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ON G-RADICAL SUPPLEMENT SUBMODULES

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Abstract. In this work, we give some new properties of Rad-supplement and g-radical supplement submodules. Let V be a g-radical supplement of U in M and U or V be essential submodule of M. Then $Rad_g V = V \cap Rad_g M$. Let V be a g-radical supplement of U in M, U or V be essential submodule of M and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$. In this work, some relations between Rad-supplement, g-radical supplement, β^* and β^*_g relations are also studied. Let $X \beta^*_g Y$ in M. If V is a g-radical supplement of X in M and $V \trianglelefteq M$, then V is also a g-radical supplement of Y in M. Let M be an R-module. It is proved that M is semilocal (g-semilocal) if every submodule of M β^* equivalent to a Rad-supplement (g-radical supplement) submodule in M.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. A submodule N of an R-module M is called an *essential submodule* of M, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap K = 0$ implies that $K = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) *submodule* of M if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$ (in [13], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V << V$, then V is called a *supplement* of U in M. M is called a *supplemented module* if every submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. M is said to be *semilocal* if $M/RadM$ is semisimple, i. e. every submodule of $M/RadM$ is a direct summand of $M/RadM$. Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly *g-radical*) of M and denoted by Rad_gM (in [13], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. If M/Rad_gM is semisimple, i. e. every submodule of M/Rad_gM is a direct summand of M/Rad_gM, then M is called a *g-semilocal* module. Let M be an R-module. We say submodules X and Y of M are β^* equivalent, $X\beta^*Y$, if and only if $Y + K = M$ for every $K \leq M$ such that $X + K = M$ and $X + T = M$ for every $T \leq M$ such that $Y + T = M$. We say submodules X and Y of M are β_g^* equivalent, $X\beta_g^*Y$, if and only if $Y + K = M$ for every $K \trianglelefteq M$ such that $X + K = M$ and $X + T = M$ for every $T \trianglelefteq M$ such that $Y + T = M$. Let M be an R-module and $X \leq Y \leq M$. If $Y/X \ll M/X$, then we say Y *lies above* X in M.

More informations about supplemented modules are in [2, 8, 12]. More informations about g-small submodules and g-supplemented modules are in [3, 7, 10]. The definition of generalized supplemented modules and some properties of them are in [11]. The definition of g-semilocal modules and some properties of them are in [5]. The definition of β^* relation and some results of this relation are in [1]. The definition of β_g^* relation and some results of this relation are in [9].

Lemma 1. *Let M be an R-module. The following assertions hold.*

(1) *For every* $m \in Rad_gM$, $Rm \ll_g M$.
(2) *If* $N \leq M$, *then* $Rad_gN \leq Rad_gM$.
(3) $Rad_gM = \sum_{L \ll_g M} L$.

Proof. See [4, Lemma 2 and Lemma 3]. □

2. G-RADICAL SUPPLEMENT SUBMODULES

Definition 1. Let M be an R-module and $U,V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_gV$, then V is called a g-radical supplement of U in M. If every submodule of M has a g-radical supplement in M, then M is called a g-radical supplemented module. (See [4, 6].)

Clearly we can see that every g-supplemented module is g-radical supplemented. But the converse is not true in general. Every Rad-supplemented module is g-radical supplemented.

Lemma 2. *Let V be a g-radical supplement of U in M and $U \trianglelefteq M$. Then $Rad_g V = V \cap Rad_g M$.*

Proof. By Lemma 1, $Rad_g V \leq V \cap Rad_g M$. Let T be an essential maximal submodule of V. Then $U \cap V \leq Rad_g V \leq T$ holds. By $\frac{M}{U+T} = \frac{U+T+V}{U+T} \simeq \frac{V}{V \cap (U+T)} = \frac{V}{U \cap V+T} = \frac{V}{T}$ and $U+T \trianglelefteq M$, $U+T$ is an essential maximal submodule of M and $Rad_g M \leq U+T$. Hence $V \cap Rad_g M \leq V \cap (U+T) = U \cap V + T = T$. Thus $Rad_g V = V \cap Rad_g M$, as desired. □

Theorem 1. *Let V be a g-radical supplement of U in M, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof.

$\Longrightarrow$ Clear.

