

HYPERGRAPHS WITHOUT EXPONENTS

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ABSTRACT. Here we give a short, concise proof for the following result. There exists a k -uniform hypergraph H (for $k \geq 5$) without exponent, i.e., when the Turán function is not polynomial in n . More precisely, we have $\text{ex}(n, H) = o(n^{k-1})$ but it exceeds n^{k-1-c} for any positive c for $n > n_0(k, c)$.

This is an extension (and simplification) of a result of Frankl and the first author from 1987 where the case $k = 5$ was proven. We conjecture that it is true for $k \in \{3, 4\}$ as well.

1. NOTATION, THE TURÁN PROBLEM

We start with some standard notation. A k -graph (or k -uniform hypergraph) H is a pair (V, E) with $V = V(G)$ a set of vertices, and $E = E(G)$ a collection of k -sets from V , which are the hyperedges (or k -edges) of H . The s -shadow, $\partial_s H$, is the family of s -sets contained in the hyperedges of H . So $\partial_1 H$ is the set of non-isolated vertices, and $\partial_2 H$ is a graph. We write $[n]$ for $\{1, 2, \dots, n\}$. Given a set A and an integer k , we write $\binom{A}{k}$ for the set of k -sets of A . When there is no confusion, we may also use ‘edge’ for ‘ k -edge’. The complete k -graph on n vertices is the k -graph $K_n^{(k)} = ([n], \binom{[n]}{k})$. Let $I_k(i)$ denote the k -uniform hypergraph consisting of two hyperedges sharing exactly i vertices. The k -graph H is k -partite if there exists a partition $\{P_1, \dots, P_k\}$ of $V(H)$ such that for every edge $e \in E(H)$ and part P_i we have $|e \cap P_i| = 1$. The complete k -partite k -graph $K_k(P_1, \dots, P_k)$ has all of such edges, $|E(K_k(P_1, \dots, P_k))| = |P_1| \times \dots \times |P_k|$.

Given a family of k -graphs \mathcal{F} , we say that a k -graph H is \mathcal{F} -free if it contains no member of \mathcal{F} as a subgraph. We write $\text{ex}(n, \mathcal{F})$ (or $\text{ex}_k(n, \mathcal{F})$ if we want to emphasize k) for the maximum number of k -edges that can be present in an n -vertex \mathcal{F} -free k -graph. The function $\text{ex}(n, \mathcal{F})$ is referred to as the Turán number of \mathcal{F} . We leave out parentheses whenever it is possible, e.g., in case of $|\mathcal{F}| = 1$ we write $\text{ex}(n, F)$ instead of $\text{ex}(n, \{F\})$.

2. RATIONAL EXPONENTS AND NON-POLYNOMIAL TURÁN FUNCTIONS

Erdős and Simonovits (see [4, 6]) conjectured that for any rational $1 \leq \alpha \leq 2$ there exists a graph F with

$$(2.1) \quad \text{ex}_2(n, F) = \Theta(n^\alpha)$$

and conversely, for every graph F we have

$$(2.2) \quad \text{ex}_2(n, F) = \Theta(n^\alpha)$$

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for some rational α . Bukh and Conlon [3] showed that the first conjecture holds if we can forbid finite families of graphs. For a single graph, it is still unknown.

For hypergraphs Frankl [9] showed that all rationals occur as exponents of $\text{ex}_k(n, \mathcal{F})$ for some k and for some finite family \mathcal{F} of k -uniform hypergraphs. Fitch [8] showed that for a fixed k all rational numbers between 1 and k occur as exponents of $\text{ex}_k(n, \mathcal{F})$ for some family \mathcal{F} of k -uniform hypergraphs.

We say that a function $f(n) : \mathbb{N} \rightarrow \mathbb{R}$ has *no exponent* if there is no real α such that $f(n) = \Theta(n^\alpha)$. In other words, the order of magnitude of $f(n)$ is not a polynomial.

Brown, Erdős, and Sós [2] proposed the following problem. Determine (or estimate) $f_k(n, v, e)$, i.e., the maximum number of edges in a k -uniform, n -vertex hypergraph in which no v vertices span e or more edges. This is a Turán type problem: Let $\mathcal{G}_k(v, e)$ be the family of k -graphs, each member having e edges and at most v vertices, then $f_k(n, v, e) = \text{ex}_k(n, \mathcal{G}_k(v, e))$.

