STOCK RETURN DISTRIBUTIONS:
A SURVEY OF EMPIRICAL INVESTIGATIONS*

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SUMMARY

In this paper we give a brief survey of the empirical investigations of the distribution of stock returns and some detailed discussion of Hungarian and German stock returns as well as the DAX using refined methods. As a conclusion the stable law hypothesis for the stock returns is rejected and procedures requiring much weaker distributional assumptions are suggested instead of the more traditional techniques.

KEYWORDS: : Pareto-distribution; Extremal value theory; Tail index.

Financial researchers have long been interested in studying the empirical distribution of stock returns. This interest can be explained by the fact that the return distribution has a direct bearing on the descriptive validity of theoretical models in financial economics.

A still open debate concerns the analysis of the asymmetry of return distributions. The issue is relevant for portfolio theory and management, because of the importance of the distribution of stock returns in designing profitable investment strategies. The important role of skewness in explaining security returns is demonstrated by *Jean* (1971,1973), and *Levy* and *Sarnat* (1972). Several researchers (*Samuelson*, 1970; *Rubinstein*, 1973) argue that in order to ignore the third and higher moments, at least one of the following three conditions must be true:

(*i*) The return distribution has negligible variation, therefore any moments beyond the first are zero.

(*ii*) The derivatives of the applicable utility function are zero for the third and higher moments.

(*iii*) The asset returns have normal distributions, or the investors' utility functions are quadratic.

Some researchers support the use of quadratic approximation for utility functions in practical problems, assuming that the risk taken by the investor is small compared to his

total wealth. Others, however, remain skeptical of its application. *Hanoch* and *Levy* (1970) point out that the quadratic utility function implies increasing absolute risk aversion (which is contrary to the normal assumption of decreasing absolute risk aversion). As a consequence of these investigations more attention has been directed to the existence and importance of skewness in portfolio selection. In the case of restricting the investment decision to a finite time interval, portfolio selection based on the mean-variance approximation becomes inadequate and higher moments become more relevant (*Samuelson* 1970). *Jean* (1971, 1973) derives the risk premium for higher moments similar to that for the two-moment case, extending portfolio analysis to three or more parameters. Empirical studies show that the three-moment CAPM fits the return distribution better than the two-moment model (*Kraus* and *Litzenberger* 1976). *Markowitz* (1991) finds that the mean-variance model may approximately maximize expected utility for relatively small deviations in rates of return even if distributions are not normal.

Stable distributions have often been used to explain the stochastic behaviour of stock prices because of the following statistical properties: (*1*) only stable distributions have domains of attraction (generalized central limit theorem), and (*2*) stable distributions belong to their own domain of attraction (stability). These properties are consistent with economic price theory and capable of explaining the observed leptokurtosis and skewness in return distributions.

The logarithm of the characteristic function $\varphi(t)$ of any *stable* random variable X is: $\log \varphi(t) \equiv \log E\left[e^{it X}\right] = i\delta t - \gamma|t|^{\alpha}\left[1 - i\beta \operatorname{sgn}(t) \tan(\alpha \pi / 2)\right]$, where $(\alpha, \beta, \gamma, \delta)$ are the four parameters that characterize each stable distribution.

In the above formula $\alpha \in (0, 2]$ is the *exponent*, $\beta \in (-\infty, \infty)$ is the *skewness index*, $\gamma \in (0, \infty)$ is the *scale parameter*, and $\delta \in (-\infty, \infty)$ is said to be the *location parameter*.

When $\alpha = 2$, the stable distribution reduces to normal. As α decreases from 2 to 0, the tail areas of the distribution become increasingly 'fatter' than that of the normal. When $\alpha \in (1, 2)$, the stable distribution has a finite mean given by δ, but when $\alpha \in (0, 1]$, even the mean is infinite. The parameter β measures the symmetry of the stable distribution; when $\beta = 0$ the distribution is symmetric, and when $\beta < 0$ (or $\beta > 0$) the distribution is skewed to the left (or right). When $\beta = 0$ and $\alpha = 1$ we have the Cauchy distribution, and when $\alpha = 1/2$, $\beta = 1$, $\delta = 0$, and $\gamma = 1$ we have the Bernoulli distribution.

