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A characterization of the identity function

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Abstract. We prove that if a multiplicative function f satisfies the equation $f(n^2+m^2+3)=f(n^2+1)+f(m^2+2)$ for all positive integers n and m, then either $f(n)$ is the identity function or $f(n^2+m^2+3)=f(n^2+1)=f(m^2+2)=0$ for all positive integers.

Throughout this paper $\mathbf{N}$ denotes the set of positive integers and let $\mathcal{M}$ be the set of complex valued multiplicative functions f such that $f(1)=1$.

In 1992, C. Spiro [3] showed that if $f\in\mathcal{M}$ is a function such that $f(p+q)=f(p)+f(q)$ for all primes p and q, then $f(n)=n$ for all $n\in\mathbf{N}$. Recently, in the paper [2] written jointly with J. M. de Koninck and I. Kátai we proved that if $f\in\mathcal{M}$, $f(p+n^2)=f(p)+f(n^2)$ holds for all primes p and $n\in\mathbf{N}$, then $f(n)$ is the identity function. It follows from results of [1] that a completely multiplicative function f satisfies the equation $f(n^2+m^2)=f(n^2)+f(m^2)$ for all $n,m\in\mathbf{N}$ if and only if $f(2)=2$, $f(p)=p$ for all primes $p\equiv 1 \pmod 4$ and $f(q)=q$ or $f(q)=-q$ for all primes $p\equiv 3 \pmod 4$.

The purpose of this note is to prove the following

Theorem. *Assume that $f\in\mathcal{M}$ satisfies the condition*

$$f\left(n^2+m^2+3\right)=f(n^2+1)+f(m^2+2) \tag{1}$$

for all $n,m\in\mathbf{N}$. Then either

$$f(n^2+1)=f(m^2+2)=f(n^2+m^2+3)=0 \quad \text{for all} \quad n,m\in\mathbf{N}, \tag{2}$$

or $f(n)=n$ for all $n\in\mathbf{N}$.

Corollary. *If $f\in\mathcal{M}$ satisfies the condition (1) and $f(n_0^2+1)\neq 0$ for some $n_0\in\mathbf{N}$, then $f(n)$ is the identity function.*

First we prove the following lemma.

Lemma. *Assume that the conditions of Theorem 1 are satisfied. Then either (2) is satisfied for all $n\in\mathbf{N}$ or the conditions*

$$\begin{aligned} &f(n^2+1)=n^2+1, \quad f(m^2+2)=m^2+2 \quad \text{and} \\ &f(n^2+m^2+3)=n^2+m^2+3 \end{aligned} \tag{3}$$

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simultaneously hold for all $n, m \in \mathbf{N}$.

Proof. From (1), we have

$$f(n^2+1)+f(m^2+2)=f(m^2+1)+f(n^2+2)$$

for all $n, m \in \mathbf{N}$, and so

$$f(n^2+2)-f(n^2+1)=f(3)-f(2):=D \quad \text{for all} \quad n \in \mathbf{N}. \tag{4}$$

Thus, the last relation together with (1) implies that

$$f(n^2+m^2+3)=f(n^2+1)+f(m^2+1)+D \tag{5}$$

holds for all $n, m \in \mathbf{N}$. Let $S_j := f(j^2+1)$. It follows from (5) that if k, l, u and $v \in \mathbf{N}$ satisfy the condition

$$k^2+l^2=u^2+v^2,$$

then

$$f(k^2+1)+f(l^2+1)+D=f(u^2+1)+f(v^2+1)+D,$$

which shows that

$$k^2+l^2=u^2+v^2 \quad \text{implies} \quad S_k+S_l=S_u+S_v. \tag{6}$$

We shall prove that

$$S_{n+12}=S_{n+9}+S_{n+8}+S_{n+7}-S_{n+5}-S_{n+4}-S_{n+3}+S_n \tag{7}$$

holds for all $n \in \mathbf{N}$.

Since

$$(2j+1)^2+(j-2)^2=(2j-1)^2+(j+2)^2$$

and

$$(2j+1)^2+(j-7)^2=(2j-5)^2+(j+5)^2,$$

we get from (6) that

$$S_{2j+1}+S_{j-2}=S_{2j-1}+S_{j+2} \tag{8}$$

and

$$S_{2j+1}+S_{j-7}=S_{2j-5}+S_{j+5}.$$

These with (8) imply that

$$S_{j+5} - S_{j+2} + S_{j-2} - S_{j-7} = S_{2j-1} - S_{2j-5}$$
$$= S_{j+1} - S_{j-3} + S_{2j-3} - S_{2j-5} = S_{j+1} - S_{j-3} + S_j - S_{j-4},$$

which proves (7) with $n = j - 7$.