Ruzsa and Szemerédi [25] showed that if a 3-uniform hypergraph does not contain three hyperedges on six vertices, then it has $o(n^2)$ edges, and they also gave a construction with $n^{2-o(1)}$ hyperedges. The assumption on the hypergraph is equivalent to forbidding the following two sub-hypergraphs $\{123, 124\}$ (a pair covered twice) and $\{123, 345, 561\}$ (a linear triangle). They proved

$$(2.3) \quad n^{2-o(1)} < \frac{1}{10}nr_3(n) < f_3(n, 6, 3) - (n/2) \\ \leq \text{ex}_3(n, \{\{123, 124\}, \{123, 345, 561\}\}) \leq f_3(n, 6, 3) = o(n^2).$$

(For the definition of $r_3(n)$, see the paragraph containing (5.3) in Section 5). Thus they found a family of two hypergraphs such that not only its Turán number does not have a rational exponent, it does not have an exponent at all. This is the famous (6, 3)-theorem, $f_3(n, 6, 3)$ is non-polynomial.

Erdős, Frankl, and Rödl [5] extended this to every k proving $f_k(n, 3k - 3, 3) = o(n^2)$ but $\lim_{n \rightarrow \infty} f_k(n, 3k - 3, 3)/n^{2-\varepsilon} = \infty$ for all $\varepsilon > 0$ ($k \geq 3$ and ε are fixed, $n \rightarrow \infty$). The proof of the upper bound here and in (2.3) are based only on Szemerédi's regularity lemma [27].

3. SINGLE HYPERGRAPHS WITH NO EXPONENTS

Answering a question of Erdős, a single 5-uniform hypergraph with no exponent was presented in [12]:

Theorem 3.1 (Frankl and Füredi [12]). *Let $H = \{12346, 12457, 12358\}$. Then $\text{ex}_5(n, H) = o(n^4)$ but $\text{ex}_5(n, H) \neq O(n^{4-\varepsilon})$ for any $\varepsilon > 0$.*

One aim of this paper is to give a short proof for this result. The original proof heavily relied on the delta-system method, we can get rid of that. We also extend it for all $k \geq 5$. We *conjecture* that examples with no exponents should exist for $k = 3$ and 4, too.

Definition 3.2. *Let us consider three disjoint sets of vertices $A = \{a_1, \dots, a_{k-r}\}$, $B = \{b_1, \dots, b_r\}$ and $C = \{c_1, \dots, c_r\}$. Let $Q_k(r)$ denote the k -uniform hypergraph consisting of all the hyperedges of the form $A \cup (B \setminus \{b_i\}) \cup \{c_i\}$, for $1 \leq i \leq r$.*

So $|E(Q_k(r))| = r$ and $|V(Q_k(r))| = k + r$. To avoid trivialities we suppose that $r \geq 2$ since $Q_k(0)$ is an empty hypergraph and $Q_k(1)$ has only one hyperedge. In this paper we study $\text{ex}_k(n, Q_k(r))$ for every pair of values k and r , $k \geq r \geq 2$, and we either determine the order of magnitude or show that there is no exponent.

Note that $Q_k(2) = I_k(k-2)$ (two k -edges meeting in $k-2$ elements). The study of the Turán number of $I_k(i)$ has been initiated by Erdős [4]. Frankl and Füredi [11] proved that $\text{ex}_k(n, I_k(i)) = \Theta(n^{\max\{i, k-i-1\}})$ for $0 \leq i \leq k-1$. This gives $\text{ex}_k(n, Q_k(2)) = \Theta(n^{k-2})$ for $k \geq 3$ and $\text{ex}_2(n, Q_2(2)) = \Theta(n)$.

Our main result is the following theorem.

Theorem 3.3. *If $k \geq r \geq 3$ and $r \geq (k/2) + 1$, then $\text{ex}_k(n, Q_k(r)) = \Theta(n^{k-1})$.*

If $k \geq r \geq 3$ and $r \leq (k+1)/2$, then $\text{ex}_k(n, Q_k(r)) = o(n^{k-1})$ but $\text{ex}_k(n, Q_k(r)) \neq O(n^{k-1-\varepsilon})$ for any $\varepsilon > 0$.