Detailed description of stable laws can be found in *Feller* (1971) and *DuMouchel* (1971). Discussions of their applicability in economic analysis are in *Mandelbrot* (1963), *Fama* (1965) and *McCulloch* (1978).

The following part of this paper is organized as follows. Section 1 summarizes the results of previous discussions as well as the methods used for investigations. In Section 2 some more detailed discussion is presented analysing the appropriateness of stable laws for German and Hungarian stock returns using refined methods. The last section provides concluding remarks.

1. Methods and results of previous discussions

Stable laws other than the normal distribution share the features of fat tails and high peak at the mean (leptokurtosis) observed in data. Stability under addition seems to be a

necessary property for daily, weekly etc. data when succesive high frequency price changes are assumed to be independent and identically distributed random variables. The reasoning outlined in the literature seemed to be so persuasive that researchers accepted the stable laws as evidence without the further testing of fit. *Teichmoeller* (1971) and *Simkowitz* and *Beedles* (1980) examined stock returns, *McFarland, Pettit* and *Sung* (1980) and *So* (1987a) investigated exchange rate changes, *Cornew, Town* and *Crowson* (1984) and *So* (1987b) studied futures returns without the testing of fit.

Some indications for the violation of the stability-under-addition property expressing itself as time dependence of *α* over daily, weekly etc. data motivated others to query the stable law hypothesis (*Hsu, Miller* and *Wichern* 1974, *Upton* and *Shannon* 1979, *Friedman* and *Bandersteel* 1982, and *Hall, Brorson* and *Irwin* 1989).

Despite the a priori plausibility of stable distributions, several empirical studies have found some evidence against the hypothesis that stock returns can be characterized by stable distributions (*Officer* 1972, *Blattberg* and *Gonedes* 1974, *Hsu, Miller,* and *Wichern* 1974). Most of these studies have been restricted to the symmetric case because the parameter estimation as well as the economic analysis have been considerably facilitated when $\beta = 0$. More recent investigations by *Simkowitz* and *Beedles* (1980), *Rozelle* and *Fielitz* (1980) and *Fielitz* and *Rozelle* (1983) have shown, however, that empirical return distributions are in most cases significantly skewed and only the asymmetric stable laws can be used as probability models of stock returns. *Peccati* and *Tibiletti* (1993) suggest a possible reading-key to the interpretation of the skewness of stock return distributions. This key relies on the fact that the asymmetry of a sum of random variables depends not only on that of the random addenda, but also on their dependence structure. The conclusion of the empirical investigations on the skewness of stock return distribution is that the introduction of the asymmetry in the mean-variance framework serves as a useful tool for describing the ex-post equilibrium of the financial markets; however, it does not seem to be a proper ex-ante tool for selecting profitable portfolio strategies.

Akgiray and *Booth* (1988) investigate the stable-law hypothesis for stock returns discussing the empirical tail shapes instead of testing the overall fit of stable distributions to data. This approach is based on the notion that the tails of stable distributions and finite-variance distributions are distinctly different. (The rate at which the tail probability $Pr(|x| > n)$ converges to 0 as $n \to \infty$ is proportional to n^k.) For infinite-variance stable distributions $k < \alpha < 2$ and $k \geq 2$ for finite-variance distributions.

The analysis of tails relies on the following result of the extremal value theory: consider a stationary sequence of independent and identically distributed random variables $X_1, X_2, ..., X_n$ and define the order statistics

$$M_n = \max (X_1, X_2, \ldots, X_n) \,. \tag{/1/}$$

As it can be shown, the limiting distribution of M_n is appropriately scaled and converges to one of the max-stable distributions (for details see *Leadbetter* et.al., 1983).

