markov, Schur and Videnskii type polynomial inequalities various extensions of the Marcinkiewicz-Zygmund type bounds for multivariate polynomials on more general multivariate domains, which in particular include polytopes, cones, spherical sectors, toruses were verified.

We will present now a new discretization result for the integral norms of general exponential sums, see [21] for details.

Let $1 \le q < \infty, 0 < \delta \le 1, n \in \mathbb{N}, M > 1$. *Then we can explicitly give discrete sets* $Y_N = \{x_j\}_{j=1}^N \subset (a,b)$ *of cardinality*

$$N \le cqn \ln^{\frac{1}{q}+1} \frac{M}{\delta}$$

so that for each exponential sum $g \in \Omega^1(n, \delta, M)$ *we have*

$$\|g\|^q_{L^q[a,b]} \sim \sum_{1 \le j \le N-1} (x_{j+1} - x_j) |g(x_j)|^q, \tag{24}$$

where all the constants involved are absolute.

Again the estimate of the cardinality of discrete mesh is "almost" independent of the exponents μ_j since their degree M and separation parameter δ effect only the logarithmic term. Moreover, the discrete nodes constructed explicitly are equidistributed with respect to the measure (22).

The above discretization result admits a generalization to the unit cube $I^d := [0,1]^d$ in $\mathbb{R}^d$, see [21]. This requires some work because the separation condition $|\mu_{j+1} - \mu_j| \ge \delta, 1 \le j \le n-1$ for the exponents $\mu_j \in \mathbb{R}^d$ of the exponential sums $g(\mathbf{w}) = \sum_{0 \le j \le n} c_j e^{\langle \mu_j, \mathbf{w} \rangle}, \ \mu_j, \mathbf{w} \in \mathbb{R}^d$ does not necessarily extend to projections to coordinate axises, so dimension reduction will work only with proper choice of directions.

Let $1 \le q < \infty, d, n \in \mathbb{N}, 0 < \delta < 1, M > 1$. *Then there exist positive weights* $a_1, \ldots, a_N$ *and discrete point sets* $Y_N = \{\mathbf{w}_1, \ldots, \mathbf{w}_N\} \subset I^d$ *of cardinality*

$$N \le c(d,q) n^d \ln^{\frac{d}{q}+d} \frac{M}{\delta},$$

so that for every exponential sum $g \in \Omega^d(n, \delta, M)$ *we have*

$$\|g\|^q_{L_q(I^d)} \sim \sum_{1 \le i \le N} a_i |g(\mathbf{w}_i)|^q, \tag{25}$$

where all the constants involved depend only on d *and* q.

5 Discretization of integral norms of exponential sums with nonnegative coefficients

Degree and exponent independent Bernstein-Markov type inequality (18) for exponential sums

$$g(x)=\sum_{1\le j\le n} a_j e^{\lambda_j x},\ a_j\ge 0$$

with nonnegative coefficients leads to much stronger Marcinkiewicz-Zygmund type result in this case.

Denote

$$\Omega_+^d(n,M):=\{\sum_{1\le j\le n} c_j e^{\langle \mu_j,\mathbf{w}\rangle},\ \ c_j\ge 0,\ \mu_j,\mathbf{w}\in\mathbb{R}^d, |\mu_j|\le M\}.$$

This is the set of n term exponential sums with nonnegative coefficients and exponents bounded by M. Note that separation of exponents now is not required. Then the next result holds, see [21].

Let $q\in\mathbb{N}, M>1$. Then we can give discrete points sets $Y_N=\{x_1,...,x_N\}\subset[0,1]$ of cardinality $N\le c_q\ln M$ so that for every exponential sum $f\in\Omega_+^1(n,M)$ we have

$$\int_0^1 f^q(x)dx\sim\sum_{1\le j\le N}(x_{j+1}-x_j)f^q(x_j). \tag{26}$$

This provides an "almost" degree independent L^q Marcinkiewicz-Zygmund type inequality for exponential sums with nonnegative coefficients in case when $q\in\mathbb{N}$ is an integer. A slight modification leads to a similar result in case of any $q\ge 1$.

Moreover the above discretization result can be extended for convex polytopes in $\mathbb{R}^d$ [21].

Let $d,q\in\mathbb{N}, M>1$. Consider any convex polytope $K\subset\mathbb{R}^d$. Then we can give discrete points sets $Y_N=\{\mathbf{x}_1,...,\mathbf{x}_N\}\subset K$ of cardinality $N=O(\ln^d M)$ and positive weights $a_1,...,a_N$ so that for every exponential sum $f\in\Omega_+^d(n,M)$ we have

$$\|g\|^q_{L^q(K)}\sim\sum_{1\le i\le N} a_i g(\mathbf{x}_i)^q.$$

Acknowledgments The author has been supported by the Hungarian National Research, Development and Innovation Fund NKFIH - OTKA Grant K128922.

References

[1] V. V. Arestov, *On integral inequalities for trigonometric polynomials and their derivatives*, Izv. Akad. Nauk SSSR, Ser. Mat., **45** (1981), 3-22.