Note that $Q_5(3) = \{12346, 12457, 12358\}$, so this Theorem is indeed an extension of Theorem 3.1. Since $Q_k(3) \subset \dots \subset Q_k(k)$, we obviously have

$$\text{ex}_k(n, Q_k(3)) \leq \text{ex}_k(n, Q_k(4)) \leq \dots \leq \text{ex}_k(n, Q_k(k)).$$

So to prove Theorem 3.3 we need to show that for $k \geq r \geq 3$ as $n \rightarrow \infty$ we have

- (3.3.a) $\text{ex}_k(n, Q_k(k)) = O(n^{k-1})$,
- (3.3.b) $\text{ex}_k(n, Q_k(r)) = \Omega(n^{k-1})$ if $k \leq 2r - 2$,
- (3.3.c) $\text{ex}_k(n, Q_k(r)) = o(n^{k-1})$ if $k \geq 2r - 1$,
- (3.3.d) $\text{ex}_k(n, Q_k(3)) = \Omega(n^{k-1-\varepsilon})$ if $k \geq 5$, $\forall \varepsilon > 0$ fixed.

We emphasize that to prove that $Q_k(3)$ has no exponent (for $k \geq 5$), we only use the hypergraph removal lemma (Lemma 5.1) and our lower bound construction from Section 9.

Problem. Determine $\limsup_{n \rightarrow \infty} \text{ex}_k(n, Q_k(r))/n^{k-1}$ for $4 \leq k \leq 2r - 2$.

The rest of the paper is organized as follows. In Section 4 we discuss a strongly related problem, in Sections 5 and 6 the necessary tools are presented, Section 7 contains the proof of the upper bounds (3.3.a) and (3.3.c), Section 8 is a simple construction to establish the lower bound (3.3.b), and our most interesting construction for the lower bound (3.3.d) is presented in Section 9. Finally, a simple proof for (3.3.d) is presented in Section 10 for the special case $k = 2r - 1$.

4. PRINCIPAL FAMILIES

An easy averaging argument shows that $\text{ex}(n, \mathcal{F})/\binom{n}{k}$ is nonincreasing and hence tends to a limit as $n \rightarrow \infty$. This limit, denoted by $\pi(\mathcal{F})$, is the *Turán density* of \mathcal{F} . The Turán (density) problem for k -graphs is this: given a family \mathcal{F} , determine $\pi(\mathcal{F})$. This question for 2-graphs, i.e., for a family of ordinary graphs \mathcal{G} , has been completely answered by the Erdős-Stone-Simonovits Theorem, which states $\pi(\mathcal{G}) = (m-2)/(m-1)$, where m is the smallest chromatic number of graphs in \mathcal{G} . Hence

$$(4.1) \quad \pi(\mathcal{G}) = \min_{G \in \mathcal{G}} \pi(G).$$

Thus Turán density is *principal* among ordinary graphs.

By contrast very few Turán densities of k -graphs are known (although Pikhurko [20] gave infinitely many values). Nonprincipality for 3-graphs was conjectured by Mubayi and Rödl [17], and first exhibited by Balogh [1]. Mubayi and Pikhurko [18] gave the first example of a *nonprincipal pair* of 3-graphs, i.e. a pair F, F' with $\pi(F, F') < \min\{\pi(F), \pi(F')\}$. The simplest pair is due to Falgas-Ravry and Vaughan [7], who proved $\pi(K_4^-, F_{3,2}) = 5/18 = 0.2777\dots$, where $E(K_4^-) = \{123, 124, 134\}$ and $E(F_{3,2}) = \{123, 124, 125, 345\}$. On the other hand there is a lower bound $\pi(K_4^-) \geq 2/7$ from [10], and in [15] it was proved that $\pi(F_{3,2}) = 4/9$.

Equation (4.1) implies that, in case of ordinary graphs, if $\min_{G \in \mathcal{G}} \chi(G) > 2$ then always exists a $G \in \mathcal{G}$ such that

$$(4.2) \quad \text{ex}(n, \mathcal{G}) = (1 + o(1))\text{ex}(n, G).$$

When bipartite graphs are involved then such a strong principality does not hold. Erdős and Simonovits [6] proved that $\text{ex}(n, \{C_4, C_5\}) = (1 + o(1))(n/2)^{3/2}$, on the other hand we have $\text{ex}(n, C_4) = (\frac{1}{2} + o(1))n^{3/2}$ and $\text{ex}(n, C_5) = \lfloor n^2/4 \rfloor$ (for $n \geq 6$). So, instead of (4.2), Erdős and Simonovits [6] made the following *compactness conjecture* (in fact, we can call it *weak* principality), that any finite family \mathcal{G} of graphs (with $\text{ex}(n, \mathcal{G}) \neq O(1)$) contains a single graph G such that

$$(4.3) \quad \text{ex}_2(n, \mathcal{G}) = \Theta(\text{ex}_2(n, G)).$$

This conjecture with the result of Bukh and Conlon (mentioned after (2.2)) would imply conjecture (2.1).