$\Longleftarrow$ Since $Rx \ll_g M$, by Lemma 1, $Rx \leq Rad_g M$ and $x \in Rad_g M$. Then $x \in V \cap Rad_g M$. By Lemma 2, $Rad_g V = V \cap Rad_g M$. Hence $x \in Rad_g V$ and by Lemma 1, $Rx \ll_g V$. We can also prove this part as follows:

Let T be an essential maximal submodule of V. Here $U \cap V \leq Rad_g V \leq T$. Assume that $Rx \nleq T$. Then $Rx + T = V$ and $M = U + V = U + Rx + T$. Since $Rx \ll_g M$ and $U + T \trianglelefteq M$, $U + T = M$. Then $V = V \cap M = V \cap (U+T) = U \cap V + T = T$, a contradiction. Hence $Rx \leq T$ for every essential maximal submodule T of V and $Rx \leq Rad_g V$. Thus $x \in Rad_g V$ and by Lemma 1, $Rx \ll_g V$.

□

Corollary 1. *Let V be a Rad-supplement of U in M and $U \trianglelefteq M$. Then $Rad_g V = V \cap Rad_g M$.*

Proof. Clear from Lemma 2. □

Corollary 2. *Let V be a Rad-supplement of U in M, $U \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof. Clear from Theorem 1. □

Theorem 2. *Let V be a g-radical supplement of U in M, $V \trianglelefteq M$ and $x \in V$. The following assertions hold.*

(1) *$Rad_g V = V \cap Rad_g M$.*
(2) *$Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof.

(1) By Lemma 1, $Rad_gV \leq V \cap Rad_gM$. Let T be an essential maximal submodule of V. Then $U \cap V \leq Rad_gV \leq T$ holds. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then $T \trianglelefteq M$ and $U+T \trianglelefteq M$. Then by $\frac{M}{U+T} = \frac{U+T+V}{U+T} \cong \frac{V}{V\cap(U+T)} = \frac{V}{U\cap V+T} = \frac{V}{T}$, $U+T$ is an essential maximal submodule of M and $Rad_gM \leq U+T$. Hence $V \cap Rad_gM \leq V \cap (U+T) = U \cap V + T = T$. Thus $Rad_gV = V \cap Rad_gM$, as desired.

(2) $\Longrightarrow$ Clear.

$\Longleftarrow$ Since $Rx \ll_g M$, by Lemma 1, $Rx \leq Rad_gM$ and $x \in Rad_gM$. Then $x \in V \cap Rad_gM$. By Theorem 2 (1), $Rad_gV = V \cap Rad_gM$. Hence $x \in Rad_gV$ and by Lemma 1, $Rx \ll_g V$. We can also prove this part as follows:

Let T be an essential maximal submodule of V. Here $U \cap V \leq Rad_gV \leq$ $\leq T$. Assume that $Rx \nleq T$. Then $Rx+T = V$ and $M = U+V =$ $= U+Rx+T$. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then $T \trianglelefteq M$ and $U+T \trianglelefteq M$. Since $Rx \ll_g M$, $U+T = M$. Then $V = V \cap M = V \cap (U+T) =$ $= U \cap V + T = T$, a contradiction. Hence $Rx \leq T$ for every essential maximal submodule T of V and $Rx \leq Rad_gV$. Thus $x \in Rad_gV$ and by Lemma 1, $Rx \ll_g V$.

□

Corollary 3. *Let V be a Rad-supplement of U in M and $V \trianglelefteq M$. Then $Rad_gV = V \cap Rad_gM$.*

Proof. Clear from Theorem 2 (1). □

Corollary 4. *Let V be a Rad-supplement of U in M, $V \trianglelefteq M$ and $x \in V$. Then $Rx \ll_g V$ if and only if $Rx \ll_g M$.*

Proof. Clear from Theorem 2 (2). □

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. For ${}_{\mathbb{Z}}\mathbb{Z} \leq_{\mathbb{Z}} \mathbb{Q}$, $Rad_g\mathbb{Z} = Rad\mathbb{Z} = 0$. Since $Rad_g\mathbb{Q} = Rad\mathbb{Q} = \mathbb{Q}$, $\mathbb{Z} \cap Rad_g\mathbb{Q} = \mathbb{Z} \cap \mathbb{Q} = \mathbb{Z}$. Hence $Rad_g\mathbb{Z} \neq \mathbb{Z} \cap Rad_g\mathbb{Q}$.