By (8), we have

$$S_7 = S_{2\cdot 3+1} = 2S_5 - S_1,$$

$$S_9 = S_{2\cdot 4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2\cdot 5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Finally, by using (6) and the facts

$$8^2 + 1^2 = 7^2 + 4^2, \quad 10^2 + 5^2 = 11^2 + 2^2 \quad \text{and} \quad 12^2 + 1^2 = 9^2 + 8^2,$$

we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1.$$

Thus, to complete the proof of the lemma , by using (1), (4), (5) and (7), it is enough to prove that either $S_1 = S_2 = S_3 = S_4 = S_5 = S_6 = 0$ or

$$S_j = j^2 + 1 \quad \text{for} \quad j = 1, 2, 3, 4, 5, 6. \tag{9}$$

Repeated use of (1), using the multiplicativity of f, gives $S_1 = f(1^2 + 1) = f(2)$,

$$S_2 = f(2^2 + 1) = f(5) = f(1^2 + 1^2 + 3) = f(2) + f(3), \tag{10}$$

$$S_3 = f(3^2 + 1) = f(10) = f(2)f(5) = f(2)^2 + f(2)f(3). \tag{11}$$

and thus

$$f(11) = f(2^2 + 2^2 + 3) = f(5) + f(6) = f(2) + f(3) + f(2)f(3).$$

On the other hand, it follows from (4) that

$$f(11)=f(3^2+2)=f(10)+D=f(2)f(5)+D$$
$$=f(2)^2+f(2)f(3)+f(3)-f(2),$$

which, together with the last relation, implies

$$f(2)^2=2f(2), \tag{12}$$

and

$$f(13)=f(1^2+3^2+3)=f(2)+f(11)=2f(2)+f(2)f(3)+f(3).$$

Finally, the relation (10) together with the fact

$$f(8)=f(1^2+2^2+3)=f(2)+f(6)=f(5)+f(3)$$

show that

$$f(2)f(3)=2f(3). \tag{13}$$

Moreover

$$S_5=f(5^2+1)=f(26)=f(2)f(13)=4f(2)+6f(3), \tag{14}$$

$$\begin{aligned} S_6&=f(6^2+1)=f(37)=f(3^2+5^2+3)\\ &=f(11)+f(26)=5f(2)+9f(3), \end{aligned} \tag{15}$$

$$2f(17)=f(4^2+4^2+3)-D=f(35)-D=f(5)f(7)-D, \tag{16}$$

and

$$f(3)f(7)=f(21)=f(3^2+3^2+3)=2f(10)+D=3f(2)+5f(3). \tag{17}$$

The equation (12) shows that either $f(2)=0$ or $f(2)=2$. Assume that $f(2)=0$. Then (13) implies that $f(3)=0$ and so, by using (10)–(17) we have

$$S_1=S_2=S_3=S_4=S_5=S_6=0,$$

from which follows that (2) is true.

Assume now that $f(2)=2$. In this case we have $f(5)=2+f(3)$, $f(8)=2+2f(3)$. We shall prove that $f(3)=3$. It follows from (15) and using the fact

$$f(37)=f(1^2+6^2+3)-f(3)=f(5)f(8)-f(3)=2f(3)^2+5f(3)+4$$

that

$$2f(3)^2-4f(3)-6=0. \tag{17}$$

On the other hand, from (4) we infer that

$$f(6)f(11)-f(3)f(13)=f(66)-f(65)=f(3)-f(2),$$

consequently

$$3f(3)^2-7f(3)-6=0.$$

This together with (17) proves that $f(3)=3$, and so (10)–(17) imply that

$$S_j=j^2+1 \quad (j=1,2,3,4,5,6).$$

This completes the proof of (9) and so the lemma is proved.

Proof of the theorem

In the proof of the theorem, using the lemma, we can assume that (3) is satisfied, that is

$$\begin{aligned} &f(n^2+1)=n^2+1, \quad f(m^2+2)=m^2+2 \quad \text{and} \\ &f(n^2+m^2+3)=n^2+m^2+3. \end{aligned} \tag{18}$$

It is clear from (18) that $f(n)=n$ for all $n\le 7$.

Assume that $f(n)=n$ for all $n<T$, where $T>7$. We shall prove that $f(T)=T$. It is clear that T must be a prime power, that is $T=q^\alpha$ with $\alpha\in\mathbf{N}$ and some prime q .