The upper bound in the Ruzsa-Szemerédi (6, 3)-theorem (i.e., $\text{ex}_3(n, \{I_3(2), Q_3(3)\}) = o(n^2)$, see (2.3)) shows that there is no compactness for hypergraphs. Indeed, the Turán number of $I_3(2)$ is $n(n-1)/6 + O(n)$ (Steiner triple systems are extremal) and $\text{ex}(n, Q_3(3)) \geq \binom{n-1}{2}$ (because the centered family $\{f : f \in \binom{[n]}{3}, 1 \in f\}$ does not contain linear triangles). Actually, it is known [12] that $\text{ex}(n, Q_3(3)) = \binom{n-1}{2}$ for $n > n_0$, so both of these hypergraphs have quadratic Turán numbers.

5. LEMMAS AND TOOLS

The following observation, due to Erdős and Kleitman, is one of the basic tools to determine the order of magnitude of the size of a k -graph H : Every k -graph H has a k -partition of its vertices $V(H) = P_1 \cup \dots \cup P_k$ into almost equal parts ($||P_i| - |P_j|| \leq 1$) such that for the k -partite subhypergraph H' with $E(H') := E(H) \cap E(K_k(P_1, \dots, P_k))$, one has

$$(5.1) \quad \frac{k!}{k^k} |E(H)| \leq |E(H')| \leq |E(H)|.$$

Suppose $n \geq r \geq t \geq 1$ are integers. An r -graph H on n vertices is called an (n, r, t) -*packing* if $|e \cap e'| < t$ holds for every $e, e' \in E(H)$, $e \neq e'$. The maximum of $|E(H)|$ of such packings is denoted by $P(n, r, t)$. Since then $\binom{n}{t} \geq |\partial_t H| = \binom{r}{t} |E(H)|$, we have $P(n, k, t) \leq \binom{n}{t} / \binom{r}{t}$. It is known that $P(n, r, t) = (1 + o(1)) \binom{n}{t} / \binom{r}{t}$ when r and t are fixed and n tends to infinity. (Even *perfect* packings, i.e., Steiner systems $S(n, r, t)$'s, exist if some divisibility constraints hold and n is sufficiently large.) We only use the following easy statement: If r is fixed and $n \rightarrow \infty$ then

$$(5.2) \quad P(n, r, t) \geq \binom{n}{t} / \binom{r}{t}^2 = \Omega(n^t).$$

Let k and n be positive integers. A set of numbers A is called AP_k -*free* if it does not contain k distinct elements forming an arithmetic progression of length k . As usual, let $r_k(n)$ denote the maximum size of an AP_k -free sequence $A \subseteq [n]$. The celebrated Szemerédi's theorem [26] states that for a fixed k as $n \rightarrow \infty$ we have

$$(5.3) \quad r_k(n) = o(n).$$

(The case $r_3(n) = o(n)$ was proved much earlier by K. F. Roth).

Let k be an integer and p be a prime, $p > k$. We say that $S \subseteq \{0, \dots, p-1\}$ is k -good if for any $m_1, m_2, m_3 \in \{-k, -k+1, \dots, -1\} \cup \{1, \dots, k\}$ and $s_1, s_2, s_3 \in S$ the following equations hold:

$$\left. \begin{array}{l} m_1 + m_2 + m_3 = 0 \quad \text{and} \\ m_1 s_1 + m_2 s_2 + m_3 s_3 = 0 \end{array} \right\} \text{ imply } s_1 = s_2 = s_3.$$

Here addition and multiplication are taken modulo p . Let $s_k(p)$ denote the size of the largest k -good set. The following result is an easy extension of Behrend's construction, see, e.g., Ruzsa [23, 24]: There is a $c_k > 0$ such that

$$p \exp[-c_k \sqrt{\log p}] < s_k(p).$$

We only need that if k and $\varepsilon > 0$ are fixed and $p \rightarrow \infty$, then

$$(5.4) \quad s_k(p) > p^{1-\varepsilon}$$

Note that a k -good set cannot contain a (strictly increasing) arithmetic progression of length 3, so $s_k(p) \leq r_3(p)$ and $r_3(p) = o(p)$ by Roth's theorem, see (5.3).