The relevant distribution is fat-tailed without finite endpoint given by the cumulative distribution function

$$F(x) = \begin{cases} 0 & if \quad x \leq 0 \\ exp(-x^{-\alpha}) & if \quad x > 0 \end{cases} \tag{/2/}$$

This means that $Pr\{a_n(M_n - b_n) \le x\}$ converges (weakly) to $F(x)$ with normalizing constants a_n and b_n. The characteristic exponent of a stable distribution coincides with the tail index α of the limiting extremal value distribution $F(x)$. Taking into account that the tail index is not restricted to the interval (0, 2] but may assume any positive value, the characteristic exponent estimated by the fractile method (or other methods) may not coincide with the tail index when the underlying distribution is not a stable one.

ARCH processes may obey limit laws characterized by indices greater than 2. De *Haan* et al. (1989) show how the tail index depends on the parameters of ARCH processes. As an other example, the Student *t* distribution converges to a limiting extremal distribution with tail index identical to the number of degrees of freedom. It means that the tail indices for the Student family extend from 1 to infinity. All these alternatives are nested in the tail estimation procedure. The relationship between the number of existing moments and the tail index also gives some useful information for the analysis. All moments smaller than the tail index exist, whereas higher moments exhibit do not converge.

Studies of proposed estimation techniques for the tail index α have favoured the estimator introduced by *Hill* (1975) as the most effective one. This estimation procedure gives a consistent estimate of the inverse of α by calculating :

$$\gamma_H = 1/\alpha_H = \frac{1}{m}\sum_{i=1}^{n}\left[\log x_{(i)} - \log x_{(m)}\right], \qquad\qquad /3/$$

where *n* is the sample size, *m* is the number of observations located in the tail of the distribution and the elements of the sample are in descending order:

$$x_{(1)} \ge x_{(2)} \ge ... \ge x_{(m)} \ge ... \ge x_{(n)}.$$

It can be shown that the variable $(\gamma_H - \gamma)\sqrt{m}$ follows asymptotically normal distribution with zero mean and variance γ^2. This result can be used to test the hypothesis of identity of limit laws across stocks as well as the equality of lower and upper tails in the same sample. The main problem connected with the application of tail index estimations is the decision about an appropriate tail size, i.e. determining the number of observations *m* used in the calculation of α_H. The choice of tail size *m* necessarily involves judgement or maintenance of a specific hypothesis on the true α. The higher (lower) α itself is, the thinner (fatter) the tails will be and the fewer (more) elements will belong to the tail region. One can realize that choosing a too large value for *m* will result in a contamination of the tail region with elements of the central parts of the distribution when the true α assumes a relatively high value.

Tail index estimation has only recently been applied in the financial literature. *Koedijk, Schafgans* and de *Vries* (1990) and *Kähler* (1993) analyse European exchange rates quoted against the US dollar. *Koedijk* et al. cannot reject the hypothesis of a tail index within the realm of the stable laws while *Kähler's* estimates lie within the interval 3-5 allowing rejection of $\alpha < 2$. *Dewachter* and *Gielens* (1991) point to biases in the estimates of *Koedijk* et.al. and report upward corrected tail indices. *Akgiray, Booth* and *Seifert* (1988) and *Koedijk, Stork* and *de Vries* analyse Latin-American black market ex-

change rates. In the study of Akgiray et.al. (1988) a less efficient maximum likelihood estimator was used giving values within the interval 0.5 to 7. *Koedijk* et al. revised the results using the *Hill* estimator for the same data. The revision resulted a narrower interval of α values (about 1.2 to 3.2). *Koedijk* and *Kool* (1993) investigate the East European exchange rates against the US dollar finding α values within the interval (2, 3). The studies of US and German stock prices performed by *Akgiray* and *Booth* (1988) and *Akgiray*, *Booth* and *Loistl* (1989) respectively were based on maximum likelihood estimation. Jensen and *de Vries* (1991) found the α values in the range of 3.2 and 5.2 considering daily returns of 10 US stocks.