Proposition 1. *Let $X\beta_g^*Y$ in M. If V is a g-radical supplement of X in M and $V \trianglelefteq M$, then V is also a g-radical supplement of Y in M.*

Proof. By hypothesis, $M = X+V$ and $X \cap V \leq Rad_gV$. Since $X\beta_g^*Y$ and $V \trianglelefteq M$, $Y+V = M$. Let T be any essential maximal submodule of V. Since $T \trianglelefteq V$ and $V \trianglelefteq M$, then $T \trianglelefteq M$. Assume that $Y \cap V \nleq T$. Then $Y \cap V + T = V$. Here $M =$ $= Y+V = Y+Y\cap V+T = Y+T$ and since $X\beta_g^*Y$ and $T \trianglelefteq M$, $X+T = M$. Then $V =$ $V \cap M = V \cap (X+T) = V \cap X + T$ and since $X \cap V \leq Rad_gV \leq T$, $V = V \cap X + T = T$. This is a contradiction. Hence $Y \cap V \leq T$ for every essential maximal submodule of V and $Y \cap V \leq Rad_gV$. Thus V is a g-radical supplement of Y in M. □

Lemma 3. *Let $X\beta^*Y$ in M. If X and Y have Rad-supplements in M, then they have the same Rad-supplements in M.*

Proof. Let C be a Rad-supplement of X in M. Then $M = X + C$ and $X \cap C \leq RadC$. Since $X\beta^*Y$, $Y + C = M$. Let T be any maximal submodule of C. Assume that $Y \cap C \nleq T$. Then $Y \cap C + T = C$. Here $M = Y + C = Y + Y \cap C + T = Y + T$ and since $X\beta^*Y$, $X + T = M$. Then $C = C \cap M = C \cap (X + T) = X \cap C + T$ and since $X \cap C \leq RadC \leq T$, $C = X \cap C + T = T$. This is a contradiction. Hence $Y \cap C \leq T$ for every maximal submodule of C and $Y \cap C \leq RadC$. Thus C is a Rad-supplement of Y in M. Similarly, the interchanging the roles of X and Y, we can prove that every Rad-supplement of Y in M is also a Rad-supplement of X in M. □

Corollary 5. *Let X lies above Y in M. If X and Y have Rad-supplements in M, then they have the same Rad-supplements in M.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let $X\beta^*Y$ in M. If X has a g-radical supplement V in M, then V is also a g-radical supplement of Y in M.*

Proof. By hypothesis, $M = X + V$ and $X \cap V \leq Rad_gV$. Since $X\beta^*Y$, $Y + V = M$. Let T be any essential maximal submodule of V. Assume that $Y \cap V \nleq T$. Then $Y \cap V + T = V$. Here $M = Y + V = Y + Y \cap V + T = Y + T$ and since $X\beta^*Y$, $X + T = M$. Then $V = V \cap M = V \cap (X + T) = X \cap V + T$ and since $X \cap V \leq \leq Rad_gV \leq T$, $V = V \cap X + T = T$. This is a contradiction. Hence $Y \cap V \leq T$ for every essential maximal submodule of V and $Y \cap V \leq Rad_gV$. Thus V is a g-radical supplement of Y in M. □

Corollary 6. *Let X lies above Y in M. If X and Y have g-radical supplements in M, then they have the same g-radical supplements in M.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let $X\beta^*Y$ and Y be a Rad-supplement of U in M. Then $U \cap X \leq RadM$.*

Proof. Since Y is a Rad-supplement of U in M, $M = U + Y$ and $U \cap Y \leq RadY \leq \leq RadM$. Since $M = U + Y$ and $X\beta^*Y$, $M = U + X$. Let T be any maximal submodule of M. Here $U \cap Y \leq RadM \leq T$. Assume that $U \cap X \nleq T$. Then $U \cap X + T = M$ and since $M = U + X$, by [2, Lemma 1.24], $X + U \cap T = M$. Since $X\beta^*Y$, $Y + U \cap T = M$ and since $U + T = M$, by [2, Lemma 1.24] again, $U \cap Y + T = M$. Then by $U \cap Y \leq T$, $M = U \cap Y + T = T$. This is a contradiction. Hence $U \cap X \leq T$ for every maximal submodule T of M and $U \cap X \leq RadM$. □