We will also use the so-called hypergraph removal lemma. It (together with other versions of hypergraph regularity) was developed by several groups of researchers, see [16, 19, 21, 22, 28].

Lemma 5.1 (Hypergraph Removal Lemma). *For any $\varepsilon > 0$ and integers $\ell \geq k$, there exist $\delta > 0$ and an integer n_0 such that the following statement holds. Suppose F is a k -uniform hypergraph on ℓ vertices and H is a k -uniform hypergraph on $n \geq n_0$ vertices, such that H contains at most $\delta \binom{n}{\ell}$ copies of F . Then one can delete at most $\varepsilon \binom{n}{k}$ hyperedges from H such that the resulting hypergraph is F -free.*

6. SZEMERÉDI'S $r_k(n) = o(n)$ BY FRANKL AND RÖDL

Recall that $I_k(i)$ denotes the k -uniform hypergraph consisting of two hyperedges sharing exactly i vertices. Frankl and Rödl [13] generalized the lower bound of the celebrated (6, 3)-theorem (i.e., (2.3)) of Ruzsa and Szemerédi [25] as follows.

Theorem 6.1 ([13]). *For any integer $k \geq 3$ there exists a $c'_k > 0$ such that for all $n \geq k$*

$$c'_k \times r_k(n) \times n^{k-2} \leq \text{ex}_k(n, \{Q_k(k), I_k(k-1)\}).$$

They conjectured $\text{ex}_k(n, \{Q_k(k), I_k(k-1)\}) = o(n^{k-1})$ and proved the case $k = 4$ (the case $k = 3$ is part of (2.3)). In order to prove $\text{ex}_4(n, \{Q_4(4), I_4(3)\}) = o(n^3)$ they developed a hypergraph removal lemma for the 3-uniform case. They also described how the hypergraph removal lemma (Lemma 5.1) would imply the general upper bound $o(n^{k-1})$. Since then Lemma 5.1 has been proved, we have the following result.

Corollary 6.2. *For any $k \geq 2$ we have $\text{ex}_k(n, \{Q_k(k), I_k(k-1)\}) = o(n^{k-1})$.*

Note that Theorem 6.1 and Corollary 6.2 imply Szemerédi's theorem: $r_k(n) = o(n)$.

The upper bound in Corollary 6.2 supplies a non-compact pair for k -graphs. The Turán number of $I_k(k-1)$ is $\Theta(n^{k-1})$ (see (5.2)) and $\text{ex}(n, Q_k(k)) = \binom{n}{k-1} + O(n^{k-2})$ (see [12] and [14]).

Since the above corollary plays such an important role in our main result, we include its few line proof for the sake of completeness.

Proof of Corollary 6.2. Let H be a $Q_k(k)$ and $I_k(k-1)$ -free k -graph on n vertices. We will give an upper bound on its size. By (5.1) we may suppose that H is k -partite

with parts P_1, \dots, P_k . Consider its shadow ∂H , it is a $(k-1)$ -uniform hypergraph. Since H is $I_k(k-1)$ -free, each $f \in \partial H$ is contained in a unique $e(f) \in E(H)$. We get $\binom{k}{k-1}|E(H)| = |E(\partial H)|$. This already gives $|E(H)| = O(n^{k-1})$.

Every edge $e \in E(H)$ induces a complete subhypergraph $K_k^{(k-1)}$ in ∂H . We claim that these are the only cliques of size k in ∂H . Consider K a copy of $K_k^{(k-1)}$ in ∂H . Then $|P_i \cap V(K)| = 1$ for each P_i . If $e(f) = V(K)$ for some $f \in E(K)$ then K is the clique generated by $V(K) = e(f) \in E(H)$. Otherwise, when $e(f) \neq V(K)$ for each $f \in E(K)$, the k hyperedges $\{e(f) : f \in E(K)\}$ form a copy of $Q_k(k)$. This contradiction implies that ∂H is indeed the edge-disjoint union of cliques induced by the edges of H , and these are the only k -cliques in ∂H .