The next section of the paper analyses the appropriateness of stable laws for German and Hungarian stock returns. Data used for the analysis cover the period from 1 January, 1988 to 9 September, 1994 of thirty of the most frequently traded German stocks forming the DAX share price index and the period from 6 January, 1993 to 31 August, 1995 for the most frequently traded Hungarian stocks (*Lux* and *Varga*, 1996).

2. Analysis of return distributions for major German and Hungarian stocks

Returns are calculated as differences of the logarithms of daily closing prices. First chi-squared tests with 10 and 25 equiprobable cells are applied to test the fit of the estimated distributions. The 25-cell test for the German individual stocks rejects the stable Paretian hypothesis in 12 (16) cases at 1 percent (5%) significance level, whilst for the DAX index the hypothesis is not rejected.

(Nearly the same results have been obtained for the Hungarian stock market, but the sample is not as large as the German one, therefore the inference may be questionable). The 10-cell test even rejects 8 more cases at 1 percent and also rejects the stable distribution for the DAX at 5 percent level. Some of the rejections of the 10-cell test may be due to certain non-robustness against some slight misspecification of the location parameter α and therefore the 25-cell variant may be considered more reliable. The interpretation of this standard test is ambiguous: partly because for about one third of all cases the stable laws are rejected at 1 percent level, and partly because the results also show that many of the empirical distributions seem to be described quite well by stable distributions. The computational results confirm the picture available from many other previous studies that there is at least some overall similarity in the shapes of empirical distributions and that of the stable distributions and the estimated characteristic exponents lie in a relatively limited interval around 1.5.

As a counter-check the tails of the empirical distributions are considered. To investigate whether upper and lower tails are identical, the *Hill*-estimator was used to the lowest and highest 5 percent of observations. Point estimates denoted by α^+ and α^- for German stocks forming the DAX and the most often traded Hungarian stocks were calculated. The test of hypothesis $\alpha^+ = \alpha^-$ relies on the approximate normality of $1/\gamma_H$. As a consequence, the sum

$$Q = \left(\frac{\gamma^+ - \gamma}{\sigma_\gamma}\right)^2 + \left(\frac{\gamma^- - \gamma}{\sigma_\gamma}\right)^2 = \left(\alpha/\alpha^+ - 1\right)^2 m + \left(\alpha/\alpha^- - 1\right)^2 m \qquad /4/$$

follows chi-squared distribution with two degrees of freedom, and *m* is the number of observations located in the tail of the distribution as in /3/ $\left(\sigma_\gamma^2 = \gamma^2 / m\right)$. The results of the computations show that in all cases there exists a broad range of hypothetical α values for which the hypothesis $\alpha^+ = \alpha^-$ is not rejected. Simultaneously it is obtained that $\alpha^+ > \alpha^-$ in 25 out of 30 cases for German stocks. This result could raise doubts about the appropriateness of the assumption of identical tail behavior on the left and right tails of the distributions. To make this point clear a simple symmetry test can be used. The hypothesis $\alpha^+ = \alpha^-$ for all stocks implies