Corollary 7. *Let X lies above Y and Y be a Rad-supplement of U in M. Then $U \cap X \leq RadM$.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be an R-module. If every submodule of M is β^* equivalent to a Rad-supplement submodule in M, then M is semilocal.*

Proof. Let $X/RadM \leq M/RadM$. Since $X \leq M$, by hypothesis, there exists a Rad-supplement submodule Y in M such that $X\beta^*Y$. Let Y be a Rad-supplement of U in M. By Lemma 5, $U \cap X \leq RadM$. Since $X\beta^*Y$ and $Y+U=M$, $X+U=M$. Then $\frac{M}{RadM} = \frac{X+U}{RadM} = \frac{X}{RadM} + \frac{U+RadM}{RadM}$ and $\frac{X}{RadM} \cap \frac{U+RadM}{RadM} = \frac{X\cap(U+RadM)}{RadM} = \frac{U\cap X+RadM}{RadM} = \frac{RadM}{RadM} = 0$. Hence $\frac{M}{RadM} = \frac{X}{RadM} \oplus \frac{U+RadM}{RadM}$ and $M/RadM$ is semisimple. Thus M is semilocal. □

Corollary 8. *Let M be an R-module. If every submodule of M lies above a Rad-supplement submodule in M, then M is semilocal.*

Proof. Clear from Lemma 6. □

Theorem 3. *Let $X\beta^*Y$ and Y be a g-radical supplement of U in M. Then $U \cap X \leq Rad_gM$.*

Proof. Since Y is a g-radical supplement of U in M, $M = U+Y$ and $U \cap Y \leq Rad_gY \leq Rad_gM$. Since $M = U+Y$ and $X\beta^*Y$, $M = U+X$. Let T be any essential maximal submodule of M. Here $U \cap Y \leq Rad_gM \leq T$. Assume that $U \cap X \nleq T$. Then $U \cap X + T = M$ and since $M = U+X$, by [2, Lemma 1.24], $X + U \cap T = M$. Since $X\beta^*Y$, $Y + U \cap T = M$ and since $U+T = M$, by [2, Lemma 1.24] again, $U \cap Y + T = M$. Then by $U \cap Y \leq T$, $M = U \cap Y + T = T$. This is a contradiction. Hence $U \cap X \leq T$ for every essential maximal submodule T of M and $U \cap X \leq Rad_gM$. □

Corollary 9. *Let X lies above Y and Y be a g-radical supplement of U in M. Then $U \cap X \leq Rad_gM$.*

Proof. Clear from Theorem 3. □

Theorem 4. *Let M be an R-module. If every submodule of M is β^* equivalent to a g-radical supplement submodule in M, then M is g-semilocal.*

Proof. Let $X/Rad_gM \leq M/Rad_gM$. Since $X \leq M$, by hypothesis, there exists a g-radical supplement submodule Y in M such that $X\beta^*Y$. Let Y be a g-radical supplement of U in M. By Theorem 3, $U \cap X \leq Rad_gM$. Since $X\beta^*Y$ and $Y+U=M$, $X+U=M$. Then $\frac{M}{Rad_gM} = \frac{X+U}{Rad_gM} = \frac{X}{Rad_gM} + \frac{U+Rad_gM}{Rad_gM}$ and $\frac{X}{Rad_gM} \cap \frac{U+Rad_gM}{Rad_gM} = \frac{X\cap(U+Rad_gM)}{Rad_gM} = \frac{U\cap X+Rad_gM}{Rad_gM} = \frac{Rad_gM}{Rad_gM} = 0$. Hence $\frac{M}{Rad_gM} = \frac{X}{Rad_gM} \oplus \frac{U+Rad_gM}{Rad_gM}$ and M/Rad_gM is semisimple. Thus M is g-semilocal. □

Corollary 10. *Let M be an R-module. If every submodule of M lies above a g-radical supplement submodule in M, then M is g-semilocal.*

Proof. Clear from Theorem 4. □

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