Therefore, the number of copies of $K_k^{(k-1)}$ in ∂H is $O(n^{k-1}) = o(n^{|V(K)|})$. Then by the hypergraph removal lemma (Lemma 5.1) there exists a subhypergraph H' , $E(H') \subset E(\partial H)$, so that $E(H')$ meets every copy of $K_k^{(k-1)}$ in ∂H and $|E(H')| = o(n^{k-1})$. For such an H' we have $|E(H)| \leq |E(H')|$, finishing the proof. \square

7. PROOF OF THEOREM 3.3, UPPER BOUNDS

In this section we prove (3.3.a) and (3.3.c), the upper bounds for $\text{ex}_k(n, Q_k(r))$.

Let H be a $Q_k(r)$ -free k -graph on n vertices. We will give an upper bound on $|E(H)|$. By (5.1) we may suppose that H is k -partite with parts P_1, \dots, P_k . For a hyperedge $e \in E(H)$, let $D(e) \subseteq [k]$ denote the set of integers i such that there is another hyperedge $e' \in E(H)$ that differs from e only in P_i , $e \setminus P_i = e' \setminus P_i$. Note that $|D(e)| < r$ because H is $Q_k(r)$ -free.

By the pigeonhole principle there is a set $D \subset \{1, \dots, k\}$ such that there are at least $|E(H)|/2^k$ hyperedges $e \in E(H)$ with $D(e) = D$. Let H' be the k -graph of these edges, $E(H') := \{e \in E(H) : D(e) = D\}$. Set $\ell := k - |D|$, we have $\ell \geq k - r + 1$, $\ell \geq 1$.

Let T be an edge of the complete $|D|$ -partite hypergraph with parts $\{P_i : i \in D\}$, i.e., $|T| = |D|$ and $|T \cap P_i| = 1$ for each $i \in D$. (D might be the empty set). There are at most $O(n^{k-\ell})$ appropriate T . Define $H'[T]$ as the *link* of T in H' , i.e., it is an ℓ -graph with edges $\{e \setminus T : T \subset e \in E(H')\}$.

Observe first that $H'[T]$ is $I_\ell(\ell-1)$ -free. Indeed, two hyperedges of $H'[T]$ sharing $\ell-1$ vertices would mean two hyperedges in H' sharing $k-1$ vertices such that their only difference is in a part not belonging to D . So every $(\ell-1)$ -element set is contained in at most one hyperedge in $H'[T]$, thus $|H'[T]| \leq \binom{n}{\ell-1}$. Since $|E(H')| = \sum_T |E(H'[T])|$, we obtained

$$(7.1) \quad |E(H)| = O(|E(H')|) = O(n^{k-\ell}) \binom{n}{\ell-1} = O(n^{k-1}),$$

completing the proof of (3.3.a).

Finally, let us assume $k \geq 2r - 1$, i.e., $\ell \geq r$. We claim that in this case $H'[T]$ is also $Q_\ell(\ell)$ -free. Indeed, if we add T to the hyperedges of a copy of $Q_\ell(\ell)$ from $H'[T]$, we obtain a $Q_k(\ell)$ in H' . Since $Q_k(\ell)$ contains a $Q_k(r)$, this is a contradiction. Thus we have $|E(H'[T])| = o(n^{\ell-1})$ by Corollary 6.2. We complete the proof as in (7.1)

$$|E(H)| = O(|E(H')|) = O(n^{k-\ell}) \times o(n^{\ell-1}) = o(n^{k-1}). \quad \square$$

For the case $k \geq 2r - 1$ we actually proved that $\text{ex}_k(n, Q_k(r)) \leq \text{ex}_k(n, \{Q_k(k), I_k(k-1)\})$. This is $o(n^{k-1})$ by Corollary 6.2. Theorem 6.1 shows that this way the upper bound cannot be improved significantly, because $\text{ex}_k(n, \{Q_k(k), I_k(k-1)\}) = \Omega(r_k(n) \times n^{k-2})$. In Section 9 we will present the slightly weaker lower bound $\Omega(s_k(n) \times n^{k-2})$ for $\text{ex}_k(n, Q_k(r))$.

8. PROOF OF THEOREM 3.3, THE POLYNOMIAL RANGE

In this section we prove the lower bound (3.3.b) by giving a construction.

Since $k \leq 2r - 2$, we have $r - 1 \geq k + 1 - r \geq 1$. Let X and Y be two disjoint sets, $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Let H^1 be an $(|X|, r - 1, r - 2)$ -packing of maximum size, i.e., an $(r - 1)$ -uniform hypergraph such that any two hyperedges share at most $r - 3$ vertices. By (5.2) we have $|E(H^1)| = \Theta(n^{r-2})$. Let H^2 be the complete $(k - r + 1)$ -uniform hypergraph with vertex set Y . Finally, let H^3 be the k -graph with vertex set $X \cup Y$ having as hyperedges all the k -sets that are unions of a hyperedge of H^1 and a hyperedge of H^2 . Then H^3 has $\Theta(n^{k-1})$ hyperedges. We claim that H^3 is $Q_k(r)$ -free.