$$\Pr(\alpha^+ > \alpha^-) = \Pr(\alpha^+ < \alpha^-) = 0.5.$$

The number of *k* observations with $\alpha^+ > \alpha^-$ under the hypothesis $\alpha^+ = \alpha^-$ follows a binomial distribution $B(30, 0.5)$. Only individual stocks are considered because the DAX is a linear combination of its constituent elements. The probability of observing $k \leq 5$ or $k \leq 25$ under $H_0: \quad \alpha^+ = \alpha^-$ is only 0.003. It tells us that a significant asymmetry between upper and lower tail indices seems to exist considering the 30 stocks as a whole. The 'mini-crash' in October 1989 (the Gulf crisis, the Russian putsch) may be responsible for this asymmetry. Omitting relevant data and recalculating the upper and lower 5 percent tail indices reduces asymmetry and gives a statistically insignificant result. This means that the asymmetry between left and right tails was caused by an extreme event. This extreme event affected all stocks in a rather uniform way (individual stocks fell by 6 to 25 percent and the DAX declined by 13 percent that day). The conclusion that no systematic differences in the extremal behaviour of left and right tails exist can be accepted. To obtain the point estimates and some insight into variation with sample size, the two-sided Hill-estimator was computed for the stocks and the DAX and BUX at tail sizes of 15, 10, 5 and 2.5 percent. The results for the German market are shown in Table 1. Monte Carlo simulations show that the 15 percent tail size would be appropriate for stable family members with characteristic exponent $1 < \alpha < 2$, whilst the thinner tails would apply to Student distributions with 3 to 5 degrees of freedom. The point estimates are either rather uniform using different choices of the number of tail observations or tend to increase slightly. It can be seen that the point estimates are outside the region characterizing the stable distribution family in all cases. Even if the point estimates and confidence intervals given in Table 1 form already strong evidence against the Paretian model, it seems useful to investigate whether the different tail sizes chosen are all appropriate or not.

It is interesting to test whether the respective tails really follow an extreme value distribution of type /2/ with the estimated parameter α. Under distribution /2/ the random variable $u_i = \alpha \cdot i \cdot \log \frac{x_{(i+1)}}{x_{(i)}}$ follows exponential distribution with origin 0 and parameter 1, i.e. $Exp(x; 0, 1) = 1 - exp(-x)$, where $x_{(1)} \geq x_{(2)} \geq \cdots x_{(i)} \geq \cdots x_{(m)}$ denote the *m* largest observations and α is the parameter of the distribution /3/ (see *Hill*, 1975). The appropriateness of the tail size can be tested by performing the goodness-of-fit test for u_i. The rejection of the exponential distribution for u_i implies the rejection of convergence of the original sample to the limit law at the tail sizes considered. Standard chi-squared test with 20, 16, 10 and 8 equiprobable cells has been implemented for this test procedure. Only very few cases led to the rejection of the exponential distribution.