It is easily seen that if $\alpha=1$, then $q>7$ and there are positive integers $n,m\le\frac{q-1}{2}$ such that $n^2+m^2+3=qN$, $(q,N)=1$ and $N<q$. Thus, we have $f(q)=q$.

Assume now that $\alpha\ge 2$ and $q>3$. We consider the congruence

$$n^2+m^2+3\equiv 0 \pmod{q^\alpha}.$$

Let

$$\mathcal{A}_q(3):=\left\{1\le m\le q-1:\ \left(\frac{-m^2-3}{q}\right)=1\right\}.$$

Then we have

$$\begin{aligned}\sharp\mathcal{A}_q(3) = & \sum_{\substack{m=1\\(m^2+3,q)=1}}^{q-1} \frac{1}{2}\left(1+\left(\frac{-m^2-3}{q}\right)\right) = \sum_{m=0}^{q-1}\frac{1}{2}\left(1+\left(\frac{-m^2-3}{q}\right)\right)\\ & - \sum_{\substack{m=0\\q|m^2+3}}^{q-1} \frac{1}{2}\left(1+\left(\frac{-m^2-3}{q}\right)\right) - \frac{1}{2}\left(1+\left(\frac{-3}{q}\right)\right)\\ & = \frac{1}{2}\left(q-\left(\frac{-1}{q}\right)-2-2\left(\frac{-3}{q}\right)\right).\end{aligned}$$

By our assumption, the last relation implies that $\sharp\mathcal{A}_q(3) \geq 1$. Thus, there are integers $m \in \{1,\dots,q-1\}$, $1 \leq n_1 \leq q^\alpha - 1$, $(n_1, q) = 1$ and $1 \leq n_2 := q^\alpha - n_1 \leq q^\alpha - 1$ such that

$$n_i^2 + m^2 + 3 = q^\alpha N_i \quad (i = 1, 2).$$

It follows from the above relations that

$$q^\alpha(N_2 - N_1) = (q^\alpha - n_1)^2 - n_1^2 = q^{2\alpha} - 2q^\alpha n_1,$$

that is

$$N_2 - N_1 = q^\alpha - 2n_1.$$

Since $(n_1, q) = 1$, we obtain that at least one of N_1 or N_2 is coprime to q. Let $n \in \{n_1, n_2\}$ and $N \in \{N_1, N_2\}$ such that $n^2 + m^2 + 3 = q^\alpha N$, $(N, q) = 1$. Then $\alpha \geq 2$ implies that

$$N \leq \frac{1}{q^\alpha}\left[(q^\alpha - 1)^2 + (q-1)^2 + 3\right] < q^\alpha.$$

Thus,

$$Nf(q^\alpha) = f(N)f(q^\alpha) = f(Nq^\alpha) = f(n^2 + m^2 + 3) = n^2 + m^2 + 3 = Nq^\alpha,$$

which shows that $f(q^\alpha) = q^\alpha$ as we wanted to establish.

To complete the proof of the theorem, it remains to consider the cases $q = 2$ and $q = 3$. Let $q = 2$ and $T = 2^\alpha$, where $\alpha \geq 3$.

Since $-7 \equiv 1 \pmod 8$, we have -7 is a quadratic residue modulo 2^α and therefore there exists $n_\alpha \in [0, 2^{\alpha-1}-1]$ such that $n_\alpha^2+7 = n_\alpha^2+2^2+3 \equiv 0 \pmod{2^\alpha}$, and consequently, $[n_\alpha + 2^{\alpha-1}]^2 + 7 \equiv 0 \pmod{2^\alpha}$. Define N_1, and N_2 by $n_\alpha^2 + 7 = 2^\alpha N_1$ and $[n_\alpha + 2^{\alpha-1}]^2 + 7 = 2^\alpha N_2$. We easily deduce from these two equation and the fact $7 < T = 2^\alpha$ that

$$N_1 < 2^\alpha, \quad N_2 < 2^\alpha \quad \text{and} \quad N_2 - N_1 = n_\alpha + 2^{\alpha-2}.$$

It follows from the last relation and the fact 2 does not divide n_α that one of N_1 or N_2 is odd, and so $f(2^\alpha) = 2^\alpha$.

Finally, let $q = 3$ and $T = 3^\alpha$, where $\alpha > 1$. We consider the congruence

$$n^2 + 2 \equiv 0 \pmod{3^\alpha}.$$

Similarly as above, one can deduce that there are positive integers $n, N \in \mathbf{N}$ such that $n^2 + 2 = 3^\alpha N$, $(N, 3) = 1$ and $N < 3^\alpha$. Thus these together with (18) implies that $f(3^\alpha) = 3^\alpha$.

The theorem is proved.

References

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