Assume, on the contrary, that there is a copy of $Q_k(r)$ in H^3 , $E(Q_k(r)) = \{f_1, f_2, \dots, f_r\}$. Note that $|\cap f_i| = k - r < (k - r + 1) \leq r - 1$ and the symmetric differences $\{f_i \Delta f_j : 1 \leq i < j \leq r\}$ are all distinct 4-element sets. Consider, first, the case when for some $i \neq j$ we have $f_i \cap X = f_j \cap X$. Then all $f_t \cap X$ are identical. Indeed, if there exists an $f_t \cap X \neq f_i \cap X$, then these two $(r - 1)$ -sets have symmetric difference at least 4, so it should be exactly 4, and then $(f_i \cap X) \Delta (f_t \cap X)$ and $(f_j \cap X) \Delta (f_t \cap X)$ are identical 4-element sets, a contradiction. Then $|\cap f_i| \geq r - 1$, a contradiction.

From now on, we may suppose that the $(r - 1)$ -element sets $\{f_i \cap X\}$ are all distinct. Then, because $|(f_i \cap X) \Delta (f_j \cap X)| \geq 4$ we have that $f_i \cap Y = f_j \cap Y$ for all $1 \leq i < j \leq r$. Hence $|\cap f_i| \geq k - r + 1$, a final contradiction. \square

9. PROOF OF THEOREM 3.3, A NON-POLYNOMIAL LOWER BOUND

In this section we prove the lower bound (3.3.d) by giving a construction. We will show that if $n = kp$, where $k \geq 5$ and p is a prime, then $\text{ex}(n, Q_k(3)) \geq p^{k-2} s_k(p)$. As $\text{ex}(n, Q_k(3))$ is monotone in n and there is a prime between $n/2k$ and n/k , this and (5.4) give the desired bound $\Omega(n^{k-1-o(1)})$ for $\text{ex}(n, Q_k(3))$.

Let the vertex set V consist of the pairs (i, j) with $1 \leq i \leq k$ and $0 \leq j \leq p - 1$. Choose two integers $0 \leq \alpha, \beta \leq p - 1$ and a k -good set $S \subset \{0, \dots, p - 1\}$ of size $s_k(p)$. Suppose that $m_1, \dots, m_k \in \{1, \dots, k\}$ are distinct integers (i.e., a permutation of $[k]$). We define a k -partite k -graph $F = F(S, \alpha, \beta)$ on V with parts $P_i := \{(i, j) : 0 \leq j \leq p - 1\}$. A k -set $\{(1, x_1), (2, x_2), \dots, (k, x_k)\}$ is a hyperedge of F if the following two equations hold.

$$\begin{aligned} \left(\sum_{i=1}^k x_i \right) &= \alpha \pmod{p}, \\ \left(\sum_{i=1}^k m_i x_i \right) &\in S + \beta \pmod{p}. \end{aligned}$$

We have $|F(S, \alpha, \beta)| = p^{k-2} s_k(p)$. Indeed, for any $s \in S$ we can pick $k - 2$ values x_3, \dots, x_k arbitrarily, and since $m_1 \neq m_2$, the above two equations uniquely determine x_1 and x_2 .

Claim 9.1. F is $Q_k(3)$ -free.

Proof of Claim. Suppose, on the contrary, that there is a copy of $Q_k(3)$ in F , and let A, B, C be the sets of vertices as in Definition 3.2. Without loss of generality we may assume that $A = \{(i, x_i) : 4 \leq i \leq k\}$, $b_i = (i, x_i)$ ($i = 1, 2, 3$), and $c_i = (i, y_i)$ ($i = 1, 2, 3$).