Table 1

Estimates of tail indices

Stocks and the index	$\alpha_{15\%}$	$\alpha_{10\%}$	$\alpha_{5\%}$	$\alpha_{2.5\%}$
DAX	2.654 (2.326, 2.982)	2.723 (2.310, 3.137)	2.964 (2.327, 3.602)	2.622 (1.819, 3.425)
Allianz	2.601 (2.279, 2.923)	2.746 (2.330, 3.163)	2.766 (2.171, 3.137)	2.446 (1.697, 3.194)
BASF	2.508 (2.198, 2.819)	2.641 (2.241, 3.042)	2.670 (2.119, 3.281)	2.961 (2.055, 3.868)
Bayer	2.571 (2.253, 2.889)	2.845 (2.413, 3.276)	3.320 (2.606, 4.034)	2.879 (1.998, 3.760)
Bayer Hypobank	2.456* (2.153, 2.760)	2.598** (2.204, 2.993)	3.266 (2.563, 3.968)	3.696 (2.565, 4.828)
BMW	2.494 (2.186, 2.803)	2.501 (2.122, 2.881)	2.753 (2.161, 3.346)	2.784 (1.932, 3.636)
Bayer Vereinsbank	2.602 (2.280, 2.924)	3.101 (2.631, 3.571)	3.380 (2.653, 4.107)	2.852 (1.979, 3.726)
Commerzbank	2.849 (2.497, 3.202)	2.996 (2.541, 3.450)	3.018 (2.369, 3.667)	2.896 (2.010, 3.783)
Continental	2.829 (2.479, 3.179)	3.062 (2.598, 3.527)	3.312 (2.599, 4.024)	3.565 (2.474, 4.656)
Daimler	2.732* (2.394, 3.070)	3.094 (2.625, 3.563)	2.882 (2.262, 3.502)	3.099 (2.150, 4.047)
Dt. Babcock	2.622 (2.298, 2.946)	2.848 (2.416, 3.279)	3.211 (2.520, 3.902)	3.141 (2.179, 4.102)
Deutsche Bank	2.698 (2.365, 3.032)	2.917 (2.474, 3.359)	3.132 (2.458, 3.806)	3.096 (2.148, 4.044)
Degussa	2.836 (2.485, 3.187)	3.119 (2.646, 3.592)	3.294 (2.585, 4.002)	3.131 (2.173, 4.090)
Dresdner Bank	2.450 (2.191, 2.809)	2.818 (2.390, 3.245)	2.786* (2.186, 3.385)	3.625 (2.515, 4.734)
Henkel	2.699 (2.339, 2.999)	2.899 (2.459, 3.338)	3.113 (2.443, 3.783)	2.768 (1.921, 3.616)
Hoechst	2.544 (2.229, 2.858)	2.832 (2.403, 3.262)	2.708 (2.125, 3.291)	3.777 (2.621, 4.933)
Karstadt	2.647 (2.320, 2.975)	2.772 (2.351, 3.192)	2.830 (2.221, 3.439)	3.365 (2.335, 4.395)
Kaufhof	2.887 (2.529, 3.243)	3.374** (2.823, 4.370)	3.597 (2.823, 4.370)	3.852 (2.673, 5.031)
Lufthansa	2.503 (2.194, 2.813)	3.073 (2.607, 3.539)	3.335 (2.617, 4.052)	4.011 (2.783, 5.238)
Linde	2.652 (2.324, 2.980)	2.845 (2.414, 3.277)	3.941 (3.093, 4.789)	3.510 (2.435, 4.584)
MAN	2.691 (2.358, 3.024)	2.913 (2.471, 3.355)	3.054 (2.397, 3.711)	3.136 (2.176, 4.096)
Metallges.	2.371 (2.078, 2.664)	2.682 (2.275, 3.089)	2.563 (2.011, 3.114)	2.422 (1.680, 3.163)
Mannesm.	2.654 (2.326, 2.983)	2.865 (2.430, 3.299)	2.858 (2.243, 3.472)	2.756 (1.193, 3.600)
Preussag	2.207 (1.934, 2.480)	2.753 (2.335, 3.170)	3.210 (2.519, 3.900)	3.462 (2.402, 4.522)
RWE	2.526 (2.213, 2.838)	2.829 (2.400, 3.259)	3.044 (2.389, 3.699)	3.286 (2.280, 4.292)
Schering	2.495 (2.187, 2.804)	2.819 (2.391, 3.246)	3.094 (2.429, 3.760)	3.891 (2.700, 5.082)
Siemens	2.450 (2.147, 2.753)	2.947 (2.450, 3.394)	2.901 (2.277, 3.525)	3.031 (2.103, 3.959)
Thyssen	2.805 (2.458, 3.152)	3.006 (2.550, 3.462)	3.365 (2.641, 4.089)	3.524 (2.445, 4.602)
VEBA	2.550 (2.235, 2.865)	2.707 (2.297, 3.118)	2.842 (2.230, 3.453)	3.140 (2.179, 4.102)
Viag	2.460 (2.156, 2.765)	2.789 (2.366, 3.212)	3.179 (2.495, 3.863)	3.274 (2.272, 4.276)
VW	3.033 (2.658, 3.409)	3.016 (2.559, 3.474)	3.121 (2.450, 3.792)	3.160 (2.193, 4.128)
Number of observations in the tail	251	167	83	41

Note. The signs * and ** indicate rejection at 5 or 1 percent level of the extreme value distribution /2/ for the sample of observations in the tail.

In most cases cell frequencies are very close to their hypothetical values have been at all tail sizes considered, hence good convergence to the extreme value distributions can be accepted. To demonstrate the difference in prediction of extremal events, exceeding probabilities per year for certain threshold values were calculated using the estimated stable distributions as well as the semi-parametric tail index estimates. The results for the DAX using the estimated stable parameters $\alpha = 1.737$ and $c = 0.651$ are summarized in Table 2.