[2] M. Baran, *Bernstein type theorems for compact sets in $\mathbb{R}^n$ revisited*, J. Approx. Th., **79** (1994), 190-198.

[3] S. N. Bernstein, *Sur une modification de la formule d'interpolation de Lagrange*, Communications of the Kharkov Mathematical Society, **5** (1932), 1025-1050.

[4] T. Bloom, L.P. Bos, J.-P. Calvi, N. Levenberg, *Polynomial interpolation and approximation in C^d*, Ann. Polon. Math., **106** (2012) 53-81.

[5] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer, 1995.

[6] L. Bos and M. Vianello, *Tchakaloff polynomial meshes*, Ann. Polon. Math., **122** (2019), 221-231.

[7] E.W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.

[8] J.P. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Th., **152** (2008), 82-100.

[9] Feng Dai, *Multivariate polynomial inequalities with respect to doubling weights and A_∞ weights*, J. Funct. Anal., **235** (2006), 137-170.

[10] F. Dai, A. Prymak, V.N. Temlyakov, and S. Tikhonov, *Integral norm discretization and related problems*, Uspekhi Mat. Nauk, **74**, Issue 4(448) (2019), 3-58.

[11] S. De Marchi, and A. Kroó, *Marcinkiewicz-Zygmund type results in multivariate domains*, Acta Math. Hungar., **154** (2018), 69-89.

[12] T. Erdélyi, *Markov- and Bernstein-Type Inequalities for Müntz Polynomials and Exponential Sums in L_p*, J. Approx. Th. **104** (2000), 142-152.

[13] T. Erdélyi, *Notes on inequalities with doubling weights*, J. Approx. Th. **100** (1999), 60-72.

[14] K. Jetter, J. Stöckler, and J. D. Ward, *Error Estimates for Scattered Data Interpolation*, Math. Comp., **68** (1999), 733-747.

[15] A. Kroó, and Sz. Révész, *On Bernstein and Markov-Type Inequalities for Multivariate Polynomials on Convex Bodies*, J. Approx. Th., **99**(1999), 134-152.

[16] A. Kroó, *On optimal polynomial meshes*, J. Approx. Th., **163** (2011), 1107-1124.

[17] A. Kroó, *Bernstein type inequalities on starlike domains in $\mathbb{R}^d$ with application to norming sets*, Bull. Math. Sci. **3** (2013), 349-361.

[18] A. Kroó, *On Bernstein-Markov-type inequalities for multivariate polynomials in L^q-norm*, J. Approx. Th., **159**(2009), 85-96.

[19] A. Kroó, *On the existence of optimal meshes in every convex domain on the plane*, J. Approx. Theory, **238** (2019), 26-37.

[20] A. Kroó, *On discretizing uniform norms of exponential sums*, submitted.

[21] A. Kroó, *On discretizing integral norms of exponential sums*, J. Math. Anal. Appl. 507 (2022), to appear.

[22] A. Kroó, *On a refinement of Marcinkiewicz-Zygmund type inequalities*, Proceedings of Krasovskii Institute of Mathematics and Mechanics UB RAS, **26** (2020), 196-209.

[23] A. Kroó, J. Szabados, *On a refinement of Marcinkiewicz-Zygmund type inequalities in $L^p, p>0$*, in preparation.

[24] D. Lubinsky, *L_p Markov-Bernstein inequalities on arcs of the unit circle*, J. Approx. Th., **108**(2001), 1-17.

[25] J. Marcinkiewicz, and A. Zygmund, *Mean values of trigonometric polynomials*, Fund. Math. **28**(1937), 131-166.

[26] G. Mastroianni, and V. Totik, *Weighted polynomial inequalities with doubling and A_∞ weights*, Constr. Approx., **16**(2000), 37-71.

[27] H. N. Mhaskar, F. J. Narcowich, J. D. Ward, *Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature*, Math. Comp., **70**(2001), 1113-1130. Corrigendum: Math. Comp. **71**(2001), 453-454.

[28] S.M. Nikolskii, *On a certain method of covering domains and inequalities for multivariate polynomials*, Mathematica, **8** (1966), 345-356.

[29] F. Piazzon, *Optimal polynomial admissible meshes on some classes of compact subsets of* $\mathbb{R}^d$, J. Approx. Th., **207** (2016) 241-264.

[30] A. Prymak, Geometric computation of Christoffel functions on planar convex domains, J. Approx. Th., to appear.

[31] Y. Sarantopoulos, *Bounds on the derivatives of polynomials on Banach Spaces*, Math. Proc. Cambr. Philos. Soc., **110** (1991), 307-312.

[32] V.N. Temlyakov, *The Marcinkiewicz-type discretization theorems*, Constr. Approx., **48** (2018), 337-369.

[33] D. R. Wilhelmsen, *A Markov Inequality in several dimensions*, J. Approx. Th., **11** (1974), 216-220.