Then the constraints in the definition of F imply the following equations.

$$\begin{aligned} \left(\sum_{i=1}^k x_i \right) + y_1 - x_1 &= \alpha \pmod{p} \\ \left(\sum_{i=1}^k x_i \right) + y_2 - x_2 &= \alpha \pmod{p} \\ \left(\sum_{i=1}^k x_i \right) + y_3 - x_3 &= \alpha \pmod{p} \\ \left(\sum_{i=1}^k m_i x_i \right) + m_1(y_1 - x_1) &= s_1 + \beta \pmod{p} \\ \left(\sum_{i=1}^k m_i x_i \right) + m_2(y_2 - x_2) &= s_2 + \beta \pmod{p} \\ \left(\sum_{i=1}^k m_i x_i \right) + m_3(y_3 - x_3) &= s_3 + \beta \pmod{p} \end{aligned}$$

for some $s_1, s_2, s_3 \in S$. Define u and v as $u := \alpha - (\sum_{i=1}^k x_i)$ and $v := (\sum_{i=1}^k m_i x_i) - \beta$. We obtain

$$(9.1) \quad y_j - x_j = u, \pmod{p} \quad \text{for } j = 1, 2, 3$$

and

$$(9.2) \quad v + m_j u = s_j \pmod{p} \quad \text{for } j = 1, 2, 3.$$

These imply

$$(v + m_1 u - s_1)(m_2 - m_3) + (v + m_2 u - s_2)(m_3 - m_1) + (v + m_3 u - s_3)(m_1 - m_2) = 0.$$

Rearranging

$$(m_3 - m_2)s_1 + (m_1 - m_3)s_2 + (m_2 - m_1)s_3 = 0 \pmod{p}.$$

As S is a k -good set and $1 \leq |m_i - m_j| \leq k$, we have $s_1 = s_2 = s_3$. Then (9.2) gives $v + m_1 u = v + m_2 u = v + m_3 u$ implying $u = 0$. Then (9.1) gives $x_j = y_j$ (for $j = 1, 2, 3$), a contradiction. \square

10. A LOWER BOUND FOR THE CASE $k = 2r - 1$

In this section we present a simple construction implying the lower bound in (3.3.d) for the case $k = 2r - 1$. It gives $\text{ex}(n, Q_{2r-1}(r)) \geq \Omega(r_r(n)n^{k-2})$, a stronger lower bound than the one in the previous section. The construction is similar to the one in Section 8.

Start with an r -graph H^1 with a set V_1 of $\lfloor n/2 \rfloor$ vertices and $\Omega(r_r(n)n^{r-2})$ hyperedges that is both $Q_r(r)$ -free and $I_r(r-1)$ -free. The existence of such hypergraphs was proved by Frankl and Rödl [13], see Theorem 6.1. Add a set V_2 of $\lceil n/2 \rceil$ new vertices and take all k -edges containing an r -edge of H^1 and $r-1$ vertices from V_2 . This hypergraph H has $\Omega(r_r(n)n^{k-2})$ hyperedges.

We claim that H is $Q_k(r)$ -free ($k = 2r - 1$). Suppose, on the contrary, that H contains a copy of $Q_k(r)$, and let $A, B = \{b_1, \dots, b_r\}$, and $C = \{c_1, \dots, c_r\}$ be the sets of vertices as in Definition 3.2. Since $|B|, |C| > r - 1$ they both share at least one element with V_1 , say $b_i \in B \cap V_1$ and $c_j \in V_1 \cap C$. If c_i is not in V_1 , then $e_i := A \cup B \setminus \{b_i\} \cup \{c_i\}$ has less

elements in V_1 than $e_j := A \cup B \setminus \{b_j\} \cup \{c_j\}$ does. It is a contradiction as both $e_i \cap V_1$ and $e_j \cap V_1$ are hyperedges of H^1 . We obtained that $b_i \in V_1 \cap B$ implies $c_i \in V_1 \cap C$.

If there exists a $b_t \in V_2 \cap B$ then c_t also must belong to V_2 . Otherwise, $|e_t \cap V_1| > |e_i \cap V_1|$, a contradiction. In this case $b_i, c_i \in V_1$ and $b_t, c_t \in V_2$ imply that $e_t := A \cup B \setminus \{b_t\} \cup \{c_t\}$ shares $r - 1$ elements with $e_i = A \cup B \setminus \{b_i\} \cup \{c_i\}$ inside V_1 , which contradicts the $I_r(r - 1)$ -free property of H^1 .

Hence we may assume that each $b_t \in B$ belongs to V_1 . Then $C \subset V_1$, too, so $A \subset V_2$. Then the r -edges $B \setminus \{b_t\} \cup \{c_t\}$ form a copy of $Q_r(r)$ in H^1 . This final contradiction completes the proof.

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