Table 2

Annual exceeding probabilities

$|r|$	$\alpha_{5\%}$=2.964	$\alpha_{2.5\%}$=2.622	Stable d. with α_S=1.737	Number of observations in data
$|r|$>0.006	0.5254	0.8734	0.7544	6
$|r|$>0.10	0.1101	0.2372	0.4405	1
$|r|$>0.15	0.0323	0.0834	0.2329	0
$|r|$>0.20	0.0136	0.0396	0.1479	0

Exceeding probabilities are calculated from *Deckers* and *de Haan's* upper quantile estimation formula /4/, stable law probabilities are obtained by interpolation from *Du Mouchel's* tabulation using the scale parameter $c = 0.651$ (obtained for DAX), and for comparability $\alpha = 0$ is assumed.

For simplicity of comparison $\alpha = 0$, i.e. the symmetry is assumed. In the case of estimated stable distribution, the probability of at least one extreme return exceeding in absolute value a certain threshold is computed by interpolation using DuMouchel's tabulation (*DuMouchel*, 1971). For the semi-parametric estimation procedure the consistent estimator of upper quantiles proposed by *Decker* and *de Haan* (1989) has been applied. This estimator is given by:

$$x_p = \left(\frac{k \cdot m}{2p \cdot n} \right)^{\gamma H} \left(1 - 2^{-\gamma H} \right)^{-1} \left(x_{(n-m)} - x_{(n-m/2)} \right) + x_{(n-m/2)}, \qquad /5/$$

where x_p denotes the p-quantile, k is the number of observations per year (here it is 250, the number of trading days per year), n and m are the sample size and the number of observations in the tail region respectively, and γ_H is the inverse of the tail index estimate α_H. Given the x_p value, the probability can be obtained by solving equation /5/. Significant differences in the valuation of the most extreme events exist. As an example, the probability of absolute returns exceed the level of 0.20, is equal to 0.1479 under the stable law hypothesis (it was not rejected by the goodness-of-fit test). Considering the lower one of the tail index estimates, $\alpha_{2.5\%}$ =2.622, the corresponding probability is 0.0396, and for $\alpha_{5\%}$ = 2.964 this probability is 0.0136. If the stable law hypothesis holds, returns of this magnitude are expected to occur once within six to seven years, while the Hill tail index estimates predict occurence of such large absolute returns only once within twenty-five or even once within seventy-five years. This example shows that conclusions drawn from the stable model concerning large absolute returns are misleading.

The point estimates of the various stocks for a given tail size lie within a relatively limited range. Any inference related to the homogeneity or heterogeneity with respect to

the likelihood of extreme returns across stocks is of paramount interest to questions of risk management and portfolio selection. In order to answer these questions the identity of limit laws has been tested. Formally it means the test of hypothesis $\alpha_1 = \alpha_2 = \ldots = \alpha_{30} = \alpha$. Using normality of $1/\alpha_H$ the statistic

$$Q = \sum_{i=1}^{30} \left(\alpha / \alpha_{H,i} - 1 \right)^2 m \qquad \qquad /6/$$

is approximately chi-squared distributed with 30 degrees of freedom under the null hypothesis of identical α values. The test results show that the hypothesis of identical extreme value distributions cannot be rejected.

Table 3

Uniformity test of limit laws across stocks

Tail size	15	10	5	2.5
	percent			
Lower bound	2.460	2.629	2.699	2.618
Upper bound	2.722	3.089	3.364	3.610

The lower and upper bounds determine the intervals of α-values for which the hypothesis $\alpha_1 = \alpha_2 = \ldots = \alpha_{30}$ cannot be rejected at 1 percent level.

The analysis of extreme value distributions confirms that there are no stocks with more pronounced inclination for extreme changes than the average. This result may suggest that macroeconomic shocks may have similar impacts on the formation of their returns.

3. Conclusions

This paper gives a survey of the empirical studies investigating stock return distributions and the detailed analysis of the most recent results for the main German stocks and some conclusions of the investigations for the Hungarian stock market. It has been found that the stable model seems to fit well for most of the stocks when the standard goodness-of-fit test was applied. Counterchecking this result with a semi-parametric analysis of extreme value distributions led to the rejection of the stable law hypothesis. These findings are in accordance with the results reported in the literature indicating that empirical distribution shapes of stock returns are similar to the Pareto-Levy distributions at first sight, while refined methods of analysis point out that they are generated by other distributions. Stable distributions make theoretical modelling difficult. (Closed form expressions for the density functions of stable random variables are available for only three special cases: the normal, the Cauchy and the Bernoulli distributions.) Standard finance theory almost always requires finite second moments of returns, and often finite higher moments as well. Stable distributions also have some counterfactual implications. First, they imply that simple estimates of the variance and higher moments of returns will tend

to increase as the sample size increases, whereas in practice these estimates seem to converge. Secondly, they imply that long-horizon returns will be just as non-normal as short-horizon returns. (Long-horizon returns are sums of short-horizon returns, and these distributions are stable under addition). In practice the evidence for non-normality is much weaker for long-horizon returns than for short-horizon returns. We suggest that returns should be modelled as drawn from a fat-tailed distribution with finite higher moments, such as the t distribution, or as drawn from a mixture of distributions. The return might be conditionally normal, conditional on a variance parameter which in itself is random. Then the unconditional distribution of returns is a mixture of normal distributions, some with small conditional variances that concentrate mass around the mean and others with large conditional variances that put mass in the tails of the distribution. This yields a fat-tailed unconditional distribution with a finite variance and finite higher moments. Since all moments are finite, and long-horizon returns will tend to be closer to the normal distribution than short-horizon returns just as the the Central Limit Theorem implies. The most convenient and widely acceptable paradigm postulates that returns are normally distributed which means that asset prices follow lognormal distributions. Both modern portfolio theory and the Black-Scholes methodology of pricing derivative assets are founded on such a paradigm.

The uncertainty of speculative prices, as measured by the variances and covariances, are changing through time. Explicit modelling time variation in second- or higher-order moments is also proposed as an alternative to the analysis. One of the most prominent tools that has emerged for characterizing variances is the Autoregressive Conditional Heteroskedasticity (ARCH) model of *Engle* (1982) and its various extensions. Since the introduction of the ARCH model a lot of research papers applying this model strategy to financial time series data have already appeared.

The randomness of asset price changes hypothesized by the Efficient Market Hypothesis (EMH) naturally leads to questions about the behaviour of the variance of such changes. If price changes are induced by changes in information, can shocks in fundamental factors affecting the economy explain the price volatility? Or, is the variance of price changes due to other factors? The literature of this topic documents that prices are too volatile and although this evidence does not imply rejection of the EMH, it raises the question of what factors other than fundamental shocks could explain such evidence of high volatility. A nonlinear deterministic methodology, chaotic dynamics, as an alternative to linear stochastic models can clarify the relation between price variability and speculation as well as explain why the empirical studies of the time series properties of asset prices are ambiguous and inconclusive. *Baumol* and *Benhabib* (1989), and *Boldrin* and *Woodford* (1990) used various single variable chaotic maps as a metaphor to illustrate the intellectual possibilities of the deterministic approach.

Another finding of the analysis was that the uniformity of extreme value distributions across the sample indicates a high degree of cooperation among the German firms forming the DAX index, and one may conclude the same on some lower level for the stocks forming the BUX. This is the reason for reporting only the computational results of the German market analysis. It is also likely that the differences in results between previous and present investigations are caused by the previously used less efficient methods. (As

some examples: *Akgiray*, *Booth* and *Loistl* (1989) found tail indices in the range (3, 13), *Lux* and *Varga* (1996) reported the interval (2, 4)).

One may conclude that the uniformity of limit laws may be a more general phenomenon and it is worth searching for the reason of this behaviour.

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